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Contents

Part I

Higher Category Theory

Chapter 1

The Language of ∞-Categories

0001 A principal goal of algebraic topology is to understand topological spaces by means of algebraic and combinatorial invariants. Let us consider some elementary examples.

- To any topological space X, one can associate the set $\pi_0(X)$ of *path components* of X. This is the quotient of X by an equivalence relation $\simeq$, where $x \simeq y$ if there exists a continuous path $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.

- To any topological space X equipped with a base point $x \in X$, one can associate the *fundamental group* $\pi_1(X, x)$. This is a group whose elements are homotopy classes of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x = p(1)$.

For many purposes, it is useful to combine the set $\pi_0(X)$ and the fundamental groups $\{\pi_1(X,x)\}_{x \in X}$ into a single mathematical object. To any topological space X, one can associate an invariant $\pi_{\leqslant 1}(X)$ called the *fundamental groupoid of* X. The fundamental groupoid $\pi_{\leqslant 1}(X)$ is a category whose objects are the points of X, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a homotopy class of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$. The set of path components $\pi_0(X)$ can then be recovered as the set of isomorphism classes of objects of the category $\pi_{\leqslant 1}(X)$, and each fundamental group $\pi_1(X, x)$ can be identified with the automorphism group of the point x as an object of the category $\pi_{\leqslant 1}(X)$. The formalism of category theory allows us to assemble information about path components and fundamental groups into a single convenient package.

The fundamental groupoid $\pi_{\leqslant 1}(X)$ is a very important invariant of a topological space X, but is far from being a complete invariant. In particular, it does not contain any information about the *higher* homotopy groups $\{\pi_n(X,x)\}_{n \geqslant 2}$. We therefore ask the following:

0002 **Question 1.0.0.1.** Let X be a topological space. Can one devise a "category-theoretic" invariant of X, in the spirit of the fundamental groupoid $\pi_{\leqslant 1}(X)$, which contains information about *all* the homotopy groups of X?

We begin to address Question 1.0.0.1 in §1.1 by introducing the theory of *simplicial sets*. A simplicial set $S_\bullet$ is a collection of sets $\{S_n\}_{n \geqslant 0}$, which are related by *face maps* $\{d_i : S_n \to S_{n-1}\}_{0 \leqslant i \leqslant n}$ and *degeneracy maps* $\{s_i : S_n \to S_{n+1}\}_{0 \leqslant i \leqslant n}$ satisfying suitable identities (see Definition 1.1.1.9 and Exercise 1.1.1.8). Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$, called the *singular simplicial set* of X, with the property that each $\mathrm{Sing}_n(X)$ is the collection of continuous maps from the topological n-simplex into X (Construction 1.1.5.1). Moreover, the homotopy groups of X can be reconstructed from the simplicial set $\mathrm{Sing}_\bullet(X)$ by a simple combinatorial procedure (see §[?]). Kan observed that this procedure can be applied more generally to any simplicial set $S_\bullet$ satisfying the following *Kan extension condition*:

(∗) For $0 \leqslant i \leqslant n$, every map $\sigma_0 : \Lambda^n_i \to S_\bullet$ admits an extension $\sigma : \Delta^n \to S_\bullet$.

Here Δ^n denotes a certain simplicial set called the *standard n-simplex* (Construction 1.1.2.1), and Λ^n_i denotes a certain simplicial subset of Δ^n called the *ith horn* (Construction 1.1.2.9). Simplicial sets satisfying condition (∗) are called *Kan complexes*. Every simplicial set of the form $\mathrm{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.1.7.3), and the converse is true up to homotopy. More precisely, Milnor proved in [6] that the construction $X \mapsto \mathrm{Sing}_\bullet(X)$ induces an equivalence from the (geometrically defined) homotopy theory of CW complexes to the (combinatorially defined) homotopy theory of Kan complexes; we will discuss this point in Chapter [?].

The singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a natural candidate for sort of invariant requested in Question 1.0.0.1: it is a mathematical object of a purely combinatorial nature which contains complete information about the homotopy groups of X and their interrelationship (from which we can even reconstruct X up to homotopy equivalence, provided that X has the homotopy type of a CW complex). But in order to see that it qualifies as a complete answer, we must address the following:

Question 1.0.0.2. Let X be a topological space. To what extent does the simplicial set $\mathrm{Sing}_\bullet(X)$ behave like a category? What is the relationship between $\mathrm{Sing}_\bullet(X)$ with the fundamental groupoid of X? 0003

Our answer to Question 1.0.0.2 begins with the observation that the theory of simplicial sets is closely related to category theory. To every category $\mathcal{C}$, one can associate a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$, called the *nerve of* $\mathcal{C}$ (we will review the construction of $\mathrm{N}_\bullet(\mathcal{C})$ in §1.2; see Construction 1.2.1.1). The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ is fully faithful (Proposition 1.2.2.1): in particular, a category $\mathcal{C}$ is determined (up to canonical isomorphism) by the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Throughout much of this book, we will abuse notation by not distinguishing between a category $\mathcal{C}$ and its nerve $\mathrm{N}_\bullet(\mathcal{C})$: that is, we will view a category as a special kind of simplicial set. These simplicial sets have a simple characterization: according to

Proposition 1.2.3.1, a simplicial set $S_\bullet$ has the form $\mathrm{N}_\bullet(\mathcal{C})$ (for some category $\mathcal{C}$) if and only it it satisfies the following variant of the Kan extension condition (Proposition 1.2.3.1):

$(*')$ For $0 < i < n$, every map $\sigma_0 : \Lambda^n_i \to S_\bullet$ admits a unique extension $\sigma : \Delta^n \to S_\bullet$.

The extension conditions $(*)$ and $(*')$ are closely related, but differ in two important respects. The Kan extension condition requires that *every* map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ admits an extension $\sigma : \Delta^n \to S_\bullet$. Condition $(*')$ requires the existence of an extension only only in the case $0 < i < n$, but demands that the extension is unique. Neither of these conditions implies the other: a simplicial set of the form $\mathrm{N}_\bullet(\mathcal{C})$ satisfies condition $(*)$ if and only if the category $\mathcal{C}$ is a groupoid (Proposition 1.2.4.2), and a simplicial set of the form $\mathrm{Sing}_\bullet(X)$ satisfies condition $(*')$ if and only if every continuous path $[0,1] \to X$ is constant. However, conditions $(*)$ and $(*')$ admit a common generalization. We will say that a simplicial set $S_\bullet$ is an ∞-*category* if it satisfies the following variant of $(*)$ and $(*')$, known as the *weak Kan extension condition*:

$(*'')$ For $0 < i < n$, every map $\sigma_0 : \Lambda^n_i \to S_\bullet$ admits an extension $\sigma : \Delta^n \to S_\bullet$.

The theory of ∞-categories can be viewed as a simultaneous generalization of homotopy theory and category theory. Every Kan complex is an ∞-category, and every category $\mathcal{C}$ determines an ∞-category (given by the nerve $\mathrm{N}_\bullet(\mathcal{C})$). In particular, the notion of ∞-category answers the first part of Question 1.0.0.2: simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ are almost never (the nerves of) categories, but are always ∞-categories. At this point, the reader might reasonably object that this is terminological legerdemain: to address the spirit of Question 1.0.0.2, we must demonstrate that simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ (or, more generally, all simplicial sets satisfying condition $(*'')$) really *behave* like categories. We begin in §1.3 by explaining how to extend various elementary category-theoretic ideas to the setting of ∞-categories. In particular, we can associate to each ∞-category $S_\bullet$ a collection of *objects* (these are the elements of S_0), a collection of *morphisms* (these are the elements of S_1), and a composition law on morphisms. In particular, we show that any ∞-category $S_\bullet$ determines an ordinary category $\mathrm{h}S_\bullet$, called the *homotopy category of* $S_\bullet$ (Proposition 1.3.5.2). The construction of the homotopy category allows us to answer the second part of Question 1.0.0.2: for every topological space X, the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category, whose homotopy category $\mathrm{hSing}_\bullet(X)$ is the fundamental groupoid $\pi_{\leqslant 1}(X)$ (see Example 1.3.5.5).

Roughly speaking, the difference between an ∞-category $S_\bullet$ and its homotopy category $\mathrm{h}S_\bullet$ is that the former can contain nontrivial homotopy-theoretic information (encoded by simplices of dimension $n \geqslant 2$, which can be loosely understood as “n-morphisms”) which is lost upon passage to the homotopy category $\mathrm{h}S_\bullet$. We can summarize the situation informally with the heuristic equation

$$\{\text{Categories}\} + \{\text{Homotopy Theory}\} = \{\infty\text{-Categories}\},$$

or more precisely with the diagram

$$\begin{array}{ccccc} \{\text{Categories}\} & \xhookrightarrow{N_\bullet} & \{\infty\text{-Categories}\} & \supset & \{\text{Kan Complexes}\} \\ & & \cap & & \uparrow \text{Sing}_\bullet \\ & & \{\text{Simplicial Sets}\} & & \{\text{Topological Spaces}\} \end{array}$$

1.1 Simplicial Sets

For each integer $n \geqslant 0$, we let 0004

$$|\Delta^n| = \{(t_0, t_1, \dots, t_n) \in [0,1] : t_0 + t_1 + \cdots + t_n = 1\}$$

denote the topological simplex of dimension n. For any topological space X, we will refer to a continuous map $\sigma : |\Delta^n| \to X$ as a *singular n-simplex in X*. Every singular n-simplex σ determines a finite collection of singular $(n-1)$-simplices $\{d_i\sigma\}_{0 \leqslant i \leqslant n}$, called the *faces* of σ, which are given explicitly by the formula

$$(d_i\sigma)(t_0, \dots, t_{n-1}) = \sigma(t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

Let $\text{Sing}_n(X) = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$ denote the set of singular n-simplices of X. Many important algebraic invariants of X can be directly extracted from the sets $\{\text{Sing}_n(X)\}_{n \geqslant 0}$ and the face maps $\{d_i : \text{Sing}_n(X) \to \text{Sing}_{n-1}(X)\}_{0 \leqslant i \leqslant n}$.

Example 1.1.0.1 (Singular Homology)**.** For any topological space X, the *singular homology* 0005
groups $\text{H}_*(X; \mathbf{Z})$ are defined as the homology groups of a chain complex

$$\cdots \xrightarrow{\partial} \mathbf{Z}[\text{Sing}_2(X)] \xrightarrow{\partial} \mathbf{Z}[\text{Sing}_1(X)] \xrightarrow{\partial} \mathbf{Z}[\text{Sing}_0(X)],$$

where $\mathbf{Z}[\text{Sing}_n(X)]$ denotes the free abelian group generated by the set $\text{Sing}_n(X)$ and the differential is given on generators by the formula

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i d_i\sigma.$$

For some other algebraic invariants, it is convenient to keep track of a bit more structure. A singular n-simplex $\sigma : |\Delta^n| \to X$ also determines a collection of singular $(n+1)$-simplices $\{s_i\sigma\}_{0 \leqslant i \leqslant n}$, given by the formula

$$(s_i\sigma)(t_0, \dots, t_{n+1}) = \sigma(t_0, t_1, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}).$$

The resulting constructions $s_i : \text{Sing}_n(X) \to \text{Sing}_{n+1}(X)$ are called *degeneracy maps*, because singular $(n+1)$-simplices of the form $s_i\sigma$ factor through the linear projection $|\Delta^{n+1}| \to |\Delta^n|$. For example, the map $s_0 : \text{Sing}_0(X) \to \text{Sing}_1(X)$ carries each point $x \in X \simeq \text{Sing}_0(X)$ to the constant map $\underline{x} : |\Delta^1| \to X$ taking the value x.

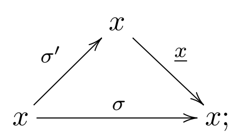
0006 **Example 1.1.0.2** (The Fundamental Group)**.** Let X be a topological space equipped with a base point $x \in X \simeq \mathrm{Sing}_0(X)$. Then continuous paths $p : [0,1] \to X$ satisfying $p(0) = x = p(1)$ can be identified with elements of the set $\{\sigma \in \mathrm{Sing}_1(X) : d_0(\sigma) = x = d_1(\sigma)\}$. The *fundamental group* $\pi_1(X, x)$ can then be described as the quotient

$$\{\sigma \in \mathrm{Sing}_1(X) : d_0(\sigma) = x = d_1(\sigma)\}/ \simeq,$$

where $\simeq$ is the equivalence relation on $\mathrm{Sing}_1(X)$ described by

$$(\sigma \simeq \sigma') \Leftrightarrow (\exists \tau \in \mathrm{Sing}_2(X))[d_0(\tau) = s_0(x) \text{ and } d_1(\tau) = \sigma \text{ and } d_2(\tau) = \sigma'].$$

The datum of a 2-simplex τ satisfying these conditions is equivalent to the datum of a continuous map $|\Delta^2| \to X$ with boundary behavior as indicated in the diagram



such a map can be identified with a homotopy between the paths determined by σ and σ'.

Motivated by the preceding examples, we can ask the following:

0007 **Question 1.1.0.3.** Given a topological space X, what can we say about the collection of sets $\{\mathrm{Sing}_n(X)\}_{n \geqslant 0}$, together with the face and degeneracy maps

$$d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X) \qquad s_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n+1}(X)?$$

What sort of mathematical structure do they form?

In [2], Eilenberg and Zilber supplied an answer to Question 1.1.0.3 by introducing what they called *complete semi-simplicial complexes*, which are now more commonly known as *simplicial sets*. Roughly speaking, a simplicial set $S_\bullet$ is a collection of sets $\{S_n\}_{n \geqslant 0}$ indexed by the nonnegative integers, equipped with face and degeneracy operators $\{d_i : S_n \to S_{n-1}, s_i : S_n \to S_{n+1}\}_{0 \leqslant i \leqslant n}$ satisfying a short list of identities. These identities can be summarized conveniently by saying that a simplicial set is a presheaf on the *simplex category* $\mathbf{\Delta}$, whose definition we review in §1.1.1.

Simplicial sets are connected to algebraic topology by two closely related constructions:

- For every topological space X, the face and degeneracy operators defined above endow the collection $\{\mathrm{Sing}_n(X)\}_{n \geqslant 0}$ with the structure of a simplicial set. We denote this simplicial set by $\mathrm{Sing}_\bullet(X)$ and refer to it as the *singular simplicial set of* X (see Construction 1.1.5.1). These simplicial sets tend to be quite large: in any nontrivial example, the sets $\mathrm{Sing}_n(X)$ will be uncountable for every nonnegative integer n.

- Any simplicial set $S_\bullet$ can be regarded as a "blueprint" for constructing a topological space $|S_\bullet|$ called the *geometric realization* of $S_\bullet$, which can be obtained as a quotient of the disjoint union $\coprod_{n \geqslant 0} S_n \times |\Delta^n|$ by an equivalence relation determined by the face and degeneracy operators on $S_\bullet$. Many topological spaces of interest (for example, any space which admits a finite triangulation) can be realized as a geometric realization of a simplicial set $S_\bullet$ having only finitely many nondegenerate simplices; we will discuss some elementary examples in §1.1.2.

These constructions determine adjoint functors

$$\mathrm{Set}_\Delta \underset{\mathrm{Sing}_\bullet}{\overset{|\,|}{\rightleftarrows}} \mathrm{Top}$$

relating the category Set_Δ of simplicial sets to the category Top of topological spaces. We review the constructions of these functors in §1.1.5 and §1.1.6, viewing them as instances of a general paradigm (Variant 1.1.5.3 and Proposition 1.1.6.18) which will appear repeatedly in Chapter [?].

For any (pointed) topological space X, Examples 1.1.0.1 and 1.1.0.2 show that the singular homology and fundamental group of X can be recovered from the simplicial set $\mathrm{Sing}_\bullet(X)$. In fact, one can say more: under mild assumptions, the entire homotopy type of X can be recovered from $\mathrm{Sing}_\bullet(X)$. More precisely, there is always a canonical map $|\mathrm{Sing}_\bullet(X)| \to X$ (given by the counit of the adjunction described above), and Giever proved that it is always a weak homotopy equivalence (hence a homotopy equivalence when X has the homotopy type of a CW complex). Consequently, for the purpose of studying homotopy theory, nothing is lost by replacing X by $\mathrm{Sing}_\bullet(X)$ and working in the setting of simplicial sets, rather than topological spaces. In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. However, not every simplicial set $S_\bullet$ behaves like the singular complex of a space; it is therefore necessary to single out a class of "good" simplicial sets to work with. In §1.1.7 we introduce a special class of simplicial sets, called *Kan complexes* (Definition 1.1.7.1). By a theorem of Milnor ([6]), the homotopy theory of Kan complexes is equivalent to the classical homotopy theory of CW complexes; we will return to this point in Chapter [?].

1.1.1 Simplicial and Cosimplicial Objects

We begin with some preliminaries. 0008

Notation 1.1.1.1. For every positive integer n, we let $[n]$ denote the linearly ordered set $\{0 < 1 < 2 < \cdots < n-1 < n\}$. 0009

Definition 1.1.1.2 (The Simplex Category)**.** Let $\mathbf{\Delta}$ denote the category whose objects are sets of the form $[n]$, where n is a nonnegative integer, where a morphism $\alpha : [m] \to [n]$ is 000A a nondecreasing map of linearly ordered sets (that is, a map which satisfies the condition $\alpha(i) \leqslant \alpha(j)$ whenever $i \leqslant j$). We refer to $\mathbf{\Delta}$ as *the simplex category*.

000B **Remark 1.1.1.3.** The category $\mathbf{\Delta}$ is equivalent to the category of *all* nonempty finite linearly ordered sets, with morphisms given by nondecreasing maps. In fact, we can say something even better: for every nonempty finite linearly ordered set I, there is a *unique* order-preserving bijection $I \simeq [n]$, for some $n \geqslant 0$.

000C **Definition 1.1.1.4.** Let $\mathcal{C}$ be any category. A *simplicial object of* $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. Dually, a *cosimplicial object of* $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

000D **Notation 1.1.1.5.** We will often use the expression $C_\bullet$ to denote a simplicial object of a category $\mathcal{C}$. In this case, we write C_n for the value of the functor $C_\bullet$ on the object $[n] \in \mathbf{\Delta}$. Similarly, we use the notation $C^\bullet$ to indicate a cosimplicial object of $\mathcal{C}$, and C^n for its value on $[n] \in \mathbf{\Delta}$.

To a first degree of approximation, a simplicial object $C_\bullet$ of a category $\mathcal{C}$ can be identified with the collection of objects $\{C_n\}_{n \geqslant 0}$. However, these objects are equipped with additional structure, arising from the morphisms in the simplex category $\mathbf{\Delta}$. We now spell this out more concretely.

000E **Notation 1.1.1.6.** Let n be a positive integer. For $0 \leqslant i \leqslant n$, we let $\delta^i : [n-1] \to [n]$ denote the unique strictly increasing function whose image does not contain the element i, given concretely by the formula

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i. \end{cases}$$

If $C_\bullet$ is a simplicial object of a category $\mathcal{C}$, then we can evaluate $C_\bullet$ on the morphism δ^i to obtain a morphism from C_n to C_{n-1}. We will denote this map by $d_i : C_n \to C_{n-1}$ and refer to it as the *ith face map.*

Dually, if $C^\bullet$ is a cosimplicial object of a category $\mathcal{C}$, then the evaluation on $C^\bullet$ on the morphism δ^i determines a map $d^i : C^{n-1} \to C^n$, which we refer to as the *ith coface map.*

000F **Notation 1.1.1.7.** For every pair of nonnegative integers i and j, we let $\sigma^i : [n+1] \to [n]$ denote the function given by the formula

$$\sigma^i(j) = \begin{cases} j & \text{if } j \leqslant i \\ j-1 & \text{if } j > i. \end{cases}$$

If $C_\bullet$ is a simplicial object of a category $\mathcal{C}$, then we can evaluate $C_\bullet$ on the morphism σ^i to obtain a morphism from C_n to C_{n+1}. We will denote this map by $s_i : C_n \to C_{n+1}$ and refer to it as the *ith degeneracy map.*

Dually, if $C^{\bullet}$ is a cosimplicial object of a category $\mathcal{C}$, then the evaluation on $C^{\bullet}$ on the morphism σ^i determines a map $s^i : C^{n+1} \to C^n$, which we refer to as the *ith codegeneracy map*.

Exercise 1.1.1.8. Let $C_{\bullet}$ be a simplicial object of a category $\mathcal{C}$. Show that the face and degeneracy maps of Notations 1.1.1.6 and 1.1.1.7 satisfy the *simplicial identities* 000G

(1) For $n \geqslant 2$ and $0 \leqslant i < j \leqslant n$, we have $d_i \circ d_j = d_{j-1} \circ d_i$ (as a map from C_n to C_{n-2}).

(2) For $0 \leqslant i \leqslant j \leqslant n$, we have $s_i \circ s_j = s_{j+1} \circ s_i$ (as a map from C_n to C_{n+2}).

(3) For $0 \leqslant i, j \leqslant n$, we have

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \mathrm{id}_{C_n} & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1. \end{cases}$$

Conversely, show that any collection of objects $\{C_n\}_{n \geqslant 0}$ and maps $\{d_i : C_n \to C_{n-1}\}_{0 \leqslant i \leqslant n}$, $\{s_i : C_n \to C_{n+1}\}_{0 \leqslant i \leqslant n}$ satisfying (1), (2), and (3) determines a (unique) simplicial object of $\mathcal{C}$.

We will be primarily interested in the following special case of Definition 1.1.1.4:

Definition 1.1.1.9. Let Set denote the category of sets. A *simplicial set* is a simplicial object of Set: that is, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. If $S_{\bullet}$ is a simplicial set, then we will refer to elements of S_n as *n-simplices of $S_{\bullet}$*. We will also refer to the elements of S_0 as *vertices of $S_{\bullet}$*, and to the elements of S_1 as *edges* of $S_{\bullet}$. 000H

We let $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ denote the category of functors from $\mathbf{\Delta}^{\mathrm{op}}$ to Set. We refer to Set_{Δ} as *the category of simplicial sets*.

Remark 1.1.1.10. Since the category of sets has all (small) limits and colimits, the category of simplicial sets also has all (small) limits and colimits. Moreover, these limits and colimits are computed levelwise: for any functor 000J

$$S_{\bullet} : \mathcal{C} \to \mathrm{Set}_{\Delta} \qquad (C \in \mathcal{C}) \mapsto S_{\bullet}(C),$$

and any nonnegative integer n, we have canonical bijections

$$(\varinjlim_{C \in \mathcal{C}} S)_n(C) \simeq \varinjlim_{C \in \mathcal{C}} (S_n(C)) \qquad (\varprojlim_{C \in \mathcal{C}} S)_n)(C) \simeq \varprojlim_{C \in \mathcal{C}} (S_n(C)).$$

1.1.2 Simplices and Horns

000K We now consider some elementary examples of simplicial sets.

000L **Construction 1.1.2.1** (The Standard Simplex)**.** Let $n \geqslant 0$ be an integer. We let Δ^n denote the simplicial set given by the construction

$$([m] \in \mathbf{\Delta}) \mapsto \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]).$$

We will refer to Δ^n as the *standard n-simplex*. By convention, we extend this construction to the case $n = -1$ by setting $\Delta^{-1} = \varnothing$.

000M **Example 1.1.2.2.** The standard 0-simplex Δ^0 is a final object of the category of simplicial sets: that is, it carries each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ to a set having a single element.

000N **Remark 1.1.2.3.** For each $n \geqslant 0$, the standard n-simplex Δ^n is characterized by the following universal property: for every simplicial set $X_\bullet$, Yoneda's lemma supplies a bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X_\bullet) \simeq X_n.$$

We will often invoke this bijection implicitly to identify n-simplices of $X_\bullet$ with maps of simplicial sets $\sigma : \Delta^n \to X_\bullet$.

000P **Remark 1.1.2.4.** Let $S_\bullet$ be a simplicial set. Suppose that, for every integer $n \geqslant 0$, we are given a subset $T_n \subseteq S_n$, and that the face and degeneracy maps

$$d_i : S_n \to S_{n-1} \qquad s_i : S_n \to S_{n+1}$$

carry T_n into T_{n-1} and T_{n+1}, respectively. Then the collection $\{T_n\}_{n \geqslant 0}$ inherits the structure of a simplicial set $T_\bullet$. In this case, we will say that $T_\bullet$ is a *simplicial subset* of $S_\bullet$ and write $T_\bullet \subseteq S_\bullet$.

000Q **Example 1.1.2.5.** Let $S_\bullet$ be a simplicial set and let v be a vertex of $S_\bullet$. Then v can be identified with a map of simplicial sets $\Delta^0 \to S_\bullet$. This map is automatically a monomorphism (note that Δ^0 has only a single n-simplex for every $n \geqslant 0$), whose image is a simplicial subset of $S_\bullet$. It will often be convenient to denote this simplicial subset by $\{v\}$. For example, we can identify vertices of the standard n-simplex Δ^n with integers i satisfying $0 \leqslant i \leqslant n$; every such integer i determines a simplicial subset $\{i\} \subseteq \Delta^n$ (whose k-simplices are the constant maps $[k] \to [n]$ taking the value i).

It will be useful to consider some other simplicial subsets of the standard n-simplex.

Construction 1.1.2.6 (The Boundary of Δ^n)**.** Let $n \geqslant 0$ be an integer. We define a simplicial set $(\partial \Delta^n) : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ by the formula 000R

$$(\partial \Delta^n)([m]) = \{\alpha \in \mathrm{Hom}_{\mathbf{\Delta}}([m], [n]) : \alpha \text{ is not surjective}\}.$$

Note that we can regard $\partial \Delta^n$ as a simplicial subset of the standard n-simplex Δ^n of Construction 1.1.2.1. We will refer to $\partial \Delta^n$ as the *boundary of* Δ^n.

Example 1.1.2.7. The simplicial set $\partial \Delta^0$ is empty. 000S

Exercise 1.1.2.8. Let $n \geqslant 0$ be an integer. For $0 \leqslant j \leqslant n$, the map $\delta^j : [n-1] \to [n]$ of Notation 1.1.1.6 determines a map of simplicial sets $\Delta^{n-1} \to \Delta^n$ which factors through the simplicial subset $\partial \Delta^n \subseteq \Delta^n$. We therefore obtain a map of simplicial sets $\Delta^{n-1} \to \partial \Delta^n$, which we will also denote by δ^j. Show that, for any simplicial set $S_\bullet$, the construction 000T

$$(f : \partial \Delta^n \to S_\bullet) \mapsto \{f \circ \delta^j\}_{0 \leqslant j \leqslant n}$$

determines an injective map

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^n, S_\bullet) \to \prod_{j \in [n]} S_{n-1},$$

whose image is the collection of sequences of $(n-1)$-simplices $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ satisfying the identities $d_j(\sigma_k) = d_{k-1}(\sigma_j)$ for $0 \leqslant j < k \leqslant n$.

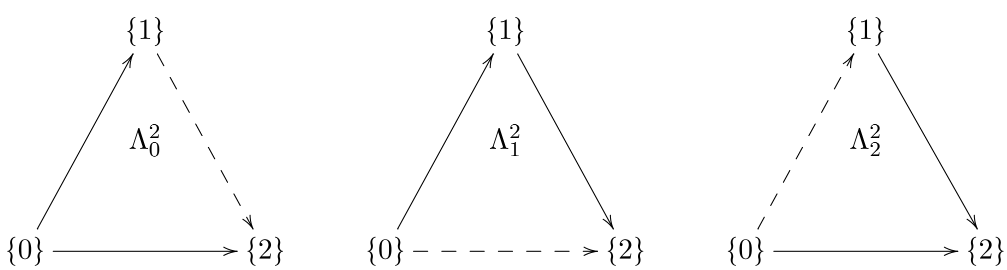
Construction 1.1.2.9 (The Horn Λ^n_i)**.** Suppose we are given a pair of integers $0 \leqslant i \leqslant n$. We define a simplicial set $\Lambda^n_i : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ by the formula 000U

$$(\Lambda^n_i)([m]) = \{\alpha \in \mathrm{Hom}_{\mathbf{\Delta}}([m], [n]) : [n] \nsubseteq \alpha([m]) \cup \{i\}\}.$$

We regard Λ^n_i as a simplicial subset of the boundary $\partial \Delta^n \subseteq \Delta^n$. We will refer to Λ^n_i as the *ith horn in* Δ^n. We will say that Λ^n_i is an *inner horn* if $0 < i < n$, and an *outer horn* if $i = 0$ or $i = n$.

Remark 1.1.2.10. Roughly speaking, one can think of the horn Λ^n_i as obtained from the n-simplex Δ^n by removing its interior together with the face opposite its ith vertex (see Example 1.1.6.13). 000V

Example 1.1.2.11. The horns contained in Δ^2 are depicted in the following diagram: 000W



Here the dotted arrows indicate edges of Δ^2 which are not contained in the corresponding horn.

000X **Example 1.1.2.12.** The horns Λ^1_0 and Λ^1_1 are both isomorphic to Δ^0, and the inclusion maps $\Lambda^1_0 \hookrightarrow \partial\Delta^1 \hookleftarrow \Lambda^1_1$ induce an isomorphism $\Delta^0 \amalg \Delta^0 \simeq \partial\Delta^1$.

000Y **Example 1.1.2.13.** The horn Λ^0_0 coincides with the 0-simplex Δ^0.

000Z **Exercise 1.1.2.14.** Let $0 \leqslant i \leqslant n$ be integers. For $j \in [n]\backslash\{i\}$, we can regard the map δ^j of Exercise 1.1.2.8 as a map of simplicial sets from Δ^{n-1} to the horn $\Lambda^n_i \subseteq \Delta^n$. Show that, for any simplicial set $S_\bullet$, the construction

$$(f : \partial\Delta^n \to S_\bullet) \mapsto \{f \circ \delta^j\}_{j \in [n]\backslash\{i\}}$$

determines an injection $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S_\bullet) \to \prod_{j\in[n]\backslash\{i\}} S_{n-1}$, whose image consists of 'incomplete" sequences $(\sigma_0, \ldots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \ldots, \sigma_n)$ satisfying $d_j(\sigma_k) = d_{k-1}(\sigma_j)$ for $j, k \in [n]\backslash\{i\}$ with $j < k$.

1.1.3 The Skeletal Filtration

0010 Roughly speaking, one can think of the simplicial sets Δ^n of Construction 1.1.2.1 as elementary building blocks out of which more complicated simplicial sets can be constructed. In this section, we make this idea more precise by introducing the *skeletal filtraton* of a simplicial set. This filtration allows us to write every simplicial set $S_\bullet$ as the union of an increasing sequence of simplicial subsets

$$\mathrm{sk}_0(S_\bullet) \subseteq \mathrm{sk}_1(S_\bullet) \subseteq \mathrm{sk}_2(S_\bullet) \subseteq \mathrm{sk}_3(S_\bullet) \subseteq \cdots,$$

where each $\mathrm{sk}_n(S_\bullet)$ is obtained from $\mathrm{sk}_{n-1}(S_\bullet)$ by attaching copies of Δ^n (see Proposition 1.1.3.11 below for a precise statement). We will need some preliminaries.

0011 **Proposition 1.1.3.1.** *Let $S_\bullet$ be a simplicial set and let $\sigma \in S_n$ be an n-simplex of $S_\bullet$ for some $n > 0$, which we will identify with a map of simplicial sets $\sigma : \Delta^n \to S_\bullet$. The following conditions are equivalent:*

(1) *The simplex σ belongs to the image of the degeneracy map $s_i : S_{n-1} \to S_n$ for some $0 \leqslant i \leqslant n-1$ (see Notation 1.1.1.7).*

(2) *The map σ factors as a composition $\Delta^n \xrightarrow{f} \Delta^{n-1} \to S_\bullet$, where f corresponds to a surjective map of linearly ordered sets $[n] \to [n-1]$.*

(3) *The map σ factors as a composition $\Delta^n \xrightarrow{f} \Delta^m \to S_\bullet$, where $m < n$ and f corresponds to a surjective map of linearly ordered sets $[n] \to [m]$.*

(4) *The map σ factors as a composition $\Delta^n \to \Delta^m \to S_\bullet$, where $m < n$.*

Proof. The implications (1) ⇔ (2) ⇒ (3) ⇒ (4) are immediate. We will complete the proof by showing that (4) implies (1). Assume that σ factors as a composition $\Delta^n \xrightarrow{f} \Delta^m \xrightarrow{\sigma'} S_\bullet$, where $m < n$. Let us abuse notation by identifying f with a map of linearly ordered sets $[n] \to [m]$. Since $m < n$, this map cannot be injective. It follows that we can find some $i < n$ such that $f(i) = f(i+1)$. It follows that f factors through the map $\sigma^i : [n] \to [n-1]$ of Notation 1.1.1.7, so that σ belongs to the image of the degeneracy map s_i. □

Definition 1.1.3.2. Let $S_\bullet$ be a simplicial set and let $\sigma : \Delta^n \to S_\bullet$ be an n-simplex of $S_\bullet$. We will say that σ is *degenerate* if $n > 0$ and σ satisfies the equivalent conditions of Proposition 1.1.3.1. We say that σ is *nondegenerate* if it is not degenerate (in particular, every 0-simplex of $S_\bullet$ is nondegenerate). 0012

Remark 1.1.3.3. Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets. If σ is a degenerate n-simplex of $S_\bullet$, then $f(\sigma)$ is a degenerate n-simplex of $T_\bullet$. The converse holds if f is a monomorphism of simplicial sets (for example, if $S_\bullet$ is a simplicial subset of $T_\bullet$). 0013

Proposition 1.1.3.4. *Let $\sigma : \Delta^n \to S_\bullet$ be a map of simplicial sets. Then σ can be factored as a composition* 0014

$$\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet,$$

where α corresponds to a surjective map of linearly ordered sets $[n] \to [m]$ and τ is a nondegenerate m-simplex of $S_\bullet$. Moreover, this factorization is unique.

Proof. Let m be the smallest nonnegative integer for which σ can be factored as a composition $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet$. It follows from the minimality of m that α must induce a surjection of linearly ordered sets $[n] \to [m]$ (otherwise, we could replace $[m]$ by the image of α) and that the m-simplex τ is nondegenerate. This proves the existence of the desired factorization.

To establish uniqueness, let us suppose we are given another factorization of σ as a composition $\Delta^n \xrightarrow{\alpha'} \Delta^{m'} \xrightarrow{\tau'} S_\bullet$. By assumption, α and α' determine surjections of linearly ordered sets $[n] \to [m]$ and $[n] \to [m']$, and therefore admit sections which we will denote by β and β', respectively. The equality $\sigma = \tau \circ \alpha$ then gives

$$\tau = \sigma \circ \beta = \tau' \circ \alpha' \circ \beta.$$

Our assumption that τ is nondegenerate then guarantees that the map $\alpha' \circ \beta : [m] \to [m']$ is injective, so that $m \leqslant m'$. The same argument shows that $m' \leqslant m$, so we must have $m = m'$. Since the only nondecreasing injection from $[m]$ to itself is the identity map, we conclude that $\alpha' \circ \beta = \mathrm{id}_{[m]}$. The desired uniqueness now follows from the calculations

$$\tau = \tau' \circ \alpha' \circ \beta = \tau' \qquad \alpha = \alpha' \circ \beta \circ \alpha = \alpha'.$$

□

0015 **Construction 1.1.3.5.** Let $S_\bullet$ be a simplicial set, let $k \geqslant -1$ be an integer, and let $\sigma : \Delta^n \to S_\bullet$ be an n-simplex of $S_\bullet$. The proof of Proposition 1.1.3.4 shows that the following conditions are equivalent:

(a) Let $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet$ be the factorization of Proposition 1.1.3.4 (so that α induces a surjection $[n] \to [m]$, the map τ is nondegenerate, and $\sigma = \tau \circ \alpha$). Then $m \leqslant k$.

(b) There exists a factorization $\Delta^n \to \Delta^{m'} \to S_\bullet$ of σ for which $m' \leqslant k$.

For each $n \geqslant 0$, we let $\mathrm{sk}_k(S_n)$ denote the subset of S_n consisting of those n-simplices which satisfy conditions (a) and (b). From characterization (b), we see that the collection of subsets $\{\mathrm{sk}_k(S_n) \subseteq S_n\}_{n \geqslant 0}$ is stable under the face and degeneracy operators of $S_\bullet$, and therefore determine a simplicial subset of $S_\bullet$ (Remark 1.1.2.4). We will denote this simplicial subset by $\mathrm{sk}_k(S_\bullet)$ and refer to it as the *k-skeleton of $S_\bullet$*.

0016 **Remark 1.1.3.6.** Let $S_\bullet$ be a simplicial set and let $k \geqslant -1$. If $n \geqslant k$, then $\mathrm{sk}_k(S_\bullet)$ contains every n-simplex of $S_\bullet$. In particular, the union $\bigcup_{k \geqslant -1} \mathrm{sk}_k(S_\bullet)$ is equal to $S_\bullet$.

0017 **Remark 1.1.3.7.** Let $S_\bullet$ be a simplicial set and let σ be a *nondegenerate* n-simplex of $S_\bullet$. Then σ is contained in $\mathrm{sk}_n(S_\bullet)$ if and only if $n \geqslant k$.

0018 **Example 1.1.3.8.** For any simplicial set $S_\bullet$, the (-1)-skeleton $\mathrm{sk}_{-1}(S_\bullet)$ is empty.

We now show that the k-skeleton of a simplicial set $S_\bullet$ can be characterized by a universal property.

0019 **Definition 1.1.3.9.** Let $S_\bullet$ be a simplicial set and let $k \geqslant -1$ be an integer. We will say that $S_\bullet$ *has dimension* $\leqslant k$ if, for $n > k$, every n-simplex of $S_\bullet$ is degenerate. If $k \geqslant 0$, we say that $S_\bullet$ *has dimension k* if it has dimension $\leqslant k$ but does not have dimension $(k-1)$.

001A **Proposition 1.1.3.10.** *Let $S_\bullet$ be a simplicial set and let $k \geqslant -1$ be an integer. Then:*

(a) *The simplicial set* $\mathrm{sk}_k(S_\bullet)$ *has dimension* $\leqslant k$.

(b) *For every simplicial set $T_\bullet$ of dimension $\leqslant k$, composition with the inclusion map* $\mathrm{sk}_k(S_\bullet) \hookrightarrow S_\bullet$ *induces a bijection*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(T_\bullet, \mathrm{sk}_k(S_\bullet)) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(T_\bullet, S_\bullet).$$

In other words, the image of any map $T_\bullet \to S_\bullet$ is contained in $\mathrm{sk}_k(S_\bullet)$.

Proof. Assertion (a) follows from Remark 1.1.3.7. To prove (b), suppose that $f : T_\bullet \to S_\bullet$ is a map of simplicial sets, where $T_\bullet$ has dimension $\leqslant k$. We wish to show that f carries every n-simplex σ of $T_\bullet$ to an n-simplex of $\mathrm{sk}_k(S_\bullet)$. Using Proposition 1.1.3.4, we can reduce to the case where σ is a nondegenerate n-simplex of $T_\bullet$. In this case, our assumption that $T_\bullet$ has dimension $\leqslant k$ guarantees that $n \leqslant k$, so that $f(\sigma)$ belongs to $\mathrm{sk}_k(S_\bullet)$ by virtue of Remark 1.1.3.6. □

Let $S_\bullet$ be a simplicial set. For each $k \geqslant 0$, we let S_k^{nd} denote the collection of all *nondegenerate* k-simplices of $S_\bullet$. Every element $\sigma \in S_k^{\mathrm{nd}}$ determines a map of simplicial sets $\Delta^k \to \mathrm{sk}_k(S_\bullet)$. Since the boundary $\partial \Delta^k \subseteq \Delta^k$ has dimension $\leqslant k-1$, this map carries $\partial \Delta^k$ into the $(k-1)$-skeleton $\mathrm{sk}_{k-1}(S_\bullet)$.

Proposition 1.1.3.11. *Let $S_\bullet$ be a simplicial set and let $k \geqslant 0$. Then the construction outlined above determines a pushout square* 001B

$$\begin{array}{ccc} \coprod_{\sigma \in S_k^{\mathrm{nd}}} \partial \Delta^k & \longrightarrow & \coprod_{\sigma \in S_k^{\mathrm{nd}}} \Delta^k \\ \downarrow & & \downarrow \\ \mathrm{sk}_{k-1}(S_\bullet) & \longrightarrow & \mathrm{sk}_k(S_\bullet) \end{array}$$

in the category Set_Δ *of simplicial sets.*

Proof. Unwinding the definitions, we must prove the following:

(∗) Let τ be an n-simplex of $\mathrm{sk}_k(S_\bullet)$ which is not contained in $\mathrm{sk}_{k-1}(S_\bullet)$. Then τ factors uniquely as a composition

$$\Delta^n \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} S_\bullet,$$

where σ is a nondegenerate simplex of $S_\bullet$ and α does not factor through the boundary $\partial \Delta^k$ (in other words, α induces a surjection of linearly ordered sets $[n] \to [k]$).

Proposition 1.1.3.4 implies that *any* n-simplex of $S_\bullet$ admits a unique factorization $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\sigma} S_\bullet$, where α is surjective and σ is nondegenerate. Our assumption that τ belongs to the $\mathrm{sk}_k(S_\bullet)$ guarantees that $m \leqslant k$, and our assumption that τ does not belong to $\mathrm{sk}_{k-1}(S_\bullet)$ guarantees that $m \geqslant k$. □

Example 1.1.3.12 (Simplicial Sets of Dimension $\leqslant 0$)**.** Let $S_\bullet$ be a simplicial set. Then the 0-skeleton $\mathrm{sk}_0(S_\bullet)$ can be identified with the coproduct $\coprod_{v \in S_0} \{v\}$, indexed by the collection of all vertices of $S_\bullet$. In particular, the simplicial set $S_\bullet$ has dimension $\leqslant 0$ if and only if it is isomorphic to a coproduct of copies of Δ^0. We therefore obtain an equivalence of categories 001C

$$\{\text{Simplicial Sets of Dimension} \leqslant 0\} \simeq \{\text{Sets}\}.$$

1.1.4 Directed Graphs as Simplicial Sets

We now generalize Example 1.1.3.12 to obtain a concrete description of simplicial sets having dimension $\leqslant 1$ (Proposition 1.1.4.9). 001D

Definition 1.1.4.1. A *directed graph* G consists of the following data: 001E

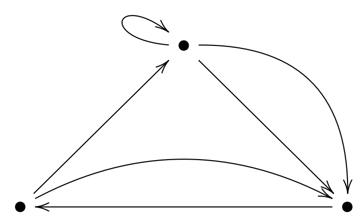
- A set $\mathrm{Vert}(G)$, whose elements we refer to as *vertices of G*.

- A set $\mathrm{Edge}(G)$, whose elements we refer to as *edges of G*.

- A pair of functions $s, t : \mathrm{Edge}(G) \to \mathrm{Vert}(G)$ which assign to each edge $e \in \mathrm{Edge}(G)$ a pair of vertices $s(e)$, $t(e) \in \mathrm{Vert}(G)$ that we refer to as the *source* and *target* of e, respectively.

001F **Warning 1.1.4.2.** The terminology of Definition 1.1.4.1 is not standard. Note that a directed graph G can have distinct edges $e \neq e'$ having the same source $s(e) = s(e')$ and target $t(e) = t(e')$ (for this reason, directed graphs in the sense of Definition 1.1.4.1 are sometimes called *multigraphs*). Definition 1.1.4.1 also allows graphs which contain loops: that is, edges e satisfying $s(e) = t(e)$.

001G **Remark 1.1.4.3.** It will sometimes be convenient to represent a directed graph G by a diagram, having a node for each vertex v of G and an arrow for each edge e of G, directed from the source of e to the target of e. For example, the diagram



represents a directed graph with three vertices and six edges.

001H **Example 1.1.4.4.** To every simplicial set $X_\bullet$, we can associate a directed graph $G(X_\bullet)$ as follows:

- The vertex set $\mathrm{Vert}(G(X_\bullet))$ is the set of 0-simplices of the simplicial set $X_\bullet$.

- The edge set $\mathrm{Edge}(G(X_\bullet))$ is the set of *nondegenerate* 1-simplices of the simplicial set $X_\bullet$.

- For every edge $e \in \mathrm{Edge}(G(X_\bullet)) \subseteq S_1$, the source $s(e)$ is the vertex $d_1(e)$, and the target $t(e)$ is the vertex $d_0(e)$ (here $d_0, d_1 : S_1 \to S_0$ are the face maps of Notation 1.1.1.6).

It will be convenient to construe Example 1.1.4.4 as providing a functor from the category of simplicial sets to the category of directed graphs. First, we need an appropriate definition for the latter category.

001J **Definition 1.1.4.5.** Let G and G' be directed graphs (in the sense of Definition 1.1.4.1). A *morphism* from G to G' is a function $f : \mathrm{Vert}(G) \amalg \mathrm{Edge}(G) \to \mathrm{Vert}(G') \amalg \mathrm{Edge}(G')$ which satisfies the following conditions:

(a) For each vertex $v \in \mathrm{Vert}(G)$, the image $f(v)$ belongs to $\mathrm{Vert}(G')$.

(b) Let $e \in \mathrm{Edge}(G)$ be an edge of G with source $v = s(e)$ and target $w = t(e)$. Then exactly one of the following conditions holds:

- The image $f(e)$ is an edge of G' having source $s(f(e)) = f(v)$ and target $t(f(e)) = f(w)$.
- The image $f(e)$ is a vertex of G' satisfying $f(v) = f(e) = f(w)$.

We let Graph denote the category whose objects are directed graphs and whose morphisms are morphisms of directed graphs (with composition defined in the evident way).

Warning 1.1.4.6. Note that part (b) of Definition 1.1.4.5 allows the possibility that a morphism of directed graphs $G \to G'$ can "collapse" edges of G to vertices of G'. Many other notions of morphism between (directed) graphs appear in the literature; we single out Definition 1.1.4.5 because of its close connection with the theory of simplicial sets (see Proposition 1.1.4.7 below). 001K

Let $X_\bullet$ be a simplicial set and let $G(X_\bullet)$ be the directed graph of Example 1.1.4.4. Then the disjoint union $\mathrm{Vert}(G(X_\bullet)) \amalg \mathrm{Edge}(G(X_\bullet))$ can be identified with the set X_1 of *all* 1-simplices of $X_\bullet$ (where we identify $\mathrm{Vert}(G(X_\bullet))$ with the collection of degenerate 1-simplices via the degeneracy map $s_0 : X_0 \to X_1$).

Proposition 1.1.4.7. *Let $f : X_\bullet \to Y_\bullet$ be a map of simplicial sets. Then the induced map* 001L

$$\mathrm{Vert}(G(X_\bullet)) \amalg \mathrm{Edge}(G(X_\bullet)) \simeq X_1 \xrightarrow{f} Y_1 \simeq \mathrm{Vert}(G(Y_\bullet)) \amalg \mathrm{Edge}(G(Y_\bullet))$$

is a morphism of directed graphs from $G(X_\bullet)$ to $G(Y_\bullet)$, in the sense of Definition 1.1.4.5.

Proof. Since f commutes with the degeneracy operator s_0, it carries degenerate 1-simplices of $X_\bullet$ to degenerate 1-simplices of $Y_\bullet$, and therefore satisfies requirement (a) of Definition 1.1.4.5. Requirement (b) follows from the fact that f commutes with the face operators d_0 and d_1. □

It follows from Proposition 1.1.4.7 that we can regard the construction $X_\bullet \mapsto G(X_\bullet)$ as a functor from the category Set_Δ of simplicial sets to the category Graph of directed graphs.

Proposition 1.1.4.8. *Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. If $X_\bullet$ has dimension $\leqslant 1$, then the canonical map* 001M

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(X_\bullet, Y_\bullet) \to \mathrm{Hom}_{\mathrm{Graph}}(G(X_\bullet), G(Y_\bullet))$$

is bijective.

Proof. If $X_\bullet$ has dimension $\leqslant 1$, then Proposition 1.1.3.11 provides a pushout diagram

$$\begin{array}{ccc} \coprod_{e \in \mathrm{Edge}(G(X_\bullet))} \partial \Delta^1 & \longrightarrow & \coprod_{e \in \mathrm{Edge}(G(X_\bullet))} \Delta^1 \\ \downarrow & & \downarrow \\ \coprod_{v \in \mathrm{Vert}(G(X_\bullet))} & \longrightarrow & X_\bullet . \end{array}$$

It follows that, for any simplicial set $Y_\bullet$, we can identify $\mathrm{Hom}_{\mathrm{Set}_\Delta}(X_\bullet, Y_\bullet)$ with the fiber product

$$(\prod_{e \in \mathrm{Edge}(G(X_\bullet))} Y_1) \times_{\prod_{e \in \mathrm{Edge}(G(X_\bullet))}(Y_0 \times Y_0)} (\prod_{v \in \mathrm{Vert}(G(X_\bullet))} Y_0),$$

which is precisely the set of morphisms of directed graphs from $G(X_\bullet)$ to $G(Y_\bullet)$. □

It follows from Proposition 1.1.4.8 that the theory of simplicial sets of dimension $\leqslant 1$ is essentially equivalent to the theory of directed graphs.

001N **Proposition 1.1.4.9.** *Let* Set_Δ *denote the category of simplicial sets and let* $\mathcal{C} \subseteq \mathrm{Set}_\Delta$ *denote the full subcategory spanned by the simplicial sets of dimension* $\leqslant 1$. *Then the construction* $X_\bullet \mapsto G(X_\bullet)$ *induces an equivalence of categories* $\mathcal{C} \to \mathrm{Graph}$.

Proof. It follows from Proposition 1.1.4.9 that the functor $X_\bullet \mapsto G(X_\bullet)$ is fully faithful when restricted to simplicial sets of dimension $\leqslant 1$. It will therefore suffice to show that is essentially surjective. Let G be any directed graph, and form a pushout diagram of simplicial sets

$$\begin{array}{ccc} \coprod_{e \in \mathrm{Edge}(G)} \partial \Delta^1 & \longrightarrow & \coprod_{e \in \mathrm{Edge}(G)} \Delta^1 \\ \downarrow{\scriptstyle s,t} & & \downarrow \\ \coprod_{v \in \mathrm{Vert}(G)} & \longrightarrow & X_\bullet . \end{array}$$

Then $X_\bullet$ is a simplicial set of dimension $\leqslant 1$, and the directed graph $G(X_\bullet)$ is isomorphic to G. □

001P **Remark 1.1.4.10.** The proof of Proposition 1.1.4.9 gives an explicit description of the inverse equivalence $\mathrm{Graph} \simeq \mathcal{C} \hookrightarrow \mathrm{Set}_\Delta$: it carries a directed graph G to the 1-dimensional simplicial set given by the pushout

$$(\coprod_{v \in \mathrm{Vert}(G)}) \amalg_{\coprod_{e \in \mathrm{Edge}(G)} \partial \Delta^1} (\coprod_{e \in \mathrm{Edge}(G)} \Delta^1).$$

1.1.5 The Singular Simplicial Set of a Topological Space

Topology provides an abundant supply of examples of simplicial sets. 001Q

Construction 1.1.5.1. Let X be a topological space. We define a simplicial set $\mathrm{Sing}_\bullet(X)$ as follows: 001R

- To each object $[n] \in \mathbf{\Delta}$, we assign the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of singular n-simplices in X.
- To each non-decreasing map $\alpha : [m] \to [n]$, we assign the map $\mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$ given by precomposition with the continuous map

$$|\Delta^m| \to |\Delta^n|$$
$$(t_0, t_1, \ldots, t_m) \mapsto (\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, \ldots, \sum_{\alpha(i)=n} t_i).$$

We will refer to $\mathrm{Sing}_\bullet(X)$ as the *singular simplicial set of* X. We view the construction $X \mapsto \mathrm{Sing}_\bullet(X)$ as a functor from the category of topological spaces to the category of simplicial sets, which we will denote by $\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$.

Example 1.1.5.2. Let X be a topological space and let $\mathrm{Sing}_\bullet(X)$ be its singular simplicial set. The vertices of $\mathrm{Sing}_\bullet(X)$ can be identified with points of X. The edges of $\mathrm{Sing}_\bullet(X)$ can be identified with continuous paths $p : [0,1] \to X$. 001S

It will be convenient to consider a generalization of Construction 1.1.5.1.

Variant 1.1.5.3. Let $\mathcal{C}$ be any category and let $Q^\bullet$ be a cosimplicial object of $\mathcal{C}$, which we view as a functor $Q : \mathbf{\Delta} \to \mathcal{C}$. For every object $X \in \mathcal{C}$, the construction $([n] \in \mathbf{\Delta}) \mapsto \mathrm{Hom}_{\mathcal{C}}(Q([n]), X)$ determines a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets, which we can view as a simplicial set. We will denote this simplicial set by $\mathrm{Sing}^Q_\bullet(X)$, so that we have canonical bijections $\mathrm{Sing}^Q_n(X) \simeq \mathrm{Hom}_{\mathcal{C}}(Q^n, X)$. We view the construction $X \mapsto \mathrm{Sing}^Q_\bullet(X)$ as a functor from the category $\mathcal{C}$ to the category of simplicial sets, which we denote by $\mathrm{Sing}^Q_\bullet : \mathcal{C} \to \mathrm{Set}_\Delta$. 001T

Example 1.1.5.4. The construction $[n] \mapsto |\Delta^n|$ determines a functor from the simplex category $\mathbf{\Delta}$ to the category Top of topological spaces, which assigns to each morphism $\alpha : [m] \to [n]$ the continuous map 001U

$$|\Delta^m| \to |\Delta^n| \qquad (t_0, \ldots, t_m) \mapsto (\sum_{\alpha(i)=0} t_i, \ldots, \sum_{\alpha(i)=n} t_i).$$

We regard this functor as a cosimplicial topological space, which we denote by $|\Delta^\bullet|$. Applying Variant 1.1.5.3 to this cosimplicial space yields a functor $\mathrm{Sing}^{|\Delta|}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$, which coincides with the singular simplicial set functor $\mathrm{Sing}_\bullet$ of Construction 1.1.5.1.

001V **Example 1.1.5.5.** The construction $[n] \mapsto \Delta^n$ determines a functor from the simplex category $\mathbf{\Delta}$ to the category $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets (this is the *Yoneda embedding* for the simplex category $\mathbf{\Delta}$). We regard this functor as a cosimplicial object of Set_{Δ}, which we denote by $\Delta^{\bullet}$. Applying Variant 1.1.5.3 to this cosimplicial object, we obtain a functor from the category of simplicial sets to itself, which is canonically isomorphic to the identity functor $\mathrm{id}_{\mathrm{Set}_{\Delta}} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ (see Remark 1.1.2.3).

001W **Remark 1.1.5.6.** The cosimplicial space $|\Delta^{\bullet}|$ of Example 1.1.5.4 can be described more informally as follows:

- To each nonempty finite linearly ordered set I, it assigns a topological simplex $|\Delta^I|$ whose vertices are the elements of I: that is, the convex hull of the set I inside the real vector space $\mathbf{R}[I]$ generated by I.
- To every nondecreasing map $\alpha : I \to J$, the induced map $|\Delta^I| \to |\Delta^J|$ is given by the restriction of the $\mathbf{R}$-linear map $\mathbf{R}[I] \to \mathbf{R}[J]$ determined by α. Equivalently, it is the unique affine map which coincides with α on the vertices of the simplex $|\Delta^I|$.

1.1.6 The Geometric Realization of Simplicial Set

001X Let X be a topological space. By definition, n-simplices of the simplicial set $\mathrm{Sing}_{\bullet}(X)$ are continuous maps $|\Delta^n| \to X$. This observation determines a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X) \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{Sing}_{\bullet}(X)).$$

We now consider a generalization of this construction, which can be applied to simplicial sets other than Δ^n.

001Y **Definition 1.1.6.1.** Let $S_{\bullet}$ be a simplicial set and let Y be a topological space. We will say that a map of simplicial sets $u : S_{\bullet} \to \mathrm{Sing}_{\bullet}(Y)$ *exhibits Y as a geometric realization of* $S_{\bullet}$ if, for every topological space X, the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(Y), \mathrm{Sing}_{\bullet}(X)) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S_{\bullet}, \mathrm{Sing}_{\bullet}(X))$$

is bijective.

001Z **Example 1.1.6.2.** For each $n \geqslant 0$, the identity map $\mathrm{id} : |\Delta^n| \simeq |\Delta^n|$ determines an n-simplex of the simplicial set $\mathrm{Sing}_{\bullet}(|\Delta^n|)$, which we can identify with a map of simplicial sets $\Delta^n \to \mathrm{Sing}_{\bullet}(|\Delta^n|)$ which exhibits $|\Delta^n|$ as a geometric realization of Δ^n.

0020 **Notation 1.1.6.3.** Let $S_{\bullet}$ be a simplicial set. It follows immediately from the definitions that if there exists a map $u : S_{\bullet} \to \mathrm{Sing}_{\bullet}(Y)$ which exhibits Y as a geometric realization of $S_{\bullet}$, then the topological space Y is determined up to homeomorphism and depends functorially on $S_\bullet$. We will emphasize this dependence by writing $|S_\bullet|$ to denote a geometric realization of $S_\bullet$ (by virtue of Example 1.1.6.2, this is compatible with our existing notation in the case where $S_\bullet$ is the standard n-simplex).

Every simplicial set admits a geometric realization:

Proposition 1.1.6.4. *For every simplicial set $S_\bullet$, there exists a topological space Y and a map $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ which exhibits Y as a geometric realization of $S_\bullet$.* 0021

Corollary 1.1.6.5. *The singular simplicial set functor $\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$ admits a left adjoint, given by the geometric realization construction $S_\bullet \mapsto |S_\bullet|$.* 0022

Our starting point is the following formal observation:

Lemma 1.1.6.6. *Let $\mathcal{J}$ be a small category equipped with a functor $F : \mathcal{J} \to \mathrm{Set}_\Delta$, which we will denote by $(J \in \mathcal{J}) \mapsto (F(J)_\bullet \in \mathrm{Set}_\Delta)$. Let $S_\bullet = \varinjlim_{J \in \mathcal{J}} F(J)_\bullet$ be a colimit of F. If each of the simplicial sets $F(J)_\bullet$ admits a geometric realization $|F(J)_\bullet|$, then $S_\bullet$ also admits a geometric realization, given by the colimit $Y = \varinjlim_{J \in \mathcal{J}} |F(J)_\bullet|$.* 0023

Proof. For each $J \in \mathcal{J}$, choose a map $u_J : F(J)_\bullet \to \mathrm{Sing}_\bullet(|F(J)_\bullet|)$ which exhibits $|F(J)_\bullet|$ as a geometric realization of $F(J)_\bullet$. We can then amalgamate the composite maps

$$F(I)_\bullet \xrightarrow{u_I} \mathrm{Sing}_\bullet(|F(I)_\bullet|) \to \mathrm{Sing}_\bullet(Y)$$

to a single map of simplicial sets $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$. We claim that u exhibits Y as a geometric realization of the simplicial set $S_\bullet$. Let X be any topological space; we wish to show that the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sing}_\bullet(Y), \mathrm{Sing}_\bullet(X)) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}_\bullet(X))$$

is bijective. This is clear, since this composite map can be written as an inverse limit of the bijections $\mathrm{Hom}_{\mathrm{Top}}(|F(J)_\bullet|, X) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(F(J)_\bullet, \mathrm{Sing}_\bullet(X))$ determined by u_J. □

It is possible to deduce Proposition 1.1.6.4 and Corollary 1.1.6.5 in a completely formal way from Lemma 1.1.6.6, since every simplicial set can be presented as a colimit of simplices (see Proposition 1.1.6.18 below). However, we will instead give a less direct argument which yields some additional information about the structure of the topological spaces $|S_\bullet|$. We begin by studying simplicial subsets of the standard simplex Δ^n.

Notation 1.1.6.7. Let $n \geqslant 0$ be an integer and let $\mathcal{U}$ be a collection of nonempty subsets of $[n] = \{0, 1, \ldots, n\}$. We will say that $\mathcal{U}$ is *downward closed* if $\emptyset \neq I \subseteq J \in \mathcal{U}$ implies that $I \in \mathcal{U}$. If this condition is satisfied, we let $\Delta^n_{\mathcal{U}}$ denote the simplicial subset of Δ^n whose m-simplices are nondecreasing maps $\alpha : [m] \to [n]$ for which the image of α is an element of $\mathcal{U}$. Similarly, we set 0024

$$|\Delta^n|_{\mathcal{U}} = \{(t_0, \ldots, t_n) \in |\Delta^n| : \{i \in [n] : t_i \neq 0\} \in \mathcal{U}\}.$$

0025 **Example 1.1.6.8.** For each $n \geqslant 0$, the boundary $\partial \Delta^n$ of Construction 1.1.2.6 is given by $\Delta^n_{\mathcal{U}}$, where $\mathcal{U}$ is the collection of all nonempty proper subsets of $[n]$.

0026 **Example 1.1.6.9.** For $0 \leqslant i \leqslant n$, the horn Λ^n_i of Construction 1.1.2.9 is given by $\Delta^n_{\mathcal{U}}$, where $\mathcal{U}$ is the collection of all nonempty subsets of $[n]$ which are distinct from $[n]$ and $[n] \backslash \{i\}$.

0027 **Exercise 1.1.6.10.** Show that every simplicial subset of the standard n-simplex Δ^n has the form $\Delta^n_{\mathcal{U}}$, where $\mathcal{U}$ is some (uniquely determined) downward closed collection of nonempty subsets of $[n]$.

0028 **Proposition 1.1.6.11.** *Let n be a nonnegative integer and let $\mathcal{U}$ be a downward closed collection of nonempty subsets of $[n]$. Then the canonical map $\Delta^n \to \operatorname{Sing}_\bullet(|\Delta^n|)$ restricts to a map of simplicial sets $f_{\mathcal{U}} : \Delta^n_{\mathcal{U}} \to \operatorname{Sing}_\bullet(|\Delta^n|_{\mathcal{U}})$, which exhibits the topological space $|\Delta^n|_{\mathcal{U}}$ as a geometric realization of $\Delta^n_{\mathcal{U}}$.*

Proof. We proceed by induction on the cardinality of $\mathcal{U}$. If $\mathcal{U}$ is empty, then the simplicial set $\Delta^n_{\mathcal{U}}$ and the topological space $|\Delta^n|_{\mathcal{U}}$ are both empty, in which case there is nothing to prove. We may therefore assume that $\mathcal{U}$ is nonempty. Choose some $S \in \mathcal{U}$ whose cardinality is as large as possible. Set

$$\mathcal{U}_0 = \mathcal{U} \backslash \{S\} \qquad \mathcal{U}_1 = \{T \subseteq S : T \neq \varnothing\} \qquad \mathcal{U}_{01} = \mathcal{U}_0 \cap \mathcal{U}_1 .$$

Our inductive hypothesis implies that the maps $f_{\mathcal{U}_0}$ and $f_{\mathcal{U}_{01}}$ exhibit $|\Delta^n|_{\mathcal{U}_0}$ and $|\Delta^n|_{\mathcal{U}_{01}}$ as geometric realizations of $\Delta^n_{\mathcal{U}_0}$ and $\Delta^n_{\mathcal{U}_{01}}$, respectively. Moreover, if $S = \{i_0 < i_1 < \cdots < i_m\} \subseteq [n]$, then we can identify $f_{\mathcal{U}_1}$ with the tautological map $\Delta^m \to \operatorname{Sing}_\bullet(|\Delta^m|)$, so that $f_{\mathcal{U}_1}$ exhibits $|\Delta^n|_{\mathcal{U}_1}$ as a geometric realization of $\Delta^n_{\mathcal{U}_1}$ by virtue of Example 1.1.6.2. It follows immediately from the definitions that the diagram of simplicial sets

$$\begin{array}{ccc} \Delta^n_{\mathcal{U}_{01}} & \longrightarrow & \Delta^n_{\mathcal{U}_0} \\ \downarrow & & \downarrow \\ \Delta^n_{\mathcal{U}_1} & \longrightarrow & \Delta^n_{\mathcal{U}} \end{array}$$

is a pushout square. By virtue of Lemma 1.1.6.6, we are reduced to proving that the diagram of topological spaces

$$\begin{array}{ccc} |\Delta^n|_{\mathcal{U}_{01}} & \longrightarrow & |\Delta^n|_{\mathcal{U}_0} \\ \downarrow & & \downarrow \\ |\Delta^n|_{\mathcal{U}_1} & \longrightarrow & |\Delta^n|_{\mathcal{U}} \end{array}$$

is also a pushout square. This is clear, since $|\Delta^n|_{\mathcal{U}_0}$ and $|\Delta^n|_{\mathcal{U}_1}$ are closed subsets of $|\Delta^n|$ whose union is $|\Delta^n|_{\mathcal{U}}$ and whose intersection is $|\Delta^n|_{\mathcal{U}_{01}}$. □

Example 1.1.6.12. Let n be a nonnegative integer. Combining Example 1.1.6.8 with 0029
Proposition 1.1.6.11, we see that the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ induces a homeomorphism from $|\partial \Delta^n|$ to the *boundary* of the topological n-simplex $|\Delta^n|$, given by

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j\}.$$

Example 1.1.6.13. Let $0 \leqslant i \leqslant n$. Combining Example 1.1.6.9 with Proposition 1.1.6.11, 002A
we see that the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ induces a homeomorphism from $|\Lambda^n_i|$ to the subset of $|\Delta^n|$ given by

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\}.$$

Proof of Proposition 1.1.6.4. Let $S_\bullet$ be a simplicial set. We first show that for each $n \geqslant -1$, the skeleton $\mathrm{sk}_n(S_\bullet)$ admits a geometric realization. The proof proceeds by induction on n, the case $n = -1$ being trivial (since $\mathrm{sk}_{-1}(S_\bullet)$ is empty). Let S^{nd}_n denote the collection of nondegenerate n-simplices of S. we note that Proposition 1.1.3.11 provides a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma \in S^{\mathrm{nd}}_n} \partial \Delta^n & \longrightarrow & \coprod_{\sigma \in S^{\mathrm{nd}}_n} \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{sk}_{n-1}(S_\bullet) & \longrightarrow & \mathrm{sk}_n(S_\bullet). \end{array}$$

Combining our inductive hypothesis, Example 1.1.6.2, Example 1.1.6.12, and Lemma 1.1.6.6, we deduce that $\mathrm{sk}_n(S_\bullet)$ admits a geometric realization $|\mathrm{sk}_n(S_\bullet)|$ which fits into a pushout diagram of topological spaces

$$\begin{array}{ccc} \coprod_{\sigma \in S^{\mathrm{nd}}_n} |\partial \Delta^n| & \longrightarrow & \coprod_{\sigma \in S^{\mathrm{nd}}_n} |\Delta^n| \\ \downarrow & & \downarrow \\ |\mathrm{sk}_{n-1}(S_\bullet)| & \longrightarrow & |\mathrm{sk}_n(S_\bullet)|. \end{array}$$

Combining the equality $S_\bullet = \bigcup_n \mathrm{sk}_n(S_\bullet)$ of Remark 1.1.3.6 with Lemma 1.1.6.6, we deduce that the simplicial set $S_\bullet$ also admits a geometric realization, given by the direct limit $\varinjlim_n |\mathrm{sk}_n(S_\bullet)|$. □

Remark 1.1.6.14. The proof of Proposition 1.1.6.4 shows that the geometric realization 002B
$|S_\bullet|$ of a simplicial set $S_\bullet$ has a canonical realization as a CW complex, having one cell of dimension n for each nondegenerate n-simplex σ of $S_\bullet$; this cell is can be described explicitly as the image of the map

$$|\Delta^n| \backslash |\partial \Delta^n| \hookrightarrow |\Delta^n| \xrightarrow{\sigma} |S_\bullet|.$$

The proof of Proposition 1.1.6.4 also yields the following fact, which we will use repeatedly throughout this book:

002C **Lemma 1.1.6.15.** *Let $\mathcal{U}$ be a full subcategory of the category* Set_Δ *of simplicial sets. If $\mathcal{U}$ is closed under small colimits and contains the standard n-simplex Δ^n for each $n \geqslant 0$, then* $\mathcal{U} = \mathrm{Set}_\Delta$.

002D **Remark 1.1.6.16.** We can state Lemma 1.1.6.15 more informally as follows: the category Set_Δ of simplicial sets is generated, under small colimits, by objects of the form Δ^n. In fact, one can say more: it is *freely* generated (under small colimits) by the essential image of the Yoneda embedding

$$\mathbf{\Delta} \hookrightarrow \mathrm{Set}_\Delta \qquad [n] \mapsto \Delta^n.$$

This is a general fact about presheaf categories, which we will return to in §[?].

We give two proofs of Lemma 1.1.6.15: one using the strategy of Proposition 1.1.6.4, and another using the formal properties of presheaf categories.

First Proof of Lemma 1.1.6.15. Set $S_\bullet$ be a simplicial set; we wish to show that $S_\bullet$ belongs to $\mathcal{U}$. By virtue of Remark 1.1.3.6, we can identify $S_\bullet$ with the colimit $\varinjlim_n \mathrm{sk}_n(S_\bullet)$. Consequently, it will suffice to show that each skeleton $\mathrm{sk}_n(S_\bullet)$ belongs to $\mathcal{U}$. We may therefore assume without loss of generality that $S_\bullet$ has dimension $\leqslant n$, for some integer n. We proceed by induction on n, the case $n = -1$ being trivial (note that the empty simplicial set is the colimit of an empty diagram). To carry out the inductive step, we invoke Proposition 1.1.3.11 to choose a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma \in S_n^{\mathrm{nd}}} \partial \Delta^n & \longrightarrow & \coprod_{\sigma \in S_n^{\mathrm{nd}}} \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{sk}_{n-1}(S_\bullet) & \longrightarrow & S_\bullet. \end{array}$$

It will therefore suffice to show that the simplicial sets $|\,\mathrm{sk}_{n-1}(S_\bullet)|$, $\coprod_{\sigma \in S_n^{\mathrm{nd}}} |\,\partial \Delta^n\,|$, and $\coprod_{\sigma \in S_n^{\mathrm{nd}}} |\Delta^n|$ belong to $\mathcal{U}$. In the first two cases, this follows from our inductive hypothesis. In the third, it follows from our assumption that Δ^n belongs to $\mathcal{U}$ and that $\mathcal{U}$ is closed under coproducts. □

Second Proof of Lemma 1.1.6.15. Let $S_\bullet$ be a simplicial set. We define a category $\mathbf{\Delta}_S$ as follows:

- The objects of $\mathbf{\Delta}_S$ are pairs $([n], \sigma)$, where $[n]$ is an object of $\mathbf{\Delta}$ and σ is an n-simplex of $S_\bullet$.

- A morphism from $([n], \sigma)$ to $([n'], \sigma')$ in the category $\mathbf{\Delta}_S$ is a nondecreasing function $f : [n] \to [n']$ with the property that the induced map $S_{n'} \to S_n$ carries σ' to σ.

Via the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{Set}_{\Delta}$, we can identify $\mathbf{\Delta}_S$ with the category whose objects are simplicial sets of the form Δ^n (for some $n \geqslant 0$), which are equipped with a map of simplicial sets $\Delta^n \to S_\bullet$. In particular, we have a canonical map of simplicial sets $\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_S} \Delta^n \to S_\bullet$. To prove Lemma 1.1.6.15, it suffices to observe that this map is an isomorphism. This is an elementary calculation which we leave to the reader (see §[?] for more details). □

Remark 1.1.6.17. Each of our proofs of Lemma 1.1.6.15 gives additional information that the other does not. Our first proof shows that every simplicial set $S_\bullet$ can be built as a colimit of standard simplices in a very specific way: namely, by forming pushouts along boundary inclusions $\partial \Delta^n \hookrightarrow \Delta^n$ (for a more precise assertion, see the proof of Proposition 1.4.5.12). This extra information was used in the proof of Proposition 1.1.6.4 to show that the geometric realization $|S_\bullet|$ is a CW complex (and not merely a topological space which is colimit of disks). On the other hand, our second proof shows that every simplicial set $S_\bullet$ can be built in a single step as the colimit of a diagram of standard simplices (which can be chosen in a specific, canonical way). 002E

In Chapter [?], we will encounter a number of variants of the geometric realization construction $S_\bullet \mapsto |S_\bullet|$, which arise as special cases of the following:

Proposition 1.1.6.18. *Let $\mathcal{C}$ be a category, let $Q^\bullet$ be a cosimplicial object of $\mathcal{C}$, and let $\mathrm{Sing}^Q_\bullet : \mathcal{C} \to \mathrm{Set}_\Delta$ be the functor of Variant 1.1.5.3. If the category $\mathcal{C}$ admits small colimits, then the functor $\mathrm{Sing}^Q_\bullet$ admits a left adjoint $\mathrm{Set}_\Delta \to \mathcal{C}$, which we will denote by $S_\bullet \mapsto |S_\bullet|^Q$.* 002F

Proof. Let us say that a simplicial set $S_\bullet$ is *good* if the functor

$$(C \in \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}^Q_\bullet(C))$$

is corepresentable by an object of the category $\mathcal{C}$ (in which case we denote the corepresenting object by $|S_\bullet|^Q$). It follows from Yoneda's lemma that the standard n-simplex Δ^n is good for each $n \geqslant 0$, with $|\Delta^n|^Q \simeq Q^n$. If $\mathcal{C}$ admits small colimits, then the proof of Lemma 1.1.6.6 shows that the collection of good simplicial sets is closed under small colimits. It now suffices to observe that every simplicial set $S_\bullet$ can be written as a small colimit of simplices (Lemma 1.1.6.15). □

1.1.7 Kan Complexes

We close this section by articulating an important property enjoyed by simplicial sets of the form $\mathrm{Sing}_\bullet(X)$. 002G

Definition 1.1.7.1. Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is a *Kan complex* if it satisfies the following condition: 002H

($*$) For $0 \leqslant i \leqslant n$, any map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$. Here $\Lambda^n_i \subseteq \Delta^n$ denotes the ith horn (see Construction 1.1.2.9).

002J **Exercise 1.1.7.2.** Show that for $n > 0$, the standard simplex Δ^n is not a Kan complex (for a more general statement, see Proposition 1.2.4.2).

002K **Proposition 1.1.7.3.** *Let X be a topological space. Then the singular simplicial set* $\mathrm{Sing}_\bullet(X)$ *is a Kan complex.*

Proof. Let $\sigma_0 : \Lambda^n_i \to \mathrm{Sing}_\bullet(X)$ be a map of simplicial sets; we wish to show that σ_0 can be extended to an n-simplex of X. Using the geometric realization functor, we can identify σ_0 with a continuous map of topological spaces $f_0 : |\Lambda^n_i| \to X$; we wish to show that f_0 factors as a composition

$$|\Lambda^n_i| \to |\Delta^n| \xrightarrow{f} X.$$

Using Example 1.1.6.13, we can identify $|\Lambda^n_i|$ with the subset

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\} \subseteq |\Delta^n|.$$

In this case, we can take f to be the composition $f_0 \circ r$, where r is any continuous retraction of $|\Delta^n|$ onto the subset $|\Lambda^n_i|$. For example, we can take r to be the map given by the formula

$$r(t_0, \ldots, t_n) = (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \ldots, t_n - c)$$

$$c = \min\{t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}.$$

□

1.2 The Nerve of a Category

002L In §1.1, we reviewed the theory of simplicial sets and its relationship to the theory of topological spaces. Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$ (Construction 1.1.5.1), and simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ have a special property: they are Kan complexes (Proposition 1.1.7.3). In this section, we will study a different class of simplicial sets, which arise instead from the theory of categories. In §1.2.1, we associate to every category $\mathcal{C}$ a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$, called the *nerve of* $\mathcal{C}$. We show in §1.2.2 that the construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ is fully faithful (Proposition 1.2.2.1). In §1.2.3, we show that a simplicial set $S_\bullet$ belongs to the essential image of the functor $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ if and only if it satisfies a certain lifting condition (Proposition 1.2.3.1). This lifting condition is similar to the Kan extension condition (Definition 1.1.7.1), but not identical to it: in §1.2.4, we show that a simplicial set of the form $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex if and only if every morphism in $\mathcal{C}$ is invertible (Proposition 1.2.4.2).

1.2.1 Construction of the Nerve

We begin with a few definitions. 002M

Construction 1.2.1.1. For every integer $n \geqslant 0$, let us view the linearly ordered set $[n] = \{0 < 1 < \cdots < n-1 < n\}$ as a category (where there is a unique morphism from i to j when $i \leqslant j$). For any category $\mathcal{C}$, we let $\mathrm{N}_n(\mathcal{C})$ denote the set of all functors from $[n]$ to $\mathcal{C}$. Note that for any nondecreasing map $\alpha : [m] \to [n]$, precomposition with α determines a map of sets $\mathrm{N}_n(\mathcal{C}) \to \mathrm{N}_m(\mathcal{C})$. We can therefore view the construction $[n] \mapsto \mathrm{N}_n(\mathcal{C})$ as a simplicial set. We will denote this simplicial set by $\mathrm{N}_\bullet(\mathcal{C})$ and refer to it as the *nerve of* $\mathcal{C}$. 002N

Remark 1.2.1.2 (The Classifying Space of a Category)**.** Let $\mathcal{C}$ be a category. Then the topological space $|\mathrm{N}_\bullet(\mathcal{C})|$ is called the *classifying space* of the category $\mathcal{C}$. 002P

Remark 1.2.1.3. Let $\mathcal{C}$ be a category and let $n \geqslant 0$. Elements of $\mathrm{N}_n(\mathcal{C})$ are given by 002Q

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$. In other words, we can identify elements of $\mathrm{N}_n(\mathcal{C})$ with n-tuples $(f_1, \ldots, f_n)$ of morphisms of $\mathcal{C}$ having the property that, for $0 < i < n$, the domain of f_{i+1} coincides with the codomain of f_i.

Example 1.2.1.4. Let $\mathcal{C}$ be a category. Then: 002R

- Vertices of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with objects of the category $\mathcal{C}$.
- Edges of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with morphisms in the category $\mathcal{C}$.
- Let $f : X \to Y$ be a morphism in $\mathcal{C}$, regarded as an edge of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Then the faces of f are given by the codomain $d_0 f = Y$ and the domain $d_1 f = X$, respectively.
- Let X be an object of $\mathcal{C}$, which we regard as a vertex of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Then the degenerate edge $s_0(X)$ is the identity morphism $\mathrm{id}_X : X \to X$.

Remark 1.2.1.5 (Face Operators on $\mathrm{N}_\bullet(\mathcal{C})$)**.** Let $\mathcal{C}$ be a category and suppose we are given an n-simplex σ of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ for some $n > 0$, which we identify with a diagram 002S

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n.$$

Then:

- The 0th face $d_0\sigma \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

  $$C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3 \to \cdots \xrightarrow{f_n} C_n$$

  obtained from σ by "deleting" the object C_0 (and the morphism f_1 with domain C_0).

- The nth face $d_n\sigma \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

$$C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_{n-2} \xrightarrow{f_{n-1}} C_{n-1}$$

obtained from σ by "deleting" the object C_n (and the morphism f_n with codomain f_n).

- For $0 < i < n$, the ith face $d_i\sigma \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

$$C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_{i-1} \xrightarrow{f_{i+1}\circ f_i} C_{i+1} \to \cdots \xrightarrow{f_n} C_n$$

obtained by "deleting" the object C_i (and composing the morphisms f_i and f_{i+1}).

002T **Remark 1.2.1.6** (Degeneracy Operators on $\mathrm{N}_\bullet(\mathcal{C})$)**.** Let $\mathcal{C}$ be a category and suppose we are given an n-simplex σ of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ which we identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n.$$

Then, for $0 \leqslant i \leqslant n$, we can identify $s_i\sigma \in \mathrm{N}_{n+1}(\mathcal{C})$ with the diagram

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \to \cdots \xrightarrow{f_n} C_n$$

obtained from σ by "inserting" the identity morphism id_{C_i}.

002U **Remark 1.2.1.7.** Let $\mathcal{C}$ be a category and let σ be an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$, corresponding to a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n.$$

Then σ is degenerate (Definition 1.1.3.2) if and only if some f_i is an identity morphism of $\mathcal{C}$ (in which case we must have $C_{i-1} = C_i$).

002V **Remark 1.2.1.8.** Let I be a set equipped with a partial ordering $\leqslant_I$. Then we can regard I as a category whose objects are the elements of I, with morphisms given by

$$\mathrm{Hom}_I(i,j) = \begin{cases} * & \text{if } i \leqslant_I j \\ \varnothing & \text{otherwise.} \end{cases}$$

We will denote the nerve of this category by $\mathrm{N}_\bullet(I)$, and refer to it as the *nerve of the partially ordered set* I. For each $n \geqslant 0$, we can identify n-simplices of $\mathrm{N}_\bullet(I)$ with monotone functions $[n] \to I$: that is, with nondecreasing sequences $(i_0 \leqslant_I i_1 \leqslant_I \cdots \leqslant_I i_n)$ of elements of I.

002W **Example 1.2.1.9.** For each $n \geqslant 0$, the nerve $\mathrm{N}_\bullet([n])$ can be identified with the standard n-simplex Δ^n of Construction 1.1.2.1.

Remark 1.2.1.10. The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ determines a functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ 002X
from the category Cat of (small) categories to the category Set_Δ of simplicial sets. This is a special case of the construction described in Variant 1.1.5.3. More precisely, we can identify $\mathrm{N}_\bullet$ with the functor $\mathrm{Sing}^Q_\bullet$, where $Q : \mathbf{\Delta} \to \mathrm{Cat}$ is the functor which carries each object $[n] \in \mathbf{\Delta}$ to itself, regarded as a category. It follows from Proposition 1.1.6.18 that this functor admits a left adjoint, which we will study in §1.3.6.

1.2.2 Recovering a Category from its Nerve

Passage from a category $\mathcal{C}$ to the nerve $\mathrm{N}_\bullet(\mathcal{C})$ does not lose any information: 002Y

Proposition 1.2.2.1. *The nerve functor* $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ *is fully faithful.* *002Z*

Throughout this book, we will often abuse terminology by identifying a category $\mathcal{C}$ with its nerve $\mathrm{N}_\bullet(\mathcal{C})$. By virtue of Proposition 1.2.2.1, this is essentially harmless: the nerve construction allows us to identify categories with certain kinds of simplicial sets.

Proof of Proposition 1.2.2.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. We wish to show that the nerve functor $\mathrm{N}_\bullet$ induces a bijection

$$\theta : \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{C}')).$$

Here the domain of θ is the *set* of all functors from $\mathcal{C}$ to $\mathcal{C}'$. We first note that θ is injective: a functor $F : \mathcal{C} \to \mathcal{C}'$ is determined by its behavior on the objects and morphisms of $\mathcal{C}$, and therefore by the behavior of $\theta(F)$ on the vertices and edges of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ (see Example 1.2.1.4). Let us prove the surjectivity of θ. Let $f : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C}')$ be a morphism of simplicial sets; we wish to show that there exists a functor $F : \mathcal{C} \to \mathcal{C}'$ such that $f = \theta(F)$. For each $n \geqslant 0$, the morphism f determines a map of sets $\mathrm{N}_n(\mathcal{C}) \to \mathrm{N}_n(\mathcal{C}')$, which we will also denote by f. In the case $n = 0$, this map carries each object $C \in \mathcal{C}$ to an object of $\mathcal{C}'$, which we will denote by $F(C)$. For every pair of objects $C, D \in \mathcal{C}$, the map f carries each morphism $u : C \to D$ to a morphism $f_u)$ in the category $\mathcal{C}'$. Since f commutes with face maps, the morphism $f(u)$ has domain $F(C)$ and codomain $F(D)$ (see Example 1.2.1.4), and can therefore be regarded as an element of $\mathrm{Hom}_{\mathcal{C}'}(F(C), F(D))$; we denote this element by $F(u)$. We will complete the proof by verifying the following:

(a) The preceding construction determines a functor $F : \mathcal{C} \to \mathcal{C}'$.

(b) We have an equality $f = \theta(F)$ of maps from $\mathrm{N}_\bullet(\mathcal{C})$ to $\mathrm{N}_\bullet(\mathcal{C}')$.

To prove (a), we first note that the compatibility of f with degeneracy maps implies that we have $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for each $C \in \mathcal{C}$ (see Example 1.2.1.4). It will therefore suffice to show that for every pair of composable morphisms $u : C \to D$ and $v : D \to E$ in the category $\mathcal{C}$, we have $F(v) \circ F(u) = F(v \circ u)$ as elements of the set $\mathrm{Hom}_{\mathcal{C}'}(F(C), F(E))$. For this, we observe that the diagram $C \xrightarrow{u} D \xrightarrow{v} E$ can be identified with a 2-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})$. Using the equality $d_i(f(\sigma)) = f(d_i(\sigma))$ for $i = 0, 2$, we see that $f(\sigma)$ corresponds to the diagram $F(C) \xrightarrow{F(u)} F(D) \xrightarrow{F(v)} F(E)$ in $\mathcal{C}'$. We now compute

$$F(v) \circ F(u) = d_1(f(\sigma)) = f(d_1(\sigma)) = F(v \circ u).$$

This completes the proof of (a). To prove (b), we must show that $f(\tau) = \theta(F)(\tau)$ for each n-simplex τ of $\mathrm{N}_\bullet(\mathcal{C})$. This follows by construction in the case $n \leqslant 1$, and follows in general since an n-simplex of $\mathrm{N}_\bullet(\mathcal{C}')$ is determined by its 1-dimensional faces (see Remark 1.2.1.3). □

1.2.3 Characterization of Nerves

0030 We now describe the essential image of the functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$.

0031 **Proposition 1.2.3.1.** *Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is isomorphic to the nerve of a category if and only if it satisfies the following condition:*

$(*')$ *For every pair of integers $0 < i < n$, and every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$, there exists a unique map $\sigma : \Delta^n \to S_\bullet$ such that $\sigma_0 = \sigma|_{\Delta^n}$.*

The proof of Proposition 1.2.3.1 will require some preliminaries. We begin by establishing the necessity of condition $(*')$.

0032 **Lemma 1.2.3.2.** *Let $\mathcal{C}$ be a category. Then the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ satisfies condition $(*')$ of Proposition 1.2.3.1.*

Proof. Choose integers $0 < i < n$ together with a map of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathrm{N}_\bullet(\mathcal{C})$; we wish to show that σ_0 can be extended uniquely to a n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$. For $0 \leqslant j \leqslant n$, let $C_j \in \mathcal{C}$ denote the image under σ_0 of the jth vertex of Δ^n (which belongs to the horn Λ^n_i). We first consider the case where $n \geqslant 3$. In this case, Λ^n_i contains every edge of Δ^n. For $0 \leqslant j \leqslant k \leqslant n$, let $f_{j,k} : C_j \to C_k$ denote the 1-simplex of $\mathrm{N}_\bullet(\mathcal{C})$ obtained by evaluating σ_0 on the edge of Δ^n corresponding to the pair (j, k). We claim that the construction

$$j \mapsto C_j \qquad (j \leqslant k) \mapsto f_{j,k}$$

determines a functor $[n] \to \mathcal{C}$, which we can then with an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$ having the desired properties. It is easy to see that $f_{j,j} = \mathrm{id}_{C_j}$ for each $0 \leqslant j \leqslant n$, so it will suffice to show that $f_{k,\ell} \circ f_{j,k} = f_{j,\ell}$ for every triple $0 \leqslant j \leqslant k \leqslant \ell \leqslant n$. The triple (j, k, ℓ) determines a 2-simplex τ of Δ^n. If τ is contained in Λ^n_i, then $\tau' = \sigma_0(\tau)$ is a 2-simplex of $\mathrm{N}_\bullet(\mathcal{C})$ satisfying $d_0(\tau') = f_{k,\ell}$, $d_1(\tau') = f_{j,\ell}$, and $d_2(\tau') = f_{k,\ell}$, so that τ' "witnesses" the identity

$f_{k,\ell} \circ f_{j,k} = f_{j,\ell}$. It will therefore suffice to treat the case where the simplex τ does *not* belong to the Λ^n_i. In this case, our assumption that $n \geqslant 3$ guarantees that we must have $\{j,k,\ell\} = [n] \backslash \{i\}$. It follows that $n = 3$, so that either $i = 1$ or $i = 2$. We will treat the case $i = 1$ (the case $i = 2$ follows by an similar argument). Note that Λ^3_1 contains all of the nondegenerate 2-simplices of Δ^3 other than τ; applying the map σ_0, we obtain 2-simplices of $\mathrm{N}_\bullet(\mathcal{C})$ which witness the identities

$$f_{0,3} = f_{1,3} \circ f_{0,1} \qquad f_{1,3} = f_{2,3} \circ f_{1,2} \qquad f_{0,2} = f_{1,2} \circ f_{0,1}.$$

We now compute

$$f_{0,3} = f_{1,3} \circ f_{0,1} = (f_{2,3} \circ f_{1,2}) \circ f_{0,1} = f_{2,3} \circ (f_{1,2} \circ f_{0,1}) = f_{2,3} \circ f_{0,2}$$

so that $f_{j,\ell} = f_{k,\ell} \circ f_{j,k}$, as desired.

It remains to treat the case $n = 2$, so that we must also have $i = 1$. In this situation, the map $\sigma_0 : \Lambda^n_i \to \mathrm{N}_\bullet(\mathcal{C})$ determines a pair of composable morphisms $f_{0,1} : C_0 \to C_1$ and $f_{1,2} : C_1 \to C_2$. This data extends uniquely to a 2-simplex σ of $\mathcal{C}$ satisfying $d_1(\sigma) = f_{1,2} \circ f_{0,1}$ (see Remark 1.2.1.3). □

Lemma 1.2.3.3. *Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets. Assume that f induces bijections $S_0 \to T_0$ and $S_1 \to T_1$, and that both $S_\bullet$ and $T_\bullet$ satisfy condition $(*')$ of Proposition 1.2.3.1. Then f is an isomorphism.* 0033

Proof. We claim that, for every simplicial set $K_\bullet$, composition with f induces a bijection

$$\theta_{K_\bullet} : \mathrm{Hom}_{\mathrm{Set}_\Delta}(K_\bullet, S_\bullet) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(K_\bullet, T_\bullet).$$

Writing $K_\bullet$ as a union of its skeleta $\mathrm{sk}_n\, K_\bullet$, we can reduce to the case where K has dimension $\leqslant n$, for some integer $n \geqslant -1$ (see Definition 1.1.3.9). We now proceed by induction on n. The case $n = -1$ is trivial (since a simplicial set of dimension $\leqslant -1$ is empty). Let us therefore assume that $n \geqslant 0$, so that Proposition 1.1.3.11 supplies a pushout diagram of simplicial sets

$$\begin{array}{ccc} \coprod \partial \Delta^n & \longrightarrow & \coprod \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{sk}_{n-1}\, K_\bullet & \longrightarrow & K_\bullet. \end{array}$$

It follows from out inductive hypothesis that the maps $\theta_{\partial \Delta^n}$ and $\theta_{\mathrm{sk}_{n-1} K_\bullet}$ are bijective. Consequently, to show that $\theta_{K_\bullet}$ is bijective, it will suffice to show that θ_{Δ^n} is bijective: that is, that f induces a bijection $S_n \to T_n$. For $n \leqslant 1$, this follows from our hypothesis. To handle the case $n \geqslant 2$, we observe that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S_\bullet) & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_1, S_\bullet) \\ \downarrow{\scriptstyle \theta_{\Delta^n}} & & \downarrow{\scriptstyle \theta_{\Lambda^n_1}} \\ \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, T_\bullet) & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_1, T_\bullet). \end{array}$$

Here the right vertical map is bijective by virtue of our inductive hypothesis, and the horizontal maps are bijective by virtue of our assumption that both $S_\bullet$ and $T_\bullet$ satisfy assumption $(*')$. It follows that the left vertical map is also bijective, as desired. □

Proof of Proposition 1.2.3.1. Let $S_\bullet$ be a simplicial set satisfying condition $(*')$ of Proposition 1.2.3.1; we will show that there is a category $\mathcal{C}$ and an isomorphism of simplicial sets $f : S_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ (the converse assertion follows from Lemma 1.2.3.2). It follows from Proposition 1.2.2.1 that the category $\mathcal{C}$ is uniquely determined (up to isomorphism), and from the proof of Proposition 1.2.2.1 we can extract an explicit construction of $\mathcal{C}$:

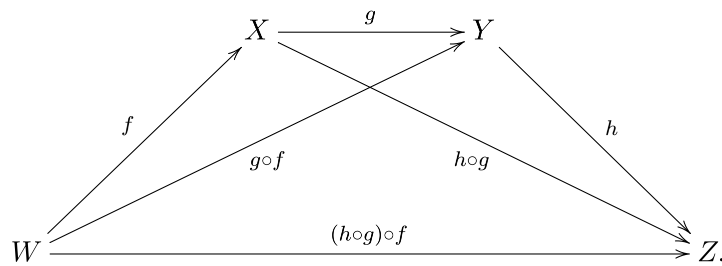
- The objects of $\mathcal{C}$ are the vertices of $S_\bullet$.
- Given a pair of objects $C, D \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(C, D)$ denote the collection of edges e of $S_\bullet$ satisfying $d_0(e) = D$ and $d_1(e) = C$.
- For each object $C \in \mathcal{C}$, we define the identity morphism $\mathrm{id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ to be the degenerate edge $s_0(C)$.
- Given a triple of objects $C, D, E \in \mathcal{C}$ and a pair of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(D, E)$, we can apply hypothesis $(*')$ (in the special case $n = 2$ and $i = 1$) to conclude that there is a unique 2-simplex σ of $S_\bullet$ satisfying $d_2(\sigma) = f$ and $d_0(\sigma) = g$. We define the composition $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(C, E)$ to be the edge $d_1(\sigma)$.

We claim that $\mathcal{C}$ is a category. For this, we must check the following:

- The composition law on $\mathcal{C}$ is unital: for every morphism $f : C \to D$ in $\mathcal{C}$, we have equalities
$$\mathrm{id}_D \circ f = f = f \circ \mathrm{id}_C .$$
Let us verify the identity on the left; the proof in the other case is similar. For this, we must construct a 2-simplex σ of $S_\bullet$ such that $d_0(\sigma) = \mathrm{id}_D$ and $d_1(\sigma) = d_2(\sigma) = f$. The degenerate 2-simplex $s_1(f)$ has these properties.
- The composition law on $\mathcal{C}$ is associative. That is, for every triple of composable morphisms
$$f : W \to X \qquad g : X \to Y \qquad h : Y \to Z$$
in $\mathcal{C}$, we have an identity $h \circ (g \circ f) = (h \circ g) \circ f$ in the category $\mathcal{C}$. Applying condition $(*')$ repeatedly, we deduce the following:
  - There is a unique 2-simplex σ_0 of $\mathcal{C}$ satisfying $d_0(\sigma_0) = h$ and $d_2(\sigma_0) = g$ (it follows that $d_1(\sigma_0) = h \circ g$).
  - There is a unique 2-simplex σ_3 of $\mathcal{C}$ satisfying $d_0(\sigma_3) = g$ and $d_2(\sigma_3) = f$ (it follows that $d_1(\sigma_3) = g \circ f$).

– There is a unique 2-simplex σ_2 of $\mathcal{C}$ satisfying $d_0(\sigma_2) = h \circ g$ and $d_0(\sigma_2) = f$ (it follows that $d_1(\sigma_2) = (h \circ g) \circ f$).
– There is a unique 3-simplex τ of $\mathcal{C}$ satisfying $d_0(\tau) = \sigma_0$, $d_2(\tau) = \sigma_2$, and $d_3(\tau) = \sigma_3$ (this follows by applying $(*')$ to the horn inclusion $\Lambda^3_1 \hookrightarrow \Delta^3$).

The 3-simplex τ can be depicted in the following diagram



Set $\sigma_1 = d_1(\tau)$. Then σ_1 is a 2-simplex of $S_\bullet$ satisfying $d_0(\sigma_1) = h$, $d_1(\sigma_2) = (h \circ g) \circ f$, and $d_2(\sigma_2) = g \circ f$. It follows that σ_1 "witnesses" the identity $h \circ (g \circ f) = (h \circ g) \circ f$.

Note that every n-simplex $\sigma : \Delta^n \to S_\bullet$ determines a functor $[n] \to \mathcal{C}$, given on objects by the values of σ on the vertices of Δ^n and on morphisms by the values of σ on the edges of Δ^n. This construction determines a map of simplicial sets $f : S_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$, which is clearly bijective on simplices of dimension $\leqslant 1$. Since the simplicial sets $S_\bullet$ and $\mathrm{N}_\bullet(\mathcal{C})$ both satisfy condition $(*')$ (Lemma 1.2.3.2), it follows from Lemma 1.2.3.3 that f is an isomorphism. □

Remark 1.2.3.4. The characterization of Proposition 1.2.3.1 has many variants. For 0034
example, one can replace condition $(*')$ by the following *a priori* weaker condition:

$(*'_0)$ For every $n \geqslant 2$ and every map of simplicial sets $\sigma_0 : \Lambda^n_1 \to S_\bullet$, there exists a unique map $\sigma : \Delta^n \to S_\bullet$ satisfying $\sigma_0 = \sigma|_{\Lambda^n_1}$.

1.2.4 The Nerve of a Groupoid

According to Proposition 1.2.2.1, every category $\mathcal{C}$ can be recovered, up to canonical 0035
isomorphism, from the nerve $\mathrm{N}_\bullet(\mathcal{C})$. In particular, any isomorphism-invariant condition on a category $\mathcal{C}$ can be reformulated as a condition on the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. We now illustrate this principle with a simple example.

Definition 1.2.4.1. Let $\mathcal{C}$ be a category. We say that a morphism $f : C \to D$ in $\mathcal{C}$ is an 0036
isomorphism if there exists a morphism $g : D \to C$ satisfying the identities

$$f \circ g = \mathrm{id}_D \qquad g \circ f = \mathrm{id}_C .$$

In this case, the morphism g is uniquely determined and we write $g = f^{-1}$. We say that $\mathcal{C}$ is a *groupoid* if every morphism in $\mathcal{C}$ is invertible.

0037 **Proposition 1.2.4.2.** *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a groupoid (Definition 1.2.4.1) if and only if the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex (Definition 1.1.7.1).*

0038 **Example 1.2.4.3** (The Milnor Construction)**.** Let G be a group. Then we can form a category $\mathcal{C}_G$ having a single object X, where $\mathrm{Hom}_{\mathcal{C}}(X, X) = G$ (and the composition of morphisms in $\mathcal{C}_G$ is given by multiplication in the group G). We will denote the nerve of the category $\mathcal{C}_G$ by BG. The geometric realization $|\,\mathrm{BG}\,|$ is a topological space called the *classifying space* of G. It can characterized (up to homotopy equivalence) by the fact that it is a CW complex with either of the following properties:

- The space $|\,\mathrm{BG}\,|$ is connected, and its homotopy groups (with respect to any choice of base point) are given by the formula

$$\pi_*(|BG|) \simeq \begin{cases} G \text{ if } * = 1 \\ 0 & \text{if } * > 1. \end{cases}$$

- For any paracompact topological space X, there is a canonical bijection

$$\{\text{Continuous maps } f : X \to |\,\mathrm{BG}\,|\}/\text{homotopy} \simeq \{G\text{-torsors } P \to X\}/\text{isomorphism.}$$

We refer the reader to [5] for a more detailed discussion (including an extension to the setting of topological groups).

Proof of Proposition 1.2.4.2. Suppose first that $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex; we wish to show that $\mathcal{C}$ is a groupoid. Let $f : C \to D$ be a morphism in $\mathcal{C}$. Using the surjectivity of the map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^2, \mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^2_2, \mathrm{N}_\bullet(\mathcal{C}))$, we see that there exists a 2-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})$ satisfying $d_0(\sigma) = f$ and $d_1(\sigma) = \mathrm{id}_D$. Setting $g = d_2(\sigma)$, we conclude that $f \circ g = \mathrm{id}_D$: that is, g is a left inverse to f. Similarly, the surjectivity of the map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^2, \mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^2_0, \mathrm{N}_\bullet(\mathcal{C}))$ allows us to construct a map $h : D \to C$ satisfying $h \circ f = \mathrm{id}_C$. The calculation

$$g = \mathrm{id}_C \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ \mathrm{id}_D = h$$

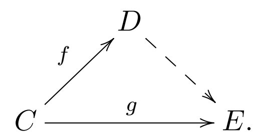
then shows that $g = h$ is an inverse of f, so that f is invertible as desired.

Now suppose that $\mathcal{C}$ is a groupoid. We wish to show that, for $0 \leqslant i \leqslant n$, every map $\sigma_0 : \Lambda^n_i \to \mathrm{N}_\bullet(\mathcal{C})$ can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$. For $0 < i < n$, this follows from Lemma 1.2.3.2 (and does not require the assumption that $\mathcal{C}$ is a groupoid). We will treat the case where $i = 0$; the case $i = n$ follows by similar reasoning. We consider several cases:

- In the case $n = 0$, we have $\Lambda^n_0 = \Delta^n$, so we can take $\sigma = \sigma_0$.

- In the case $n = 1$, the map $\sigma_0 : \Lambda^n_0 \to \mathcal{C}$ can be identified with an object $C \in \mathcal{C}$. In this case, we can take σ to be an edge of $\mathrm{N}_\bullet(\mathcal{C})$ corresponding to any morphism with codomain C (for example, we can take σ to be the identity map id_C).

- In the case $n = 2$, we can identify σ_0 with a pair of morphisms in $\mathcal{C}$ having the same domain, which we can depict as a diagram



  Our assumption that $\mathcal{C}$ is a groupoid guarantees that we can extend this diagram to a 2-simplex of $\mathcal{C}$, whose 0th face is given by the morphism $g \circ f^{-1} : D \to E$.

- In the case $n \geqslant 3$, the map σ_0 determines a collection of objects $\{C_i\}_{0 \leqslant i \leqslant n}$ and morphisms $f_{i,j} : C_i \to C_j$ for $i \leqslant j$ (as in the proof of Lemma 1.2.3.2). We wish to show that these morphisms determine a functor $[n] \to \mathcal{C}$ (which we can then identify with an n-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})$ satisfying $\sigma|_{\Lambda^n_0} = \sigma_0$). For this, we must verify the identity $f_{j,k} \circ f_{i,j} = f_{i,k}$ for $0 \leqslant i \leqslant j \leqslant k \leqslant n$. Note that this identity is satisfied whenever the triple $(i \leqslant j \leqslant k)$ determines a 2-simplex of Δ^n belonging to the horn Λ^n_0. This is automatic unless $n = 3$ and $(i, j, k) = (1, 2, 3)$. To handle this exceptional case, we compute

$$\begin{aligned} (f_{2,3} \circ f_{1,2}) \circ f_{0,1} &= f_{2,3} \circ (f_{1,2} \circ f_{0,1}) \\ &= f_{2,3} \circ f_{0,2} \\ &= f_{0,3} \\ &= f_{1,3} \circ f_{0,1}. \end{aligned}$$

  Since $\mathcal{C}$ is a groupoid, composing with $f^{-1}_{0,1}$ on the right yields the desired identity $f_{2,3} \circ f_{1,2} = f_{1,3}$.

□

1.3 ∞-Categories

In §1.1 and §1.2, we considered two closely related conditions on a simplicial set $S_\bullet$: 0039

(∗) For $0 \leqslant i \leqslant n$, every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$.

(∗′) For $0 < i < n$, every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended uniquely to a map $\sigma : \Delta^n \to S_\bullet$.

Simplicial sets satisfying $(*)$ are called Kan complexes and form the basis for a combinatorial approach to homotopy theory, while simplicial sets satisfying $(*')$ can be identified with categories (Propositions 1.2.2.1 and 1.2.3.1). These notions admit a common generalization:

003A **Definition 1.3.0.1.** An *∞-category* is a simplicial set $S_\bullet$ which satisfies the following condition:

$(*'')$ For $0 < i < n$, every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$.

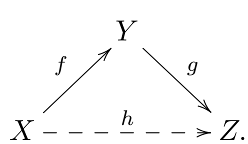
003B **Remark 1.3.0.2.** Condition $(*'')$ is commonly known as the *weak Kan extension condition*. It was introduced by Boardman and Vogt in [1], who refer to ∞-categories as *weak Kan complexes*. The theory was developed further by Joyal ([3] and [4]), who refers to ∞-categories as *quasicategories*.

003C **Example 1.3.0.3.** Every Kan complex is an ∞-category. In particular, if X is a topological space, then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category.

003D **Example 1.3.0.4.** For every category $\mathcal{C}$, the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is an ∞-category.

003E **Remark 1.3.0.5.** We will often abuse terminology by identifying a category $\mathcal{C}$ with its nerve $\mathrm{N}_\bullet(\mathcal{C})$ (this abuse is essentially harmless by virtue of Proposition 1.2.2.1). Adopting this convention, we can state Example 1.3.0.4 more simply: every category is an ∞-category. To minimize the possibility of confusion, we will sometimes refer to categories as *ordinary categories*.

Throughout this book, we will generally use calligraphic letters (like $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$) to denote ∞-categories, and we will generally describe them using terminology borrowed from category theory. For example, if $\mathcal{C} = S_\bullet$ is an ∞-category, then we will refer to *vertices* of the simplicial set $S_\bullet$ as *objects* of the ∞-category $\mathcal{C}$, and to *edges* of the simplicial set $S_\bullet$ as *morphisms* of the ∞-category $\mathcal{C}$ (see §1.3.1). One of the central themes of this book is that ∞-categories behave much like ordinary categories. In particular, for any ∞-category $\mathcal{C}$, there is notion of composition for morphisms of $\mathcal{C}$, which we study in §1.3.4. Given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$ (corresponding to edges $f, g \in S_1$ satisfying $d_0(f) = d_1(g)$), the pair (f, g) defines a map of simplicial sets $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$. Applying condition $(*'')$, we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$, which we can think of heuristically as a commutative diagram



In this case, we will refer to the morphism $h = d_1(\sigma)$ as a *composition of f and g*. However, this comes with a caveat: the extension σ is usually not unique, so the morphism h is not completely determined by f and g. However, we will show that it is unique up to a certain notion of *homotopy* which we study in §1.3.3. We apply this observation in §1.3.5 to extract an ordinary category $\mathrm{h}\mathcal{C}$ called the *homotopy category* of $\mathcal{C}$, whose morphisms are homotopy classes of morphisms of $\mathcal{C}$ (Definition 1.3.5.3). This is a special case of a more general construction which can be applied to any simplicial set $S_\bullet$, which we describe in §1.3.6.

1.3.1 Objects and Morphisms

We begin by introducing some terminology. 003F

Definition 1.3.1.1. Let $\mathcal{C} = S_\bullet$ be an ∞-category. An *object of* $\mathcal{C}$ is a vertex of the simplicial set $S_\bullet$ (that is, an element of the set S_0). A *morphism of* $\mathcal{C}$ is an edge of the simplicial set $S_\bullet$ (that is, an element of S_1). If $f \in S_1$ is a morphism of $\mathcal{C}$, we will refer to the object $X = d_1(f)$ as the *domain* of f and to the object $Y = d_0(f)$ as the *codomain* of f. In this case, we will say that f is a *morphism from X to Y*. For any object X of $\mathcal{C}$, we can regard the degenerate edge $s_0(X)$ as a morphism from X to itself; we will denote this morphism by id_X and refer to it as the *identity morphism* of X. 003G

Notation 1.3.1.2. Let $\mathcal{C}$ be an ∞-category. We will often write $X \in \mathcal{C}$ to indicate that X is an object of $\mathcal{C}$. We use the phrase "$f : X \to Y$ is a morphism of $\mathcal{C}$" to indicate that f is a morphism of $\mathcal{C}$ having domain X and codomain Y. 003H

Example 1.3.1.3. Let $\mathcal{C}$ be an ordinary category, and regard the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ as an ∞-category. Then: 003J

- The objects of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The morphisms of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ are the morphisms of $\mathcal{C}$. Moreover, the domain and codomain of a morphism of $\mathcal{C}$ coincide with the domain and codomain of the corresponding morphism in $\mathrm{N}_\bullet(\mathcal{C})$.
- For every object $X \in \mathcal{C}$, the identity morphism id_X does not depend on whether we view X as an object of the category $\mathcal{C}$ or the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$.

Example 1.3.1.4. Let X be a topological space, and regard the simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then: 003K

- The objects of $\mathrm{Sing}_\bullet(X)$ are the points of X.
- The morphisms of $\mathrm{Sing}_\bullet(X)$ are continuous paths $f : [0,1] \to X$. The domain of a morphism f is the point $f(0)$, and the codomain is the point $f(1)$.
- For every point $x \in X$, the identity morphism id_x is the constant path $[0,1] \to X$ taking the value x.

1.3.2 The Opposite of an ∞-Category

003L Let $\mathcal{C}$ be an ordinary category. Then we can construct a new category $\mathcal{C}^{\mathrm{op}}$, called the *opposite category of* $\mathcal{C}$, as follows:

- The objects of the opposite category $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$.
- For every pair of objects $C, D \in \mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C, D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$.
- Composition of morphisms in $\mathcal{C}^{\mathrm{op}}$ is given by the composition of morphisms in $\mathcal{C}$, with the order reversed.

The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ admits a straightforward generalization to the setting of ∞-categories. In fact, it can be extended to arbitrary simplicial sets.

003M **Notation 1.3.2.1.** Let Lin denote the category whose objects are linearly ordered sets and whose morphisms are nondecreasing functions. Let I be an object of Lin, regarded as a set with a linear ordering $\leqslant_I$. We let I^{op} denote the same set with the opposite ordering, so that

$$(i \leqslant_{I^{\mathrm{op}}} j) \Leftrightarrow (j \leqslant_I i).$$

The construction $I \mapsto I^{\mathrm{op}}$ determines an equivalence from the category Lin to itself.

Recall that the simplex category $\mathbf{\Delta}$ of Definition 1.1.1.2 is the full subcategory of Lin spanned by objects of the form $[n] = \{0 < 1 < \cdots < n\}$, and is equivalent to Lin itself. There is a unique functor $\mathrm{Op} : \mathbf{\Delta} \to \mathbf{\Delta}$ for which the diagram

$$\begin{array}{ccc} \mathbf{\Delta} & \longrightarrow & \mathrm{Lin} \\ \downarrow{\scriptstyle \mathrm{Op}} & & \downarrow{\scriptstyle I \mapsto I^{\mathrm{op}}} \\ \mathbf{\Delta} & \longrightarrow & \mathrm{Lin} \end{array}$$

commutes up to isomorphism, where the horizontal maps are given by the inclusion. The functor Op can be described more concretely as follows:

- For each object $[n] \in \mathbf{\Delta}$, we have $\mathrm{Op}([n]) = [n]$ (note that the construction $i \mapsto n - i$ determines an isomorphism of $[n]$ with the opposite linear ordering $[n]^{\mathrm{op}}$).
- For each morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the morphism $\mathrm{Op}(\alpha) : [m] \to [n]$ is given by the formula $\mathrm{Op}(\alpha)(i) = n - \alpha(m - i)$.

003N **Construction 1.3.2.2.** Let $S_\bullet$ be a simplicial set, which we regard as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. We let $S_\bullet^{\mathrm{op}}$ denote the simplicial set given by the composition

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Op}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{S_\bullet} \mathrm{Set},$$

where Op is the functor described in Notation 1.3.2.1. We will refer to $S_\bullet^{\mathrm{op}}$ as the *opposite* of the simplicial set $S_\bullet$.

Remark 1.3.2.3. Let $S_\bullet$ be a simplicial set. Then the opposite simplicial set $S_\bullet^{\mathrm{op}}$ can be described more concretely as follows: 003P

- For each $n \geqslant 0$, we have $S_n^{\mathrm{op}} = S_n$.
- The face and degeneracy maps of $S_\bullet^{\mathrm{op}}$ are given by

$$(d_i : S_n^{\mathrm{op}} \to S_{n-1}^{\mathrm{op}}) = (d_{n-i} : S_n \to S_{n-1})$$
$$(s_i : S_n^{\mathrm{op}} \to S_{n+1}^{\mathrm{op}}) = (s_{n-i} : S_n \to S_{n+1}).$$

Example 1.3.2.4. Let $\mathcal{C}$ be a category. For each $n \geqslant 0$, we can identify n-simplices σ of $\mathrm{N}_\bullet(\mathcal{C})$ with diagrams 003Q

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$. Then σ determines an n-simplex σ' of $\mathrm{N}_\bullet(\mathcal{C}^{\mathrm{op}})$, given by the diagram

$$C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0$$

in the opposite category $\mathcal{C}^{\mathrm{op}}$. The construction $\sigma \mapsto \sigma'$ determines an isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{N}_\bullet(\mathcal{C}^{\mathrm{op}})$.

Example 1.3.2.5. Let X be a topological space. Then there is a canonical isomorphism of simplicial sets $\mathrm{Sing}_\bullet(X) \simeq \mathrm{Sing}_\bullet(X)^{\mathrm{op}}$, which carries each singular n-simplex $\sigma : |\Delta^n| \to X$ to the composite map 003R

$$|\Delta^n| \xrightarrow{r} |\Delta^n| \xrightarrow{\sigma} X$$

where r is denotes the homeomorphism of $|\Delta^n|$ with itself given by $r(t_0, t_1, \ldots, t_{n-1}, t_n) = (t_n, t_{n-1}, \ldots, t_1, t_0)$.

Proposition 1.3.2.6. *Let $\mathcal{C}$ be an ∞-category. Then the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an ∞-category.* *003S*

Proof. Let $\sigma_0 : \Lambda^n_i \to \mathcal{C}^{\mathrm{op}}$ be a map of simplicial sets for $0 < i < n$; we wish to show that σ_0 can be extended to an n-simplex of $\mathcal{C}^{\mathrm{op}}$. Passing to opposite simplicial sets, we are reduced to showing that the map $\sigma_0^{\mathrm{op}} : (\Lambda^n_i)^{\mathrm{op}} \to \mathcal{C}$ can be extended to a map $(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$. This follows from our assumption that $\mathcal{C}$ is an ∞-category, since there is a canonical isomorphism $(\Delta^n)^{\mathrm{op}} \simeq \Delta^n$ which carries the simplicial subset $(\Lambda^n_i)^{\mathrm{op}}$ to Λ^n_{n-i}. □

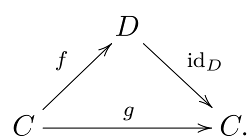
Remark 1.3.2.7. Let $\mathcal{C}$ be an ∞-category. We will refer to the ∞-category $\mathcal{C}^{\mathrm{op}}$ of Proposition 1.3.2.6 as the *opposite* of the ∞-category $\mathcal{C}$. Note that: 003T

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$.
- Given a pair of objects $X, Y \in \mathcal{C}$, the datum of a morphism from X to Y in $\mathcal{C}^{\mathrm{op}}$ is equivalent to the datum of a morphism from Y to X in $\mathcal{C}$.

1.3.3 Homotopies of Morphisms

003U For any topological space X, we can view the singular simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a continuous path $f : [0,1] \to X$ satisfying $f(0) = x$ and $f(1) = y$. For many purposes (for example, in the study of the fundamental group $\pi_1(X, x)$), it is useful to work not with paths but with *homotopy classes* of paths (having fixed endpoints). This notion can be generalized to an arbitrary ∞-category:

003V **Definition 1.3.3.1.** Let $\mathcal{C}$ be an ∞-category and let $f, g : C \to D$ be a pair of morphisms in $\mathcal{C}$ having the same domain and codomain. A *homotopy from f to g* is a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = \mathrm{id}_D$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as depicted in the diagram



We will say that f and g are *homotopic* if there exists a homotopy from f to g.

003W **Example 1.3.3.2.** Let $\mathcal{C}$ be an ordinary category. Then a pair of morphisms $f, g : C \to D$ in $\mathcal{C}$ (having the same domain and codomain) are homotopic as morphisms of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ if and only if $f = g$.

003X **Example 1.3.3.3.** Let X be a topological space. Suppose we are given points $x, y \in X$ and a pair of continuous paths $f, g : [0,1] \to X$ satisfying $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then f and g are homotopic as morphisms of the ∞-category $\mathrm{Sing}_\bullet(\mathcal{C})$ (in the sense of Definition 1.3.3.1 if and only the paths f and g are homotopic relative to their endpoints: that is, if and only if there exists a continuous function $H : [0,1] \times [0,1] \to X$ satisfying

$$H(s,0) = f(s) \quad H(s,1) = g(s) \quad H(0,t) = x \quad H(1,t) = y$$

(see Exercise 1.3.3.4 for a more precise statement).

003Y **Exercise 1.3.3.4.** Let $\pi : [0,1] \times [0,1] \to |\Delta^2|$ denote the continuous function given by the formula $\pi(s,t) = (1-s, ts, (1-t)s)$. For any topological space X, the construction $\sigma \mapsto \sigma \circ \pi$ determines a map from the set $\mathrm{Sing}_2(X)$ of singular 2-simplices of X to the set of all continuous functions $H : [0,1] \times [0,1] \to X$. Show that, if $f, g : [0,1] \to X$ are continuous paths satisfying $f(0) = g(0)$ and $f(1) = g(1)$, then the construction $\sigma \mapsto \sigma \circ \pi$ induces a bijection from the set of homotopies from f to g (in the sense of Definition 1.3.3.1) to the set of continuous functions H satisfying the requirements of Example 1.3.3.3.

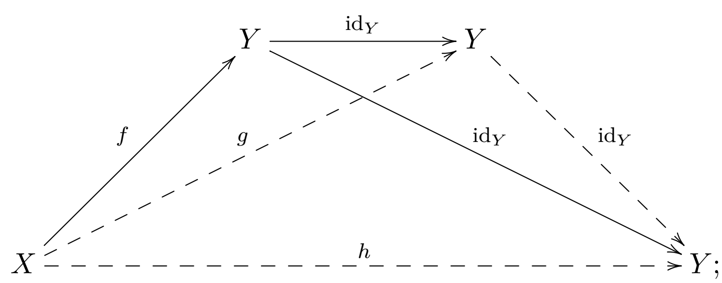
003Z **Proposition 1.3.3.5.** *Let $\mathcal{C}$ be an ∞-category containing objects $X, Y \in \mathcal{C}$, and let E denote the collection of all morphisms from X to Y in $\mathcal{C}$. Then homotopy is an equivalence relation on E.*

Proof. We first observe that for any morphism $f : X \to Y$ in $\mathcal{C}$, the degenerate 2-simplex $s_1(f)$ is a homotopy from f to itself. It follows that homotopy is a reflexive relation on E. We will complete the proof by establishing the following:

(∗) Let $f, g, h : X \to Y$ be three morphisms from X to Y. If f is homotopic to g and f is homotopic to h, then g is homotopic to h.

Let us first observe that assertion (∗) implies Proposition 1.3.3.5. Note that in the special case $f = h$, (∗) asserts that if f is homotopic to g, then g is homotopic to f (since f is always homotopic to itself). That is, the relation of homotopy is symmetric. We can therefore replace the hypothesis that f is homotopic to g in assertion (∗) by the hypothesis that g is homotopic to f, so that (∗) is equivalent to the transitivity of the relation of homotopy.

It remains to prove (∗). Let σ_2 and σ_3 be 2-simplices of $\mathcal{C}$ which are homotopies from f to g and f to h, respectively, and let σ_0 be the 2-simplex given by the constant map $\Delta^2 \to \Delta^0 \xrightarrow{X} \mathcal{C}$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (see Exercise 1.1.2.14), depicted informally by the diagram



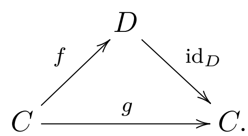
here the dotted arrows represent the boundary of the "missing" face of the horn Λ^3_1. Our hypothesis that $\mathcal{C}$ is an ∞-category guarantees that τ_0 can be extended to a 3-simplex τ of $\mathcal{C}$. We can then regard the face $d_1(\tau)$ as a homotopy from g to h. □

Note that there is a potential asymmetry in Definition 1.3.3.1: if $f, g : X \to Y$ are two morphisms in an ∞-category $\mathcal{C}$, then the datum of a homotopy from f to g in the ∞-category $\mathcal{C}$ is not equivalent to the datum of a homotopy from f to g in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Nevertheless, we have the following:

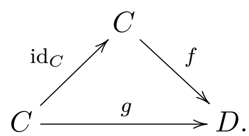
Proposition 1.3.3.6. *Let $\mathcal{C}$ be an ∞-category, and let $f, g : X \to Y$ be morphisms of $\mathcal{C}$ having the same domain and codomain. Then f and g are homotopic if and only if they are homotopic when regarded as morphisms of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. In other words, the following conditions are equivalent:* 0040

(1) *There exists a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = \mathrm{id}_D$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as*

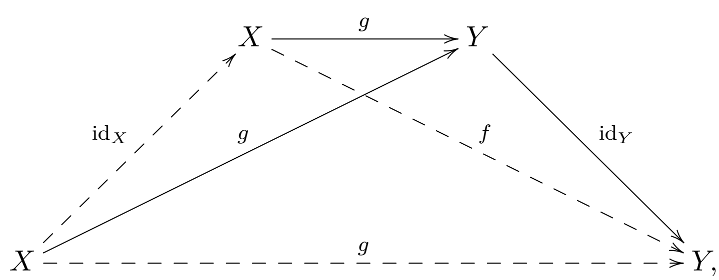
depicted in the diagram



(2) *There exists a 2-simplex τ of $\mathcal{C}$ satisfying $d_0(\tau) = f$, $d_1(\tau) = g$, and $d_2(\tau) = \mathrm{id}_X$, as depicted in the diagram*



Proof. We will show that (1) implies (2); the proof of the reverse implication is similar. Assume that f is homotopic to g. Since the relation of homotopy is symmetric (Proposition 1.3.3.5), it follows that g is also homotopic to f. Let σ be a homotopy from g to f. Then we can regard the tuple of 2-simplices $(\sigma, s_1(g), \bullet, s_0(g))$ as a map of simplicial sets $\rho_0 : \Lambda^3_2 \to \mathcal{C}$ (see Exercise 1.1.2.14), depicted informally in the diagram



where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_2. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend ρ_0 to a 3-simplex ρ of $\mathcal{C}$. Then the face $\tau = d_2(\rho)$ has the properties required by (2). □

1.3.4 Composition of Morphisms

0041 We now introduce a notion of composition for morphisms in an ∞-category.

0042 **Definition 1.3.4.1.** Let $\mathcal{C}$ be an ∞-category. Suppose we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. We will say that *h is a composition of f and g* if there exists a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$, and $d_2(\sigma) = f$. In this case, we will also say that the 2-simplex σ *witnesses h as a composition of f and g*.

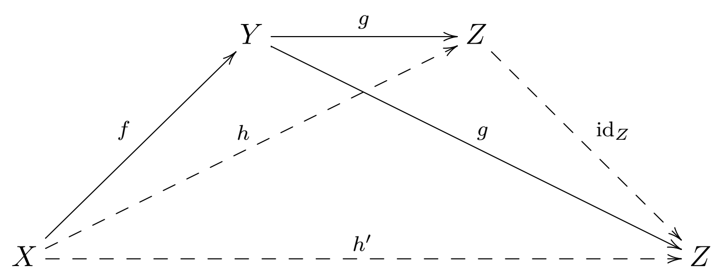
Beware that, in the situation of Definition 1.3.4.1, the morphism h is not determined by f and g. However, it is determined up to homotopy:

Proposition 1.3.4.2. *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : X \to Y$ and $g : Y \to Z$. Then:* 0043

(1) *There exists a morphism $h : X \to Z$ which is a composition of f and g.*

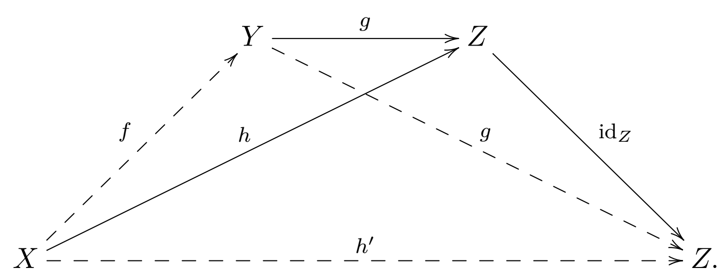
(2) *Let $h : X \to Z$ be a composition of f and g, and let $h' : X \to Z$ be another morphism in $\mathcal{C}$ having the same domain and codomain. Then h' is a composition of f and g if and only if h' is homotopic to h.*

Proof. The tuple $(g, \bullet, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$ (Exercise 1.1.2.14). Since $\mathcal{C}$ is an ∞-category, we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$. Then σ witnesses the morphism $h = d_1(\sigma)$ as a composition of f and g. This proves (1). To prove (2), let us first suppose that $h' : X \to Z$ is some other morphism in $\mathcal{C}$ which is a composition of f and g. We will show that h is homotopic to h'. Choose a 2-simplex σ' which witnesses h' as a composition of f and g. Then the tuple $(s_1(g), \bullet, \sigma', \sigma)$ determines a morphism of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram



where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex of $\mathcal{C}$. Then the face $d_1(\tau)$ is a homotopy from h to h'.

We now prove the converse. Let σ be a 2-simplex of $\mathcal{C}$ which witnesses h as a composition of f and g, and let $h' : X \to Z$ be a morphism of $\mathcal{C}$ which is homotopic to h. Let σ'' be a 2-simplex of $\mathcal{C}$ which is a homotopy from h to h'. Then the tuple $(s_1(g), \sigma'', \bullet, \sigma)$ determines a map of simplicial sets $\rho_0 : \Lambda^3_2 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram



Our assumption that $\mathcal{C}$ is an ∞-category guarantees that we can extend ρ_0 to a 3-simplex ρ of $\mathcal{C}$. Then the face $d_2(\rho)$ witnesses h' as a composition of f and g. □

0044 **Notation 1.3.4.3.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ and $g : Y \to Z$ be a pair of morphisms in $\mathcal{C}$. We will write $h = g \circ f$ to indicate that h is a composition of f and g (in the sense of Definition 1.3.4.1. In this case, it should be implicitly understood that we have chosen a 2-simplex that witnesses h as a composition of f and g. We will sometimes abuse terminology by referring to h as *the* composition of f and g. However, the reader should beware that only the homotopy class of h is well-defined (Proposition 1.3.4.2).

0045 **Example 1.3.4.4.** Let $\mathcal{C}$ be an ordinary category containing a pair of morphisms $f : X \to Y$ and $g : Y \to Z$. Then there is a unique morphism $h : X \to Z$ in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ which is a composition of f and g, given by the usual composition $g \circ f$ in the category $\mathcal{C}$.

0046 **Example 1.3.4.5.** Let X be a topological space and suppose we are given continuous paths $f, g : [0,1] \to X$ which are composable in the sense that $f(1) = g(0)$, and let $g \star f : [0,1] \to X$ denote the path obtained by concatenating f and g, given concretely by the formula

$$(g \star f)(ts) = \begin{cases} f(2s) & \text{if } 0 \leqslant s \leqslant 1/2 \\ g(2s-1) & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

Then $g \star f$ is a composition of f and g in the ∞-category $\mathrm{Sing}_\bullet(X)$. More precisely, the continuous map

$$\sigma : |\Delta^2| \to X \qquad \sigma(t_0, t_1, t_2) = \begin{cases} f(t_1 + 2t_2) & \text{if } t_0 \geqslant t_2 \\ g(t_2 - t_0) & \text{if } t_0 \leqslant t_2. \end{cases}$$

can be regarded as a 2-simplex of $\mathrm{Sing}_\bullet(X)$ which witnesses $g \circ f$ as a composition of f and g.

0047 **Warning 1.3.4.6.** In the situation of Example 1.3.4.5, the concatenation $g \star f$ is not the only path which is a composition of f and g in the ∞-category $\mathrm{Sing}_\bullet(\mathcal{C})$. Any path in X which is homotopic to $g \star f$ (with endpoints fixed) has the same property, by virtue of Proposition 1.3.4.2 (and Example 1.3.3.3). For example, we can replace $g \star f$ by a reparametrization, such as the path

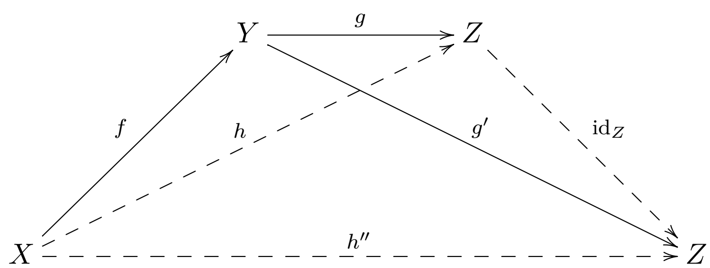
$$(s \in [0,1]) \mapsto \begin{cases} f(3s) & \text{if } 0 \leqslant s \leqslant 1/3 \\ g(\frac{3}{2}s - \frac{1}{2}) & \text{if } 1/3 \leqslant s \leqslant 1. \end{cases}$$

When viewing $\mathrm{Sing}_\bullet(X)$ as an ∞-category, all of these paths have an equal claim to be regarded as "the" composition of f and g.

We now show that composition respects the relation of homotopy:

0048 **Proposition 1.3.4.7.** *Let $\mathcal{C}$ be an ∞-category. Suppose we are given a pair of homotopic morphisms $f, f' : X \to Y$ in $\mathcal{C}$ and a pair of homotopic morphisms $g, g' : Y \to Z$ in $\mathcal{C}$. Let h be a composition of f and g, and let h' be a composition of f' and g'. Then h is homotopic to h'.*

Proof. Let h'' be a composition of f and g'. Since homotopy is an equivalence relation (Proposition 1.3.3.5), it will suffice to show that both h and h' are homotopic to h''. We will show that h is homotopic to h''; the proof that h' is homotopic to h'' is similar. Let σ_3 be a 2-simplex of $\mathcal{C}$ which witnesses h as a composition of f and g, let σ_2 be a 2-simplex of $\mathcal{C}$ which witnesses h'' as a composition of f and g', and let σ_0 be a 2-simplex of $\mathcal{C}$ which is a homotopy from g to g'. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram



where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. Then the face $d_1(\tau)$ is a homotopy from h to h''. □

1.3.5 The Homotopy Category

To any topological space X, one can associate a category $\pi_{\leqslant 1}(X)$, called the *fundemental* 0049
groupoid of X. This category can be described informally as follows:

- The objects of $\pi_{\leqslant 1}(X)$ are the points of X.
- Given a pair of points $x, y \in X$, we can identify $\mathrm{Hom}_{\pi_{\leqslant 1}(X)}(x, y)$ with the set of *homotopy classes* of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.
- Composition in $\tau_{\leqslant 1}(X)$ is given by concatenation of paths.

All of the concepts needed to define the fundamental groupoid $\pi_{\leqslant 1}(X)$ (such as points, paths, homotopies, and concatenation) can be formulated in terms of singular n-simplices of X (for $n \leqslant 2$). Consequently, one can view the fundamental groupoid $\pi_{\leqslant 1}(X)$ as an invariant of the simplicial set $\mathrm{Sing}_\bullet(X)$, rather than the topological space X. In this section, we describe an extension of this invariant, where the simplicial set $\mathrm{Sing}_\bullet(X)$ is replaced by an arbitrary ∞-category $\mathcal{C}$. In this case, the fundamental groupoid $\pi_{\leqslant 1}(X)$ is replaced by a category $\mathrm{h}\mathcal{C}$ which we call the *homotopy category* of $\mathcal{C}$ (beware that the homotopy category $\mathrm{h}\mathcal{C}$ is generally not a groupoid: in fact, we will later see that it is a groupoid if and only if $\mathcal{C}$ is a Kan complex (Theorem [?]).

004A **Construction 1.3.5.1.** Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ denote the set of homotopy classes of morphisms from X to Y in $\mathcal{C}$. For every morphism $f : X \to Y$, we let $[f]$ denote its equivalence class in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$.

It follows from Propositions 1.3.4.2 and 1.3.4.7 that, for every triple of objects $X, Y, Z \in \mathcal{C}$, there is a unique composition law

$$\circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$$

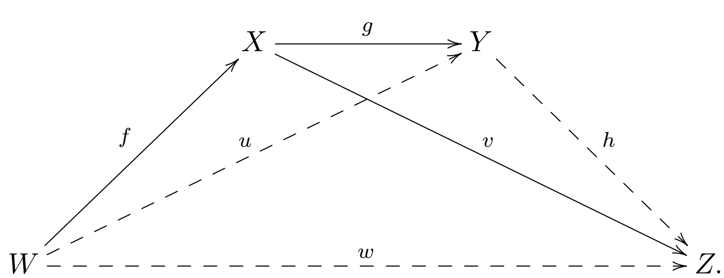
satisfying the identity $[g] \circ [f] = [h]$ whenever $h : X \to Z$ is a composition of f and g in the ∞-category $\mathcal{C}$.

004B **Proposition 1.3.5.2.** *Let $\mathcal{C}$ be an ∞-category. Then:*

(1) *The composition law of Construction 1.3.5.1 is associative. That is, for every triple of composable morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ in $\mathcal{C}$, we have an equality $([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$ in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(W, Z)$.*

(2) *For every object $X \in \mathcal{C}$, the homotopy class $[\mathrm{id}_X] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, X)$ is a two-sided identity with respect to the composition law of Construction 1.3.5.1. That is, for every morphism $f : W \to X$ in $\mathcal{C}$ and every morphism $g : X \to Y$ in $\mathcal{C}$, we have $[\mathrm{id}_X] \circ [f] = [f]$ and $[g] \circ [\mathrm{id}_X] = [g]$.*

Proof. We first prove (1). Let $u : W \to Y$ be a composition of f and g, let $v : X \to Z$ be a composition of g and h, and let $w : W \to Z$ be a composition of f and v. Then $([h] \circ [g]) \circ [f] = [w]$ and $[h] \circ ([g] \circ [f]) = [h] \circ [u]$. It will therefore suffice to show that w is a composition of u and h. Choose a 2-simplex σ_0 of $\mathcal{C}$ which witnesses v as a composition of g and h, a 2-simplex σ_2 of $\mathcal{C}$ which witnesses w as a composition of f and v, and a 2-simplex σ_3 of $\mathcal{C}$ which witnesses u as a composition of f and g. Then the sequence $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram



Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. Then the 2-simplex $d_1(\tau)$ witnesses w as a composition of u and h.

We now prove (2). Fix an object $X \in \mathcal{C}$ and a morphism $g : X \to Y$ in $\mathcal{C}$; we will show that $[g] \circ [\mathrm{id}_X] = [g]$ (the analogous identity $[\mathrm{id}_X] \circ [f] = [f]$ follows by a similar argument). For this, it suffices to observe that the degenerate 2-simplex $s_0(g)$ witnesses g as a composition of id_X and g. □

Definition 1.3.5.3 (The Homotopy Category)**.** Let $\mathcal{C}$ be an ∞-category. We define a category $\mathrm{h}\mathcal{C}$ as follows: 004C

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ denote the collection of homotopy classes of morphisms from X to Y in the ∞-category $\mathcal{C}$ (as in Construction 1.3.5.1.
- For every object $X \in \mathcal{C}$, the identity morphism from X to itself in $\mathrm{h}\mathcal{C}$ is given by the homotopy class $[\mathrm{id}_X]$.
- Composition of morphisms is defined as in Construction 1.3.5.1.

We will refer to $\mathrm{h}\mathcal{C}$ as *the homotopy category* of the ∞-category $\mathcal{C}$.

Example 1.3.5.4. Let $\mathcal{C}$ be an ordinary category. Then the homotopy category of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with $\mathcal{C}$. 004D

Example 1.3.5.5. Let X be a topological space, and regard the singular simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then the homotopy category $\mathrm{hSing}_\bullet(X)$ can be identified with the fundamental groupoid $\pi_{\leqslant 1}(X)$. More precisely, we can regard the contents of §1.3, when specialized to ∞-categories of the form $\mathrm{Sing}_\bullet(X)$, as providing a *construction* of the fundamental groupoid of X. By virtue of Exercise 1.3.3.4 and Example 1.3.4.5, the resulting category $\mathrm{hSing}_\bullet(X)$ matches the informal description of $\pi_{\leqslant 1}(X)$ given in the introduction to §1.3.5. 004E

1.3.6 The Universal Property of $\mathrm{h}\mathcal{C}$

We now give an alternative description of the homotopy category of an ∞-category. 004F

Construction 1.3.6.1. Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$. For $0 \leqslant i \leqslant n$, let C_i denote the object of $\mathcal{C}$ given by the image of the ith vertex of Δ^n. For $0 \leqslant i \leqslant j \leqslant n$, let $f_{ij} : C_i \to C_j$ denote the image under σ of the edge of Δ^n joining the ith vertex to the jth vertex, and let $[f_{ij}] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C_i, C_j)$ denote the homotopy class of f_{ij}. Then we can regard $(\{C_i\}_{0 \leqslant i \leqslant n}, \{[f_{ij}]\}_{0 \leqslant i \leqslant j \leqslant n})$ as a functor from the linearly ordered set $[n]$ to the homotopy category $\mathrm{h}\mathcal{C}$. Let $u(\sigma)$ denote the corresponding n-simplex of $\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$. Then the construction $\sigma \mapsto u(\sigma)$ determines a map of simplicial sets 004G

$$u : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}).$$

The comparison map of Construction 1.3.6.1 has the following universal property:

004H **Proposition 1.3.6.2.** *Let $\mathcal{C}$ be an ∞-category and let $u : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ be as in Construction 1.3.6.1. For every category $\mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathrm{h}\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathrm{N}_\bullet(\mathcal{D}))$$

is bijective.

Proof. Let $F : \mathcal{C} \to \mathrm{N}_\bullet(\mathcal{D})$ be a map of simplicial sets. Then F induces a functor of homotopy categories $G : \mathrm{h}\mathcal{C} \to \mathrm{hN}_\bullet(\mathcal{D}) \simeq \mathcal{D}$ (where the second identification comes from Example 1.3.5.4). By construction, the map of simplicial sets

$$\mathcal{C} \xrightarrow{u} \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}) \xrightarrow{\mathrm{N}_\bullet(G)} \mathrm{N}_\bullet(\mathcal{D})$$

agrees with F on the vertices and edges of $\mathcal{C}$, and therefore coincides with F (since a simplex of $\mathrm{N}_\bullet(\mathcal{D})$ is determined by its 1-dimensional facets; see Remark 1.2.1.3). We leave it to the reader to verify that G is the unique functor with this property. □

Using Proposition 1.3.6.2, we can extend the notion of homotopy category to more general simplicial sets.

004J **Definition 1.3.6.3.** Let $\mathcal{C}$ be a category. We will say that a map of simplicial sets $u : S_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ *exhibits $\mathcal{C}$ as the homotopy category of $S_\bullet$* if, for every category $\mathcal{D}$, the composite map

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{N}_\bullet(\mathcal{D}))$$

is bijective (note that the map on the left is always bijective, by virtue of Proposition 1.2.2.1).

004K **Notation 1.3.6.4.** Let $S_\bullet$ be a simplicial set. It follows immediately from the definition that if there exists a category $\mathcal{C}$ and a map $u : S_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of $S_\bullet$, then the category $\mathcal{C}$ is unique up to isomorphism and depends functorially on $S_\bullet$. To emphasize this dependence, we will refer to $\mathcal{C}$ as *the* homotopy category of $S_\bullet$ and denote it by $\mathrm{h}S_\bullet$.

004L **Example 1.3.6.5.** Let $\mathcal{C}$ be an ∞-category. Then the homotopy category $\mathrm{h}\mathcal{C}$ constructed in Definition 1.3.5.3 is also a homotopy category of $\mathcal{C}$ in the sense of Definition 1.3.6.3. More precisely, the map $u : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ of Construction 1.3.6.1 exhibits $\mathrm{h}\mathcal{C}$ as a homotopy category of $\mathcal{C}$, by virtue of Proposition 1.3.6.2.

004M **Proposition 1.3.6.6.** *Let $S_\bullet$ be a simplicial set. Then there exists a category $\mathcal{C}$ and a map of simplicial sets $u : S_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of $S_\bullet$.*

Proof. Let $Q^\bullet$ denote the cosimplicial object of Cat given by the inclusion $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}$. Unwinding the definitions, we see that a homotopy category of $S_\bullet$ can be identified with a realization $|S_\bullet|^Q$, whose existence is a special case of Proposition 1.1.6.18. Alternatively, we can give a direct construction of the homotopy category $\mathrm{h}S_\bullet$:

- The objects of $\mathrm{h}S_\bullet$ are the vertices of $S_\bullet$.
- Every edge e of $S_\bullet$ determines a morphism $[e]$ in $\mathrm{h}S_\bullet$, whose domain is the vertex $d_1(e)$ and whose codomain is the vertex $d_0(e)$.
- The collection of morphisms in $\mathrm{h}S_\bullet$ is generated under composition by morphisms of the form $[e]$, subject only to the relations

$$[s_0(x)] = \mathrm{id}_x \text{ for } x \in S_0 \qquad [d_1(\sigma)] = [d_0(\sigma)] \circ [d_2(\sigma)] \text{ for } \sigma \in S_2.$$

□

Corollary 1.3.6.7. *The nerve functor* $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ *admits a left adjoint, given on objects by the construction* $S_\bullet \mapsto \mathrm{h}S_\bullet$. 004N

Example 1.3.6.8. Let G be a directed graph (Definition 1.1.4.1) and let $S_\bullet$ denote the associated simplicial set of dimension $\leqslant 1$ (Proposition 1.1.4.9). Then the homotopy category $\mathrm{h}S_\bullet$ can be described explicitly as follows: 004P

- The objects of $\mathrm{h}S_\bullet$ are the vertices of the graph G.
- Given a pair of vertices $v, w \in \mathrm{Vert}(G)$, a morphism from v to w in $\mathrm{h}S_\bullet$ is given by a *path* from v to w in the directed graph G: that is, an ordered sequence of edges $(e_1, e_2, \ldots, e_n)$ satisfying $s(e_1) = v$, $t(e_n) = w$, and $t(e_i) = s(e_{i+1})$ for $0 < i < n$. Here $s, t : \mathrm{Vert}(G) \to \mathrm{Edge}(G)$ denote the source and target maps. Moreover, we allow $n = 0$ in the case $v = w$ (the empty sequence is regarded as the identity morphism from the vertex v to itself).
- Composition of morphisms in $\mathrm{h}S_\bullet$ is given by concatenation of paths. More precisely, given morphisms $f = (e_1, e_2, \ldots, e_m)$ from u to v and $g = (e'_1, e'_2, \ldots, e'_n)$ from v to w, the composition $g \circ f$ is given by the sequence $(e_1, e_2, \ldots, e_m, e'_1, \ldots, e'_n)$.

1.3.7 Equivalences

Recall that a morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ satisfying $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. This notion has an ∞-categorical analogue: 004Q

004R **Definition 1.3.7.1.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We will say that f is an *equivalence* if the homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that two objects $X, Y \in \mathcal{C}$ are *equivalent* if there exists an equivalence between X to Y (that is, if X and Y are isomorphic as objects of the homotopy category $\mathrm{h}\mathcal{C}$).

004S **Example 1.3.7.2.** Let $\mathcal{C}$ be an ordinary category. Then a morphism $f : X \to Y$ of $\mathcal{C}$ is an isomorphism if and only if it is an equivalence when regarded as a morphism of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$.

004T **Remark 1.3.7.3.** If $f : X \to Y$ is an equivalence in an ∞-category $\mathcal{C}$, then one should regard the objects $X, Y \in \mathcal{C}$ as essentially interchangeable, just as isomorphic objects of an ordinary category are essentially interchangeable. Our use of the term "equivalence" rather than "isomorphism" is motivated by the desire to avoid confusion in situations where a class of mathematical objects admits both 1-categorical and ∞-categorical descriptions. For example:

- The collection of topological spaces can be organized into an ordinary category Top, whose morphisms are continuous functions and whose isomorphisms are homeomorphisms. However, it can also be organized into an ∞-category $\mathrm{N}^{\Delta}_{\bullet}(\mathrm{Top})$ (see §[?]) in which the equivalences (in the sense of Definition 1.3.7.1) are homotopy equivalences: that is, continuous functions which admit a homotopy inverse.

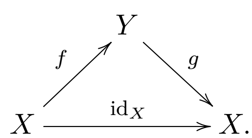
- The collection of (small) categories can be organized into an ordinary category Cat, whose morphisms are functors and whose isomorphisms are functors which are fully faithful and bijective on objects. However, it can also be organized into an ∞-category in which the equivalences (in the sense of Definition 1.3.7.1) are equivalences of categories: that is, functors which are fully faithful and bijective on *isomorphism classes* of objects.

004U **Remark 1.3.7.4** (Two-out-of-three)**.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in an ∞-category $\mathcal{C}$ and let h be a composition of f and g. If any two of the morphisms f, g, and h is an equivalence, then so is the third.

004V **Definition 1.3.7.5.** Let $\mathcal{C}$ be an ∞-category and suppose we are given a pair of morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$. We say that g is a *left homotopy inverse* of f if the identity morphism id_X is a composition of f and g: that is, if we have an equality $[\mathrm{id}_X] = [g] \circ [f]$ in the homotopy category $\mathrm{h}\mathcal{C}$. We say that g is a *right homotopy inverse* of f if the identity morphism id_Y is a composition of g and f: that is, if we have an equality $[\mathrm{id}_Y] = [f] \circ [g]$ in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that g is a *homotopy inverse* of f if it is both a left and a right homotopy inverse of f.

Remark 1.3.7.6. Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an ∞-category $\mathcal{C}$. Then 004W
the condition that g is a left homotopy inverse (right homotopy inverse, homotopy inverse) to f depends only on the homotopy classes $[f]$ and $[g]$.

Remark 1.3.7.7. Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an ∞-category $\mathcal{C}$. Then 004X
g is left homotopy inverse to f if and only if f is right homotopy inverse to g. Both of these conditions are equivalent to the existence of a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = \mathrm{id}_X$, and $d_2(\sigma) = f$, as depicted in the diagram



Remark 1.3.7.8. Let $f : X \to Y$ be a morphism in ∞-category $\mathcal{C}$. Suppose that f admits 004Y
a left homotopy inverse g and a right homotopy inverse h. Then g and h are homotopic: this follows from the calculation

$$[g] = [g] \circ [\mathrm{id}_Y] = [g] \circ ([f] \circ [h]) = ([g] \circ [f]) \circ [h] = [\mathrm{id}_Y] \circ [h] = [h].$$

It follows that both g and h are homotopy inverse to f.

Remark 1.3.7.9. Let $f : X \to Y$ be a morphism in the ∞-category $\mathcal{C}$. It follows from 004Z
Remark 1.3.7.8 that the following conditions are equivalent:

(1) The morphism f is an equivalence.

(2) The morphism f admits a homotopy inverse g.

(3) The morphism f admits both left and right homotopy inverses.

In this case, the morphism g is uniquely determined up to homotopy; moreover, any left or right homotopy inverse of f is homotopic to g. We will sometimes abuse notation by writing f^{-1} to denote a homotopy inverse to f.

Warning 1.3.7.10. Let $f : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$, and suppose that 0050
$g, h : Y \to X$ are left homotopy inverses to f. If f does not admit a right homotopy inverse, then g and h need not be homotopic.

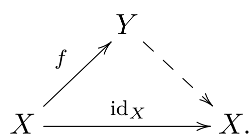
Proposition 1.3.7.11 (Two-out-of-Six)**.** *Let $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ be* 0051
morphisms in an ∞-category $\mathcal{C}$. If the morphisms $g \circ f$ and $h \circ g$ are equivalences, then f, g, and h are also equivalences.

Proof. Let u be a homotopy inverse to $g \circ f$. Then the iterated composition $g \circ (f \circ u)$ is homotopic to the identity, so that g admits a right homotopy inverse. Similarly, g admits a left homotopy inverse. It follows that g is an equivalence (Remark 1.3.7.8). Since $f \circ u$ is a right homotopy inverse to g, it is homotopy inverse to g (Remark 1.3.7.8), and is therefore also an equivalence. Applying Remark 1.3.7.4, we conclude that f is also an equivalence. A similar argument shows that h is an equivalence. □

0052 **Proposition 1.3.7.12.** *Let $\mathcal{C}$ be a Kan complex. Then every morphism in $\mathcal{C}$ is an equivalence.*

0053 **Remark 1.3.7.13.** We will see later that the converse to Proposition 1.3.7.12 is also true: if $\mathcal{C}$ is an ∞-category in which every morphism is an equivalence, then $\mathcal{C}$ is a Kan complex (Theorem [?]).

Proof of Proposition 1.3.7.12. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the tuple $(\bullet, \mathrm{id}_X, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_0 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict as



If $\mathcal{C}$ is a Kan complex, then we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$. Then σ exhibits the morphism $g = d_0(\sigma)$ as a left homotopy inverse to f. A similar argument shows that f admits a right homotopy inverse, so that f is an equivalence by virtue of Remark 1.3.7.9. □

0054 **Example 1.3.7.14.** Let X be a topological space. Then every morphism in the ∞-category $\mathrm{Sing}_\bullet(X)$ is an equivalence. In other words, the category $\pi_{\leqslant 1}(X) = \mathrm{hSing}_\bullet(X)$ is a groupoid. This follows by combining Propositions 1.3.7.12 and 1.1.7.3. However, it is also easy to see directly: if f is a morphism of $\mathrm{Sing}_\bullet(X)$, given by a continous path $f : [0,1] \to X$, then the continuous path

$$g : [0,1] \to X \qquad g(t) = f(1-t)$$

is a homotopy inverse of f in the ∞-category $\mathrm{Sing}_\bullet(X)$.

1.4 Functors of ∞-Categories

0055 Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathrm{N}_\bullet(\mathcal{C})$ and $\mathrm{N}_\bullet(\mathcal{D})$ denote the corresponding ∞-categories. According to Proposition 1.2.2.1, the nerve functor $\mathrm{N}_\bullet$ induces a bijection

$$\{\text{Functors } F : \mathcal{C} \to \mathcal{D}\} \simeq \{\text{Maps of simplicial sets } \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})\}.$$

Consequently, the notion of functor admits an obvious generalization to the setting of ∞-categories:

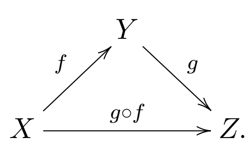
Definition 1.4.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is a map of 0056
simplicial sets $F : \mathcal{C} \to \mathcal{D}$.

This section is devoted to the study of functors between ∞-categories, in the sense of Definition 1.4.0.1. We begin in §1.4.1 with some simple examples, which illustrate the meaning of Definition 1.4.0.1 in the case of ∞-categories which arise from ordinary categories (via the construction $\mathcal{E} \mapsto \mathrm{N}_{\bullet}(\mathcal{E})$) or topological spaces (via the construction $X \mapsto \mathrm{Sing}_{\bullet}(X)$).

In ordinary category theory, one can think of a functor $F : \mathcal{C} \to \mathcal{D}$ as a kind of *commutative diagram* in $\mathcal{D}$, having vertices indexed by the objects of $\mathcal{C}$ and arrows indexed by the morphisms of $\mathcal{C}$. This perspective is quite useful: if the category $\mathcal{C}$ is sufficiently small, one can communicate the datum of a functor by drawing a graphical representation of the corresponding diagram. In §1.4.2, we discuss the notion of commutative diagram in an ∞-category (Convention 1.4.2.11) and describe some dangers associated with diagrammatic reasoning in the higher-categorical setting (Remark 1.4.2.12).

If $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories, then the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ can itself be organized into a category, which we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In §1.4.3, we describe a counterpart of this construction in the setting of ∞-categories. For every pair of simplicial sets $S_{\bullet}$ and $T_{\bullet}$, one can form a new simplicial set $\mathrm{Fun}(S_{\bullet}, T_{\bullet})$ whose vertices are maps from $S_{\bullet}$ to $T_{\bullet}$ (Construction 1.4.3.1). The main result of this section asserts that if $T_{\bullet}$ is an ∞-category, then $\mathrm{Fun}(S_{\bullet}, T_{\bullet})$ is also an ∞-category (Theorem 1.4.3.7). Moreover, our notation is consistent: in the case where $S_{\bullet}$ and $T_{\bullet}$ are isomorphic to the nerves of categories $\mathcal{C}$ and $\mathcal{D}$, the ∞-category $\mathrm{Fun}(S_{\bullet}, T_{\bullet})$ is isomorphic to the nerve of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (Proposition 1.4.3.3).

In order to prove Theorem 1.4.3.7, we will need to introduce some auxiliary ideas. Recall that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms in an ∞-category $\mathcal{C}$, then we can form a composition of f and g by choosing a 2-simplex σ of $\mathcal{C}$ which satisfies $d_0(\sigma) = g$ and $d_2(\sigma) = f$, as indicated in the diagram



We proved in §1.3.4 that the resulting morphism $g \circ f$ is well-defined up to homotopy (Proposition 1.3.4.2). In §1.4.6, we prove a variant of this assertion which asserts that the 2-simplex σ is "unique up to a contractible space of choices" (see Corollary 1.4.6.2 for a precise statement). Moreover, we show that a strong version of this uniqueness result is *equivalent* to the assumption that $\mathcal{C}$ is an ∞-category (Theorem 1.4.6.1), and deduce the existence of functor ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as a consequence (Theorem 1.4.3.7). The precise formulation and proof of Theorem 1.4.6.1 will require some general ideas about categorical lifting properties and the homotopy theory of simplicial sets, which we develop in §1.4.4 and §1.4.5, respectively.

1.4.1 Examples of Functors

0057 Let us begin by illustrating Definition 1.4.0.1 in some special cases.

0058 **Example 1.4.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. It follows from Proposition 1.2.2.1 that the formation of nerves induces a bijection

$$\begin{array}{c}\{\text{Functors of } \infty\text{-categories from } \mathrm{N}_\bullet(\mathcal{C}) \text{ to } \mathrm{N}_\bullet(\mathcal{D})\} \\ \downarrow \sim \\ \{\text{Functors of ordinary categories from } \mathcal{C} \text{ to } \mathcal{D}\}.\end{array}$$

In other words, Definition 1.4.0.1 can be regarded as a generalization of the usual notion of functor to the setting of ∞-categories.

0059 **Example 1.4.1.2.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}$ be an ordinary category. Using Proposition 1.3.6.2, we obtain a bijection

$$\begin{array}{c}\{\text{Functors of } \infty\text{-categories from } \mathcal{C} \text{ to } \mathrm{N}_\bullet(\mathcal{D})\} \\ \downarrow \sim \\ \{\text{Functors of ordinary categories from } \mathrm{h}\mathcal{C} \text{ to } \mathcal{D}\}.\end{array}$$

005A **Remark 1.4.1.3.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

(a) To each object $X \in \mathcal{C}$ the functor F assigns an object of $\mathcal{D}$, which we will denote by $F(X)$ (or sometimes more simply by FX).

(b) To each morphism $f : X \to Y$ in the ∞-category $\mathcal{C}$, the functor F assigns a morphism $F(f) : F(X) \to F(Y)$ in the ∞-category $\mathcal{D}$.

(c) For every object $X \in \mathcal{C}$, the functor F carries the identity morphism $\mathrm{id}_X : X \to X$ in $\mathcal{C}$ to the identity morphism $\mathrm{id}_{F(X)} : F(X) \to F(X)$ in $\mathcal{D}$.

(d) If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$ and h is a composition of f and g (in the sense of Definition 1.3.4.1), then the morphism $F(h) : F(X) \to F(Z)$ is a composition of $F(f)$ and $F(g)$.

005B **Warning 1.4.1.4.** To define a functor F from an ordinary category $\mathcal{C}$ to an ordinary category $\mathcal{D}$, it suffices to specify the values of F on objects and morphisms (as described in (a) and (b) of Remark 1.4.1.3) and to verify that F is compatible with the formation of composition and identity morphisms (as described in (c) and (d) of Remark 1.4.1.3). In the ∞-categorical setting, this is not enough: to give a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, one must specify its values on simplices of *all* dimensions. Roughly speaking, these values encode the requirement that F is compatible with composition "up to coherent homotopy." For example, suppose that we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. Part (d) of Remark 1.4.1.3 asserts that if h is a composition of f and g, then $F(h)$ is a composition of $F(f)$ and $F(g)$. However, we can say more: if σ is a 2-simplex of $\mathcal{C}$ which *witnesses* h as a composition of f and g, then $F(\sigma)$ is a 2-simplex of $\mathcal{D}$ which witnesses $F(h)$ as a composition of $F(f)$ and $F(g)$.

Remark 1.4.1.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If $f, g : X \to Y$ are homotopic morphisms of $\mathcal{C}$, then $F(f), F(g) : F(X) \to F(Y)$ are homotopic morphisms of $\mathcal{D}$. More precisely, the functor F carries homotopies from f to g (viewed as 2-simplices of $\mathcal{C}$) to homotopies from $F(f)$ to $F(g)$ (viewed as 2-simplices of $\mathcal{D}$). 005C

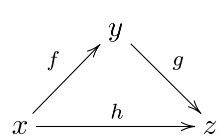
Remark 1.4.1.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If $f : X \to Y$ is a morphism in $\mathcal{C}$ and $g : Y \to X$ is a homotopy inverse to f, then $F(g)$ is a homotopy inverse to $F(f)$. In particular, if f is an equivalence in $\mathcal{C}$, then $F(f)$ is also an equivalence in $\mathcal{D}$. 005D

Example 1.4.1.7. Let X be a topological space and let $\mathcal{C}$ be an ordinary category. To specify a functor of ∞-categories $F : \mathrm{Sing}_\bullet(X) \to \mathrm{N}_\bullet(\mathcal{C})$, one must give a rule which assigns to each continuous map $\sigma : |\Delta^n| \to X$ (viewed as an n-simplex of $\mathrm{Sing}_\bullet(X)$) a diagram $F(\sigma) = (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n)$. In particular: 005E

(a) To each point $x \in X$, the functor F assigns an object $F(x) \in \mathcal{C}$.

(b) To each continuous path $f : [0,1] \to X$ starting at the point $x = f(0)$ and ending at the point $y = f(1)$, the functor F assigns a morphism $F(f) : F(x) \to F(y)$ in the category $\mathcal{C}$. The morphism $F(f)$ is automatically an isomorphism (by virtue of Proposition 1.3.7.12 and Remark 1.4.1.6).

(c) For each continuous map $\sigma : |\Delta^2| \to X$ with boundary behavior as depicted in the diagram



we have an identity $F(h) = F(g) \circ F(f)$ in $\mathrm{Hom}_{\mathcal{C}}(F(x), F(z))$.

The data of a collection of objects $\{F(x)\}_{x \in X}$ and isomorphisms $\{F(f)\}_{f : [0,1] \to X}$ satisfying

(c) is called a *$\mathcal{C}$-valued local system on X*. The preceding discussion determines a bijection

$$\begin{array}{c}\{\text{ Functors of } \infty\text{-categories from } \mathrm{Sing}_\bullet(X) \text{ to } \mathrm{N}_\bullet(\mathcal{C})\} \\ \downarrow \sim \\ \{\ \mathcal{C}\text{-valued local systems on } X\}.\end{array}$$

By virtue of Example 1.4.1.2, we can also identify local systems with functors from the fundamental groupoid $\pi_{\leqslant 1}(X)$ into $\mathcal{C}$.

005F **Remark 1.4.1.8.** Let X be a topological space and let $\mathcal{C}$ be an arbitrary ∞-category. Motivated by Example 1.4.1.7, one can define a *$\mathcal{C}$-valued local system on X* to be a functor of ∞-categories $\mathrm{Sing}_\bullet(X) \to \mathcal{C}$. Beware that this notion generally cannot be reformulated in terms of the fundamental groupoid $\pi_{\leqslant 1}(X)$.

005G **Example 1.4.1.9.** Let $\mathcal{C}$ be an ∞-category and let X be a topological space. Then we have a canonical bijection

$$\begin{array}{c}\{\text{Functors of } \infty\text{-categories from } \mathcal{C} \text{ to } \mathrm{Sing}_\bullet(X)\} \\ \downarrow \sim \\ \{\text{Maps of topological spaces from } |\mathcal{C}| \text{ to } X\}.\end{array}$$

Here $|\mathcal{C}|$ denotes the geometric realization of the simplicial set $\mathcal{C}$ (see Definition 1.1.6.1). Beware that neither side has an obvious interpretation in terms of functors between ordinary categories (even in the special case where $\mathcal{C}$ is the nerve of a category).

1.4.2 Commutative Diagrams

005H We now consider a variant of the terminology introduced in §1.4.1.

005J **Definition 1.4.2.1.** Let $\mathcal{C}$ be an ∞-category. A *diagram in $\mathcal{C}$* is a map of simplicial sets $f : K_\bullet \to \mathcal{C}$. We will also refer to a map $f : K_\bullet \to \mathcal{C}$ as a *diagram in $\mathcal{C}$ indexed by $K_\bullet$*, or a *$K_\bullet$-indexed diagram in $\mathcal{C}$*.

If $\mathcal{C}$ is an ordinary category, then a *($K_\bullet$-indexed) diagram in $\mathcal{C}$* is a ($K_\bullet$-indexed) diagram in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$.

In the special case where $K_\bullet$ is the nerve $\mathrm{N}_\bullet(I)$ of a partially ordered set I (Remark 1.2.1.8), we will refer to a map $f : K_\bullet \to \mathcal{C}$ as a *diagram in $\mathcal{C}$ indexed by I*, or an *I-indexed diagram in $\mathcal{C}$*.

005K **Remark 1.4.2.2.** In the case where $K_\bullet$ is an ∞-category, Definition 1.4.2.1 is superfluous: a $K_\bullet$-indexed diagram in $\mathcal{C}$ (in the sense of Definition 1.4.2.1) is just a functor from $K_\bullet$ to $\mathcal{C}$ (in the sense of Definition 1.4.0.1). However, the redundant terminology will be useful to signal a shift in emphasis. We will generally refer to a map $f : \mathcal{C} \to \mathcal{D}$ as a *functor* when we wish to regard the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ on an equal footing. By contrast, we will refer to a map $f : K_\bullet \to \mathcal{C}$ as a *diagram* if we are primarily interested in the ∞-category $\mathcal{C}$ (in many cases, the domain of f will be a very simple simplicial set).

Remark 1.4.2.3 (Diagrams of Dimension $\leqslant 1$)**.** Let $\mathcal{C}$ be an ∞-category and let $K_\bullet$ be a 005L
simplicial set of dimension $\leqslant 1$, corresponding to a directed graph G (Proposition 1.1.4.9). In this case, a diagram $K_\bullet \to \mathcal{C}$ can be identified with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_e\}_{e \in \mathrm{Edge}(G)})$, where each C_v is an object of the ∞-category $\mathcal{C}$ and each $f_e : C_{s(e)} \to C_{t(e)}$ is a morphism of $\mathcal{C}$ (here $s(e)$ and $t(e)$ denote the source and target of the edge e). It is often convenient to specify diagrams $K_\bullet \to \mathcal{C}$ by drawing a graphical representation of G (as in Remark 1.1.4.3), where each node is labelled by an object of $\mathcal{C}$ and each arrow is labelled by a morphism in $\mathcal{C}$ (having the indicated domain and codomain).

Example 1.4.2.4 (Non-Commuting Squares)**.** Let $K_\bullet$ denote the the boundary of the 005M
product $\Delta^1 \times \Delta^1$: that is, the simplicial subset of $\Delta^1 \times \Delta^1$ given by the union of the simplicial subsets $\partial\Delta^1 \times \Delta^1$ and $\Delta^1 \times \partial\Delta^1$. Then $K_\bullet$ is a 1-dimensional simplicial set, corresponding to a directed graph which we can depict as

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet. \end{array}$$

We can then display a $K_\bullet$-indexed diagram in an ∞-category $\mathcal{C}$ pictorially

$$\begin{array}{ccc} C_{00} & \xrightarrow{f} & C_{01} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C_{10} & \xrightarrow{f'} & C_{11}, \end{array}$$

where each C_{ij} is an object of $\mathcal{C}$, f is a morphism in $\mathcal{C}$ from C_{00} to C_{01}, g is a morhism in $\mathcal{C}$ from C_{00} to C_{10}, f' is a morphism in $\mathcal{C}$ from C_{10} to $C_1 1$, and g' is a morphism in $\mathcal{C}$ from C_{01} to C_{11}.

In classical category theory, it is useful to extend the notational conventions of Remark 1.4.2.3 to more general situations by introducing the notion of a *commutative diagram.*

Definition 1.4.2.5. Let $K_\bullet$ be a simplicial set of dimension $\leqslant 1$, which we will identify 005N
with a directed graph G (see Proposition 1.1.4.9). Assume that G satisfies the following additional conditions:

(a) For every pair of vertices $v, w \in \mathrm{Vert}(G)$, there is at most one edge of G with source v and target w. We will denote this edge (if it exists) by $(v, w) \in \mathrm{Edge}(G)$.

(b) The graph G has no directed loops. That is, if there exists a sequence of vertices $v_0, v_1, \ldots, v_n \in \mathrm{Vert}(G)$ with the property that the edges (v_{i-1}, v_i) exist for $1 \leqslant i \leqslant n$, then either $m = 0$ or $v_0 \neq v_n$.

Let $\mathcal{C}$ be an ordinary category and suppose we are given a diagram $\sigma : K_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$, which we identify with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_{v,w} : C_v \to C_w\}_{(v,w) \in \mathrm{Edge}(G)})$. We will say that the diagram σ *commutes* (or that σ *is a commutative diagram*) if the following additional condition is satisfied:

(c) Let v and w be vertices of G which are joined by directed paths $(v = v_0, v_1, \ldots, v_m = w)$ and $(v = v'_0, v'_1, \ldots, v'_n = w)$ (so that we the edges $(v_{i-1}, v_i), (v'_{j-1}, v'_j) \in \mathrm{Edge}(G)$ exist for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$). Then we have an identity

$$f_{v_{m-1}, v_m} \circ f_{v_{m-2}, v_{m-1}} \circ \cdots \circ f_{v_0, v_1} = f_{v'_{n-1}, v'_n} \circ f_{v'_{n-2}, v'_{n-1}} \circ \cdots \circ f_{v'_0, v'_1}$$

in the set $\mathrm{Hom}_{\mathcal{C}}(C_v, C_w)$.

005P **Proposition 1.4.2.6.** *Let $K_\bullet$ be a simplicial set of dimension $\leqslant 1$, corresponding to a directed graph G which satisfies conditions (a) and (b) of Definition 1.4.2.5. Let $\mathcal{C}$ be an ordinary category, and let $\sigma : K_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ be diagram. Then:*

(1) *There is a partial ordering $\leqslant$ on the vertex set $\mathrm{Vert}(G)$, where we have $v \leqslant w$ if and only if there exists a sequence of vertices $(v = v_0, v_1, \ldots, v_n = w)$ with the property that the edges $(v_{i-1}, v_i) \in \mathrm{Edge}(G)$ exist for $1 \leqslant i \leqslant n$.*

(2) *There is a unique monomorphism of simplicial sets $K_\bullet \hookrightarrow \mathrm{N}_\bullet(\mathrm{Vert}(G))$ which carries each vertex to itself.*

(3) *The diagram σ extends to a map $\overline{\sigma} : \mathrm{N}_\bullet(\mathrm{Vert}(G)) \to \mathrm{N}_\bullet(\mathcal{C})$ (that is, to a functor $\mathrm{Vert}(G) \to \mathcal{C}$) if and only if it commutative, in the sense of Definition 1.4.2.5. Moreover, if the extension $\overline{\sigma}$ exists, then it is unique.*

Proof. It follows immediately from the definitions that the relation $\leqslant$ defined in (1) is reflexive and transitive. Antisymmetry follows from our assumption that the graph G has no directed loops (condition (b) of Definition 1.4.2.5). By construction, we have $v \leqslant w$ whenever v and w are connected by an edge $(v, w) \in \mathrm{Edge}(G)$. From the description of the simplicial set $K_\bullet$ given in Remark 1.1.4.10, we immediately see that there is a unique map of simplicial sets $i : K_\bullet \to \mathrm{N}_\bullet(\mathrm{Vert}(G))$ which is the identity on vertices. It follows from assumption (a) of Definition 1.4.2.5 that the map i is a monomorphism. Let us henceforth identify $K_\bullet$ with a simplicial subset of $\mathrm{N}_\bullet(\mathrm{Vert}(G))$ given by the image of i. Let us identify

σ with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_{v,w} : C_v \to C_w\}_{(v,w) \in \mathrm{Edge}(G)})$. Suppose that the diagram σ extends to a functor $\overline{\sigma} : \mathrm{Vert}(G) \to \mathcal{C}$. If v and w are a pair of vertices of G with $v \leqslant w$, then we can choose a directed path $(v = v_0, v_1, \ldots, v_n = w)$ from v to w. The compatibility of $\overline{\sigma}$ with composition then guarantees that $\overline{\sigma}$ must carry the edge (v, w) of $\mathrm{N}_\bullet(\mathrm{Vert}(G))$ to the iterated composition $f_{v_{m-1},v_m} \circ f_{v_{m-2},v_{m-1}} \circ \cdots \circ f_{v_0,v_1} \in \mathrm{Hom}_{\mathcal{C}}(C_v, C_w)$. Since the morphism $\overline{\sigma}(v, w)$ is independent of the choice of directed path, it follows that the diagram σ is commutative. Conversely, if σ is commutative, then we can define $\overline{\sigma}$ on morphisms by the formula $\overline{\sigma}(v, w) = f_{v_{m-1},v_m} \circ f_{v_{m-2},v_{m-1}} \circ \cdots \circ f_{v_0,v_1}$ to obtain the desired extension of σ. □

Example 1.4.2.7 (Commutative Squares in a Category)**.** Let $K_\bullet = \partial(\Delta^1 \times \Delta^1)$ be as in 005Q
Example 1.4.2.4. For any ordinary category $\mathcal{C}$, we can display a diagram $\sigma : K_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ pictorially as

$$\begin{array}{ccc} C_{00} & \xrightarrow{f} & C_{01} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C_{10} & \xrightarrow{f'} & C_{11}. \end{array}$$

The diagram σ is commutative if and only if we have $g' \circ f = f' \circ g$ in $\mathrm{Hom}_{\mathcal{C}}(C_{00}, C_{11})$. In this case, Proposition 1.4.2.6 ensures that σ extends uniquely to a diagram $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet(\mathcal{C})$, or equivalently to a functor of ordinary categories $[1] \times [1] \to \mathcal{C}$.

In the setting of ∞-categories, assertion (3) of Proposition 1.4.2.6 is false in general.

Example 1.4.2.8 (Square Diagrams in an ∞-Category)**.** Let I denote the partially ordered 005R
set $[1] \times [1]$. The simplicial set $\mathrm{N}_\bullet(I) \simeq \Delta^1 \times \Delta^1$ has four vertices (given by the elements of I), five nondegenerate edges, and two nondegenerate 2-simplices. Unwinding the definitions, we see that an I-indexed diagram in an ∞-category $\mathcal{C}$ is equivalent to the following data:

- A collection of objects $\{C_{ij}\}_{0 \leqslant i,j \leqslant 1}$ in $\mathcal{C}$.

- A collection of morphisms $f : C_{00} \to C_{01}$, $g : C_{00} \to C_{10}$, $f' : C_{10} \to C_{11}$, $g' : C_{01} \to C_{11}$, and $h : C_{00} \to C_{11}$.

- A 2-simplex σ of $\mathcal{C}$ which witnesses h as a composition of f with g', and a 2-simplex τ of $\mathcal{C}$ which witnesses h as a composition of g with f'.

This data can be depicted graphically as follows:

$$\begin{array}{ccc} C_{00} & \xrightarrow{f} & C_{01} \\ \Big\downarrow g & \overset{\sigma}{\searrow^{h}} & \Big\downarrow g' \\ & \tau & \\ C_{10} & \xrightarrow{f'} & C_{11}. \end{array}$$

Beware that such a diagram is usually not determined by its restriction to the simplicial subset $K_\bullet \subseteq \mathrm{N}_\bullet(I)$ of Example 1.4.2.7.

005S **Exercise 1.4.2.9.** Let $\mathcal{C}$ be an ∞-category and let $K_\bullet \subseteq \Delta^1 \times \Delta^1$ be the simplicial subset appearing in Example 1.4.2.7. Suppose we are given a diagram $\sigma : K_\bullet \to \mathcal{C}$, which we depict graphically as

$$\begin{array}{ccc} C_{00} & \xrightarrow{f} & C_{01} \\ \Big\downarrow g & & \Big\downarrow g' \\ C_{10} & \xrightarrow{f'} & C_{11}. \end{array}$$

Composing with the unit map $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$, we obtain a diagram σ' in the homotopy category $\mathrm{h}\mathcal{C}$, which we can depict as

$$\begin{array}{ccc} C_{00} & \xrightarrow{[f]} & C_{01} \\ \Big\downarrow [g] & & \Big\downarrow [g'] \\ C_{10} & \xrightarrow{[f']} & C_{11}. \end{array}$$

Show that the diagram σ' is commutative if and only if σ can be extended to a map $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{C}$. Beware, however, that this extension is generally not unique.

005T **Warning 1.4.2.10.** Let I be a partially ordered set and let $\mathcal{C}$ be an ∞-category. In the case $I = [1] \times [1]$, Exercise 1.4.2.9 implies that every functor of ordinary categories $I \to \mathrm{h}\mathcal{C}$ can be lifted to a functor of ∞-categories $\mathrm{N}_\bullet(I) \to \mathcal{C}$. Beware that this conclusion is generally false for more complicated partially ordered sets. For example, it fails in the case $I = [1] \times [1] \times [1]$ (see Example [?]).

Example 1.4.2.8 illustrates that the notion of "commutative diagram" becomes considerably more subtle in the setting of ∞-categories. To specify an I-indexed diagram

$F : \mathrm{N}_\bullet(I) \to \mathcal{C}$ of an ∞-category $\mathcal{C}$, one generally needs to specify the values of F on *all* the simplices of the simplicial set $\mathrm{N}_\bullet(I)$. In general, it is not feasible to graphically encode *all* of this data in a comprehensible way. On the other hand, the formalism of commutative diagrams is too useful to completely abandon. We will therefore sacrifice some degree of mathematical precision in favor of clarity of exposition.

Convention 1.4.2.11. Let $\mathcal{C}$ be an ∞-category and let G be a directed graph satisfying conditions (a) and (b) of Definition 1.4.2.5, so that the vertex set $\mathrm{Vert}(G)$ inherits a partial ordering (Proposition 1.4.2.6). We will sometimes refer to the notion of a *commutative* diagram σ in $\mathcal{C}$, which we indicate graphically by a collection of objects $\{C_v\}_{v \in \mathrm{Vert}(G)}$ of $\mathcal{C}$, connected by arrows which are labelled by morphisms $\{f_e\}_{e \in \mathrm{Edge}(G)}$. In this case, it should be understood that σ is a diagram $\mathrm{N}_\bullet(\mathrm{Vert}(G)) \to \mathcal{C}$, which carries each vertex v of $\mathrm{N}_\bullet(\mathrm{Vert}(G))$ to the object $C_v \in \mathcal{C}$ and each edge $e = (v, w)$ of G to the morphism f_e in $\mathcal{C}$. Beware that in this case, the map σ need not be completely determined by the pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_e\}_{e \in \mathrm{Edge}(G)})$ (this pair can instead be identified with the restriction $\sigma|_{K_\bullet}$, where $K_\bullet$ is the 1-dimensional simplicial subset of $\mathrm{N}_\bullet(\mathrm{Vert}(G))$ corresponding to G). 005U

Remark 1.4.2.12. In the situation of Convention 1.4.2.11, suppose that $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$, where $\mathcal{C}_0$ is an ordinary category. In this case, giving a commutative diagram in the ∞-category $\mathcal{C}$ (in the sense of Convention 1.4.2.11) is equivalent to giving a commutative diagram in the ordinary category $\mathcal{C}_0$ (in the sense of Definition 1.4.2.5). In this case, commutativity is a *property* that that the underlying diagram (indexed by a 1-dimensional simplicial set) does or does not possess. For a general ∞-category $\mathcal{C}$, commutativity of a diagram in $\mathcal{C}$ is not a property but a *structure*; to promote a diagram to a commutative diagram, one must specify additional data *witness* the requisite commutativity. 005V

Example 1.4.2.13. Let $\mathcal{C}$ be an ∞-category. If we refer to a commutative diagram σ : 005W

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ X & & \xrightarrow{h} & & Z, \end{array}$$

then we mean that σ is a 2-simplex of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$, and $d_2(\sigma) = f$. In other words, we mean that σ is a 2-simplex which witnesses h as a composition of f and g, in the sense of Definition 1.3.4.1.

Example 1.4.2.14. Let $\mathcal{C}$ be an ∞-category. If we refer to a commutative diagram σ : 005X

$$\begin{array}{ccc} C_{00} & \xrightarrow{f} & C_{01} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C_{10} & \xrightarrow{f'} & C_{11}, \end{array}$$

we implicitly assume that σ is a map from the entire simplicial set $\Delta^1 \times \Delta^1$ to $\mathcal{C}$. In other words, we assume that we have specified another morphism $h : C_{00} \to C_{11}$, which is not indicated in the picture, together with a 2-simplex σ witnessing h as the composition of f and g' and a 2-simplex τ witnessing h as the composition of g and f'.

005Y **Warning 1.4.2.15.** In ordinary category theory, it is sometimes useful to refer to the commutativity of diagrams in situations which do not fit the paradigm of Definition 1.4.2.5. For example, the commutativity of a diagram

$$X \xrightarrow{f} Y \underset{v}{\overset{u}{\rightrightarrows}} Z$$

is often understood as the requirement that $u \circ f = v \circ f$. Beware that this usage is potentially ambiguous (from the shape of the diagram alone, it is not clear that commutativity should enforce the identity $u \circ f = v \circ f$, but not the identity $u = v$), so we will take special care when applying similar terminology in the ∞-categorical setting.

1.4.3 The ∞-Category of Functors

005Z Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then we can form a new category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations (Example [?]). In this section, we describe an analogous construction in the setting of ∞-categories.

0060 **Construction 1.4.3.1.** Let $S_\bullet$ and $T_\bullet$ be simplicial sets. Then the construction

$$([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times S_\bullet, T_\bullet)$$

determines a functor from the category $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets. We regard this functor as a simplicial set which we will denote by $\operatorname{Fun}(S_\bullet, T_\bullet)$.

Note that, given an n-simplex f of $\operatorname{Fun}(S_\bullet, T_\bullet)$ and an n-simplex σ of $S_\bullet$, we can construct an n-simplex $\operatorname{ev}(f, \sigma)$ of $T_\bullet$, given by the composition

$$\Delta^n \xrightarrow{\delta} \Delta^n \times \Delta^n \xrightarrow{\mathrm{id} \times \sigma} \Delta^n \times S_\bullet \xrightarrow{f} T_\bullet.$$

This construction determines a map of simplicial sets $\operatorname{ev} : \operatorname{Fun}(S_\bullet, T_\bullet) \times S_\bullet \to T_\bullet$, which we will refer to as *the evaluation map.*

0061 **Proposition 1.4.3.2.** *Let $S_\bullet$, $T_\bullet$, and $U_\bullet$ be simplicial sets. Then the composite map*

$$\begin{aligned}\theta : \operatorname{Hom}_{\mathrm{Set}_\Delta}(U_\bullet, \operatorname{Fun}(S_\bullet, T_\bullet)) &\to \operatorname{Hom}_{\mathrm{Set}_\Delta}(U_\bullet \times S_\bullet, \operatorname{Fun}(S_\bullet, T_\bullet) \times S_\bullet) \\ &\xrightarrow{\operatorname{ev} \circ} \operatorname{Hom}_{\mathrm{Set}_\Delta}(U_\bullet \times S_\bullet, T_\bullet)\end{aligned}$$

is bijective.

Proof. Let $f : U_\bullet \times S_\bullet \to T_\bullet$ be a map of simplicial sets. For each n-simplex σ of $U_\bullet$, the composite map

$$\Delta^n \times S_\bullet \xrightarrow{\sigma\times \mathrm{id}} U_\bullet \times S_\bullet \xrightarrow{f} T_\bullet$$

can be regarded as an n-simplex of $\mathrm{Fun}(S_\bullet, T_\bullet)$, which we will denote by $g(\sigma)$. The construction $\sigma \mapsto g(\sigma)$ determines a map of simplicial sets $g : U_\bullet \to \mathrm{Fun}(S_\bullet, T_\bullet)$. We leave as an exercise for the reader to verify that g is the unique map satisfying $\theta(g) = f$. □

Beware that the notation of Construction 1.4.3.1 is potentially confusing, because it conflicts with our use of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to denote the category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. However, these usages are compatible:

Proposition 1.4.3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $e : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D}$ denote the evaluation functor, given on objects by the formula $e(F, C) = F(C)$. Then the composite map* *0062*

$$\mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \times \mathrm{N}_\bullet(\mathcal{C}) \simeq \mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}) \xrightarrow{\mathrm{N}_\bullet(e)} \mathrm{N}_\bullet(\mathcal{D})$$

corresponds, under the bijection of Proposition 1.4.3.2, to an isomorphism of simplicial sets $\rho : \mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D}))$.

Proof. For each $n \geqslant 0$, the map ρ is given on n-simplices by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))) &\simeq \mathrm{Hom}_{\mathrm{Cat}}([n], \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}}([n] \times \mathcal{C}, \mathcal{D}) \\
&\xrightarrow{v} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet([n] \times \mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet([n]) \times \mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times \mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Fun}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D}))).
\end{aligned}$$

It will therefore suffice to show that v is bijective, which is a special case of Proposition 1.2.2.1. □

Passing to homotopy categories, we obtain the following weaker result:

Corollary 1.4.3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is a canonical isomorphism of categories* *0063*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{hFun}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})).$$

We can also generalize Proposition 1.4.3.3 as follows:

0064 **Corollary 1.4.3.5.** *Let $S_\bullet$ be a simplicial set having homotopy category $\mathrm{h}S_\bullet$. Then, for any category $\mathcal{D}$, the composite map*

$$\mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S_\bullet, \mathcal{D})) \times S_\bullet \to \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S_\bullet, \mathcal{D})) \times \mathrm{N}_\bullet(\mathrm{h}S_\bullet) \simeq \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S_\bullet, \mathcal{D}) \times \mathrm{h}S_\bullet) \to \mathrm{N}_\bullet(\mathcal{D})$$

induces an isomorphism of simplicial sets $\rho_{S_\bullet} : \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S_\bullet, \mathcal{D})) \simeq \mathrm{Fun}(S_\bullet, \mathrm{N}_\bullet(\mathcal{D}))$.

Proof. The construction $S_\bullet \mapsto \rho_{S_\bullet}$ carries colimits (in the category Set_Δ of simplicial sets) to limits (in the category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$ of morphisms between simplicial sets). Since the category Set_Δ is generated under colimits by objects of the form Δ^n (Lemma 1.1.6.15), it will suffice to prove Corollary 1.4.3.5 in the special case where $S_\bullet \simeq \Delta^n$. In this case, the desired result follows from Proposition 1.4.3.3, since $S_\bullet$ is isomorphic to the nerve of the category $\mathcal{C} = [n]$. □

0065 **Corollary 1.4.3.6.** *The formation of homotopy categories determines a functor $\mathrm{Set}_\Delta \to \mathrm{Cat}$ which commutes with finite products.*

Proof. Since the construction $S_\bullet \mapsto \mathrm{h}S_\bullet$ preserves final objects, it will suffice to show that for any pair of simplicial sets $S_\bullet$ and $T_\bullet$, the canonical map

$$u : \mathrm{h}(S_\bullet \times T_\bullet) \to \mathrm{h}S_\bullet \times \mathrm{h}T_\bullet$$

is an isomorphism of categories. In other words, we wish to show that for any category $\mathcal{C}$, composition with u induces a bijection

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S_\bullet \times \mathrm{h}T_\bullet, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}(S_\bullet \times T_\bullet), \mathcal{C}).$$

Unwinding the definitions, we see that this map is given by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S_\bullet \times \mathrm{h}T_\bullet, \mathcal{C}) &\simeq \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S_\bullet, \mathrm{Fun}(\mathrm{h}T_\bullet, \mathcal{C}))\\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}T_\bullet, \mathcal{C})))\\
&\xrightarrow{\rho_{T_\bullet}\circ} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Fun}(T_\bullet, \mathrm{N}_\bullet(\mathcal{C}))\\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet \times T_\bullet, \mathrm{N}_\bullet(\mathcal{C}))\\
&\simeq \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}(S_\bullet \times T_\bullet), \mathcal{C}),
\end{aligned}$$

where $\rho_{T_\bullet}$ is the isomorphism appearing in the statement of Corollary 1.4.3.5. □

We will be primarily interested in the special case of Construction 1.4.3.1 where the target simplicial set $T_\bullet$ is an ∞-category. In this case, we have the following result:

0066 **Theorem 1.4.3.7.** *Let $S_\bullet$ be a simplicial set and let $\mathcal{D}$ be an ∞-category. Then the simplicial set $\mathrm{Fun}(S_\bullet, \mathcal{D})$ is an ∞-category.*

The proof of Theorem 1.4.3.7 will require some combinatorial preliminaries; we defer the proof to §1.4.6.

Definition 1.4.3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. It follows from Theorem 1.4.3.7 that the 0067
simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also an ∞-category. We will refer to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as *the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$*.

Remark 1.4.3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. By definition, the objects of the ∞-category 0068
$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with functors from $\mathcal{C}$ to $\mathcal{D}$, in the sense of Definition 1.4.0.1 (that is, with maps of simplicial sets from $\mathcal{C}$ to $\mathcal{D}$).

Remark 1.4.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose we are given a pair of functors 0069
$F, G : \mathcal{C} \to \mathcal{D}$. We define a *natural transformation from F to G* is a map of simplicial sets $u : \Delta^1 \times \mathcal{C} \to \mathcal{D}$ satisfying $u|_{\{0\} \times \mathcal{C}} = F$ and $u|_{\{1\} \times \mathcal{C}} = G$. In other words, a natural transformation from F to G is a morphism from F to G in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 1.4.3.11. Let us abuse notation by identifying each ordinary category $\mathcal{E}$ with the 006A
∞-category $\mathrm{N}_\bullet(\mathcal{E})$. In this case, Corollary 1.4.3.5 implies that when $\mathcal{C}$ is an ∞-category and $\mathcal{D}$ is an ordinary category, then we have a canonical isomorphism $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathcal{D})$. In particular, the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an ordinary category.

1.4.4 Digression: Lifting Properties

We now review some categorical terminology which will be useful in the proof of Theorem 006B
1.4.3.7, and in several other parts of this book.

Definition 1.4.4.1. Let $\mathcal{C}$ be a category. A *lifting problem* in $\mathcal{C}$ is a commutative diagram 006C
σ :

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y \end{array}$$

in $\mathcal{C}$. A *solution to the lifting problem σ* is a morphism $h : B \to X$ in $\mathcal{C}$ satisfying $p \circ h = v$ and $h \circ f = u$, as indicated in the diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y. \end{array}$$

006D **Remark 1.4.4.2.** In the situation of Definition 1.4.4.1, we will often indicate a lifting problem by a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y, \end{array}$$

which includes a dotted arrow representing a hypothetical solution.

006E **Definition 1.4.4.3.** Let $\mathcal{C}$ be a category, and suppose we are given a morphisms $f : A \to B$ and $p : X \to Y$ in $\mathcal{C}$. We will say that f *has the left lifting property with respect to* p, or that p *has the right lifting property with respect to* f, if, for every pair of morphisms $u : A \to X$ and $v : B \to Y$ satisfying $p \circ u = v \circ f$, the associated lifting problem

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y \end{array}$$

admits a solution (that is, there exists a map $h : B \to X$ satisfying $p \circ h = v$ and $h \circ f = u$).

If S is a collection of morphisms in $\mathcal{C}$, we will say that a morphism $f : A \to B$ *has the left lifting property with respect to* S if it has the left lifting property with respect to every morphism in S. Similarly, we will say that a morphism $p : X \to Y$ *has the right lifting property with respect to* S if it has the right lifting property with respect to every morphism in S.

Let S be a collection of morphisms in a category $\mathcal{C}$. We now summarize some closure properties enjoyed by the collection of morphisms which have the left lifting property with respect to S.

006F **Definition 1.4.4.4.** Let $\mathcal{C}$ be a category which admits pushouts and let T be a collection of morphisms of $\mathcal{C}$. We will say that T is *closed under pushouts* if, for every pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \longrightarrow & B' \end{array}$$

in the category $\mathcal{C}$ where the morphism i belongs to T, the morphism i' also belongs to T.

Proposition 1.4.4.5. *Let $\mathcal{C}$ be a category which admits pushouts, let S be a collection of morphisms of $\mathcal{C}$, and let T be the collection of all morphisms of $\mathcal{C}$ having the left lifting property with respect to S. Then T is closed under pushouts.* 006G

Proof. Suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{h} & B' \end{array}$$

where f belongs to T. We wish to show that f' also belongs to T. For this, we must show that every lifting problem

$$\begin{array}{ccc} A' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & \nearrow & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{v} & Y, \end{array}$$

admits a solution, provided that the morphism p belongs to S. Using our assumption that σ is a pushout square, we are reduced to solving the associated lifting problem

$$\begin{array}{ccc} A & \xrightarrow{u \circ g} & X \\ \downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow{v \circ h} & Y, \end{array}$$

which is possible by virtue of our assumption that f has the left lifting property with respect to p. □

Definition 1.4.4.6. Let $\mathcal{C}$ be a category containing a pair of objects C and C'. We will say that C *is a retract of* C' if there exist maps $i : C \to C'$ and $r : C' \to C$ such that $r \circ i = \mathrm{id}_C$. 006H

Variant 1.4.4.7. Let $\mathcal{C}$ be a category. We will say that a morphism $f : C \to D$ of $\mathcal{C}$ is a *retract* of another morphism $f' : C' \to D'$ if it is a retract of f' when viewed as an object of the functor category $\mathrm{Fun}([1], \mathcal{C})$. In other words, we say that f is a retract of f' if there exists a commutative diagram 006J

$$\begin{array}{ccccc} C & \xrightarrow{i} & C' & \xrightarrow{r} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\bar{i}} & D' & \xrightarrow{\bar{r}} & D \end{array}$$

in the category $\mathcal{C}$, where $r \circ i = \mathrm{id}_C$ and $\overline{r} \circ \overline{i} = \mathrm{id}_D$.

We say that a collection of morphisms T of $\mathcal{C}$ is *closed under retracts* if, for every pair of morphisms f, f' in $\mathcal{C}$, if f is a retract of f' and f' belongs to T, then f also belongs to T.

006K **Exercise 1.4.4.8.** Let $\mathcal{C}$ be a category and let T be the collection of all monomorphisms in $\mathcal{C}$. Show that T is closed under retracts.

006L **Proposition 1.4.4.9.** *Let $\mathcal{C}$ be a category, let S be a collection of morphisms of $\mathcal{C}$, and let T be the collection of all morphisms of $\mathcal{C}$ having the left lifting property with respect to S. Then T is closed under retracts.*

Proof. Let f' be a morphism of $\mathcal{C}$ which belongs to T and let f be a retract of f', so that there exists a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{i} & C' & \xrightarrow{r} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\overline{i}} & D' & \xrightarrow{\overline{r}} & D \end{array}$$

with $r \circ i = \mathrm{id}_C$ and $\overline{r} \circ \overline{i} = \mathrm{id}_D$. We wish to show that f also belongs to T. Consider a lifting problem σ :

$$\begin{array}{ccc} C & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ D & \xrightarrow{v} & Y, \end{array}$$

where p belongs to S. Our assumption $f' \in T$ ensures that the associated lifting problem

$$\begin{array}{ccc} C' & \xrightarrow{r \circ u} & X \\ \downarrow{\scriptstyle f'} & \nearrow & \downarrow{\scriptstyle p} \\ D' & \xrightarrow{\overline{r} \circ v} & Y, \end{array}$$

admits a solution: that is, we can choose a morphism $h' : D' \to X$ satisfying $p \circ h' = v \circ \overline{r}$ and $h' \circ f' = u \circ r$. Then the morphism $h = h' \circ \overline{i}$ is a solution to the lifting problem σ, by virtue of the calculations

$$p \circ h = p \circ h' \circ \overline{i} = v \circ \overline{r} \circ \overline{i} = v$$

$$h \circ f = h' \circ \overline{i} \circ f = h' \circ f' \circ i = u \circ r \circ i = u.$$

□

Definition 1.4.4.10. For every ordinal number α, let $[\alpha] = \{\beta : \beta \leqslant \alpha\}$ denote the collection of all ordinal numbers which are less than or equal to α, regarded as a linearly ordered set. 006M

Let $\mathcal{C}$ be a category and let T be a collection of morphisms of $\mathcal{C}$. We will say that a morphism f of $\mathcal{C}$ is a *transfinite composition of morphisms of* T if there exists an ordinal number α and a functor $F : [\alpha] \to \mathcal{C}$, given by a collection of objects $\{C_\beta\}_{\beta \leqslant \alpha}$ and morphisms $\{f_{\beta,\gamma} : C_\beta \to C_\gamma\}_{\beta \leqslant \gamma}$ with the following properties:

(a) For every nonzero limit ordinal $\lambda \leqslant \alpha$, the functor F exhibits C_γ as a colimit of the diagram $(\{C_\beta\}_{\beta < \lambda}, \{f_{\beta,\beta'}\}_{\beta \leqslant \gamma < \lambda})$.

(b) For every ordinal $\beta < \alpha$, the morphism $f_{\beta,\beta+1}$ belongs to T.

(c) The morphism f is equal to $f_{0,\alpha} : C_0 \to C_\alpha$.

We will say that T is *closed under transfinite composition* if, for every morphism f which is a transfinite composition of morphisms of T, we have $f \in T$.

Example 1.4.4.11. Let $\mathcal{C}$ be a category and let T be a collection of morphisms of $\mathcal{C}$. Then every identity morphism of $\mathcal{C}$ is a transfinite composition of morphisms of T (take $\alpha = 0$ in Definition 1.4.4.10). In particular, if T is closed under transfinite composition, then it contains every identity morphism of $\mathcal{C}$. 006N

Example 1.4.4.12. Let $\mathcal{C}$ be a category and let T be a collection of morphisms of $\mathcal{C}$. Then every morphism of T is a transfinite composition of morphisms of T (take $\alpha 1$ in Definition 1.4.4.10). 006P

Example 1.4.4.13. Let $\mathcal{C}$ be a category and let T be a collection of morphisms of $\mathcal{C}$ which contains a pair of composable morphisms $f : C_0 \to C_1$ and $g : C_1 \to C_2$. Then the composition $g \circ f$ is a transfinite composition of morphisms of $\mathcal{C}$ (take $\alpha = 2$ in Definition 1.4.4.10). In particular, if T is closed under transfinite composition, then it is closed under composition. 006Q

Proposition 1.4.4.14. *Let $\mathcal{C}$ be a category, let S be a collection of morphisms in $\mathcal{C}$, and let T be the collection of all morphisms of $\mathcal{C}$ which have the left lifting property with respect to S. Then T is closed under transfinite composition.* 006R

Proof. Let α be an ordinal and suppose we are given a functor $[\alpha] \to \mathcal{C}$, given by $(\{C_\beta\}_{\beta \leqslant \alpha}, \{f_{\beta,\beta'}\}_{\beta \leqslant \beta' \leqslant \alpha})$, which satisfies condition (a) of Definition 1.4.4.10. Assume that each of the morphisms $f_{\beta,\beta+1}$ belongs to T. We wish to show that the morphism $f_{0,\alpha}$ also belongs to T. For this, we must show that every lifting problem σ :

$$\begin{array}{ccc} C_0 & \xrightarrow{u} & X \\ {\scriptstyle f_{0,\alpha}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ C_\alpha & \xrightarrow{v} & Y, \end{array}$$

admits a solution, provided that p belongs to S. We construct a collection of morphisms $\{u_\beta : C_\beta \to X\}_{\beta \leqslant \alpha}$, satisfying the requirements $p \circ u_\beta = v \circ f_{\beta,\alpha}$ and $u_\beta = u_\gamma \circ f_{\gamma,\beta}$ for $\beta \leqslant \gamma$, using transfinite recursion. Fix an ordinal $\gamma \leqslant \alpha$, and assume that the morphisms $\{u_\beta\}_{\beta<\gamma}$ have been constructed. We consider three cases:

- If $\gamma = 0$, we set $u_\gamma = u$.
- If γ is a nonzero limit ordinal, then our hypothesis that C_γ is the colimit of the diagram $\{C_\beta\}_{\beta<\gamma}$ guarantees that there is a unique morphism $u_\gamma : C_\gamma \to X$ satisfying $u_\beta = u_\gamma \circ f_{\beta,\gamma}$ for $\beta < \gamma$. Moreover, our assumption that the equality $p \circ u_\beta = v \circ f_{\beta,\alpha}$ holds for $\beta < \gamma$ guarantees that it also holds for $\beta = \gamma$.
- Suppose that $\gamma = \beta + 1$ is a successor ordinal. In this case, we take u_γ to be any solution to the lifting problem

$$\begin{array}{ccc} C_\beta & \xrightarrow{u_\beta} & X \\ {\scriptstyle f_{\beta,\beta+1}}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ C_{\beta+1} & \xrightarrow{v \circ f_{\beta+1,\alpha}} & Y, \end{array}$$

  which exists by virtue of our assumption that $f_{\beta,\beta+1}$ belongs to T.

We now complete the proof by observing that u_α is a solution to the lifting problem σ. □

Motivated by the preceding discussion, we introduce the following:

006S **Definition 1.4.4.15.** Let $\mathcal{C}$ be a category which admits small colimits and let T be a collection of morphisms of $\mathcal{C}$. We will say that T is *weakly saturated* if it is closed under pushouts (Definition 1.4.4.4, retracts (Variant 1.4.4.7), and transfinite composition (Definition 1.4.4.10).

006T **Proposition 1.4.4.16.** *Let $\mathcal{C}$ be a category which admits small colimits, let S be a collection of morphisms of $\mathcal{C}$, and let T be the collection of all morphisms of $\mathcal{C}$ which have the left lifting property with respect to S. Then $\mathcal{C}$ is weakly saturated.*

Proof. Combine Propositions 1.4.4.5, 1.4.4.9, and 1.4.4.14. □

Remark 1.4.4.17. Let $\mathcal{C}$ be a category and let T_0 be a collection of morphisms of $\mathcal{C}$. Then 006U
there exists a smallest collection of morphisms T of $\mathcal{C}$ such that $T_0 \subseteq T$ and T is weakly saturated (for example, we can take T to be the intersection of all the weakly saturated collections of morphisms containing T_0). We will refer to T as the *weakly saturated collection of morphisms generated by* T_0. It follows from Proposition 1.4.4.16 that if every morphism of T_0 has the left lifting property with respect to some collection of morphisms S, then every morphism of T also has the left lifting property with respect to S.

1.4.5 Trivial Kan Fibrations

We now specialize the ideas of §1.4.4 to the category of simplicial sets. 006V

Definition 1.4.5.1. Let $p : X_\bullet \to Y_\bullet$ be a map of simplicial sets. We say that p is a *trivial* 006W
Kan fibration if, for each $n \geqslant 0$, every lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X_\bullet \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y_\bullet \end{array}$$

admits a solution; here $i : \partial\Delta^n \hookrightarrow \Delta^n$ denotes the inclusion map.

Remark 1.4.5.2. Suppose we are given a pullback diagram of simplicial sets 006X

$$\begin{array}{ccc} X'_\bullet & \longrightarrow & X_\bullet \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y'_\bullet & \longrightarrow & Y_\bullet. \end{array}$$

If p is a trivial Kan fibration, then so is p' (this follows from Proposition 1.4.4.5, applied to the opposite of the category Set_Δ).

Proposition 1.4.5.3. *Let $p : X_\bullet \to Y_\bullet$ be a map of simplicial sets. The following conditions* 006Y
are equivalent:

(1) *The map p is a trivial Kan fibration (in the sense of Definition 1.4.5.1).*

(2) *The map p has the right lifting property with respect to every monomorphism of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$. In other words, every lifting problem*

$$\begin{array}{ccc} A_\bullet & \longrightarrow & X_\bullet \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B_\bullet & \longrightarrow & Y_\bullet \end{array}$$

admits a solution, provided that i is a monomorphism.

We will give the proof of Proposition 1.4.5.3 at the end of this section.

006Z **Corollary 1.4.5.4.** *Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration of simplicial sets. Then:*

(a) *The map p admits a section: that is, there is a map of simplicial sets $s : Y_\bullet \to X_\bullet$ such that the composition $p \circ s$ is the identity map $\mathrm{id}_{Y_\bullet} : Y_\bullet \to Y_\bullet$.*

(b) *Let s be any section of p. Then the composition $s \circ p : X_\bullet \to X_\bullet$ is fiberwise homotopic to the identity. That is, there exists a map of simplicial sets $h : \Delta^1 \times X_\bullet \to X_\bullet$, compatible with the projection to $Y_\bullet$, such that $h|_{\{0\} \times X_\bullet} = s \circ p$ and $h|_{\{1\} \times X_\bullet} = \mathrm{id}_{X_\bullet}$.*

Proof. To prove (a), we observe that a section of p can be described as a solution to the lifting problem

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_\bullet \\ \downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ Y_\bullet & \xrightarrow{\mathrm{id}} & Y_\bullet, \end{array}$$

which exists by virtue of Proposition 1.4.5.3. Given any section s, a fiberwise homotopy from $s \circ p$ to the identity can be identified with a solution to the lifting problem

$$\begin{array}{ccc} \partial \Delta^1 \times X_\bullet & \xrightarrow{(s \circ p, \mathrm{id})} & X_\bullet \\ \downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^1 \times X_\bullet & \longrightarrow & Y_\bullet, \end{array}$$

which again exists by virtue of Proposition 1.4.5.3. □

0070 **Corollary 1.4.5.5.** *Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration of simplicial sets and let $i : A_\bullet \to B_\bullet$ be a monomorphism of simplicial sets. Then the canonical map*

$$\theta : \mathrm{Fun}(B_\bullet, X_\bullet) \to \mathrm{Fun}(B_\bullet, Y_\bullet) \times_{\mathrm{Fun}(A_\bullet, Y_\bullet)} \mathrm{Fun}(A_\bullet, X_\bullet)$$

is also a trivial Kan fibration.

Proof. Fix an integer $n \geqslant 0$; we wish to show that every lifting problem

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & \mathrm{Fun}(B_\bullet, X_\bullet) \\ \downarrow & \nearrow & \downarrow \theta \\ \Delta^n & \longrightarrow & \mathrm{Fun}(B_\bullet, Y_\bullet) \times_{\mathrm{Fun}(A_\bullet, Y_\bullet)} \mathrm{Fun}(A_\bullet, X_\bullet). \end{array}$$

admits a solution. Unwinding the definitions, we see that this is equivalent to solving an associated lifting problem

$$\begin{array}{ccc} (\partial \Delta^n \times B_\bullet) \coprod_{\partial \Delta^n \times A_\bullet} (\Delta^n \times A_\bullet) & \longrightarrow & X_\bullet \\ \downarrow i & \nearrow & \downarrow p \\ \Delta^n \times B_\bullet & \longrightarrow & Y_\bullet. \end{array}$$

This is possible by virtue of Proposition 1.4.5.3, since p is a trivial Kan fibration and i is a monomorphism. □

Corollary 1.4.5.6. *Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration of simplicial sets. Then, for every simplicial set $B_\bullet$, the induced map $\mathrm{Fun}(B_\bullet, X_\bullet) \to \mathrm{Fun}(B_\bullet, Y_\bullet)$ is a trivial Kan fibration.* 0071

Proof. Apply Corollary 1.4.5.5 in the special case $A_\bullet = \varnothing$. □

Definition 1.4.5.7. Let $X_\bullet$ be a simplicial set. We say that $X_\bullet$ is a *contractible Kan complex* if the projection map $X_\bullet \to \Delta^0$ is a trivial Kan fibration (Definition 1.4.5.1). In other words, $X_\bullet$ is a contractible Kan complex if every map $\sigma_0 : \partial \Delta^n \to X_\bullet$ can be extended to an n-simplex of $X_\bullet$. 0072

Example 1.4.5.8. Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a contractible Kan complex if and only if the space X is *weakly contractible*: that is, if and only if every continuous map $\sigma_0 : S^{n-1} \to X$ is nullhomotopic (here $S^{n-1} \simeq |\partial \Delta^n|$ denotes the sphere of dimension $n-1$, so that σ_0 is nullhomotopic if and only if extends to a continuous map defined on the disk $D^n \simeq |\Delta^n|$). In particular, if the topological space X is contractible, then the simplicial set $\mathrm{Sing}_\bullet(X)$ is a contractible Kan complex. 0073

0074 **Remark 1.4.5.9.** Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration. Then, for every vertex y of $Y_\bullet$, the fiber $X_\bullet \times_{Y_\bullet} \{y\}$ is a contractible Kan complex (this is a special case of Remark 1.4.5.2). For a partial converse, see Proposition [?].

Applying Proposition 1.4.5.3 in the case $Y_\bullet = \Delta^0$, we obtain the following:

0075 **Corollary 1.4.5.10.** *Let $X_\bullet$ be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set $X_\bullet$ is a contractible Kan complex.*

(2) *For every monomorphism of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$ and every map of simplicial sets $f_0 : A_\bullet \to X_\bullet$, there exists a map $f : B_\bullet \to X_\bullet$ such that $f_0 = f \circ i$.*

0076 **Corollary 1.4.5.11.** *Let $X_\bullet$ be a contractible Kan complex. Then $X_\bullet$ is a Kan complex. In particular, $X_\bullet$ is an ∞-category.*

We will deduce Proposition 1.4.5.3 from the following:

0077 **Proposition 1.4.5.12.** *Let T be the collection of all monomorphisms in the category Set_Δ of simplicial sets. Then:*

(a) *The collection T is weakly saturated, in the sense of Definition 1.4.4.15.*

(b) *As a weakly saturated collection of morphisms, T is generated by collection of inclusion maps $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \geqslant 0}$ (see Remark 1.4.4.17).*

Proof. To prove (a), we must establish the following:

- The collection T is closed under pushouts. That is, if we are given a pushout diagram of simplicial sets

$$\begin{array}{ccc} A_\bullet & \longrightarrow & A'_\bullet \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B_\bullet & \longrightarrow & B'_\bullet \end{array}$$

  where f is a monomorphism, then f' is also a monomorphism. This is clear, since we have a pushout diagram

$$\begin{array}{ccc} A_n & \longrightarrow & A'_n \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & B'_n \end{array}$$

  in the category of sets for each $n \geqslant 0$ (where the left vertical map is injective, so the right vertical map is injective as well).

- The collection T is closed under retracts. This is a special case of Example 1.4.4.8.

- The collection T is closed under transfinite composition. Suppose we are given an ordinal α and a functor $S : [\alpha] \to \mathrm{Set}_\Delta$, given by a collection of simplicial sets $\{S(\beta)_\bullet\}_{\beta \leqslant \alpha}$ and transition maps $f_{\beta,\gamma} : S(\beta)_\bullet \to S(\gamma)_\bullet$. Assume that the maps $f_{\beta,\beta+1}$ are monomorphisms for $\beta < \alpha$ and that, for every nonzero limit ordinal $\lambda \leqslant \alpha$, the induced map $\varinjlim_{\beta<\lambda} S(\beta)_\bullet \to S(\lambda)_\bullet$ is an isomorphism. We must show that the map $f_{0,\alpha} : S(0)_\bullet \to S(\alpha)_\bullet$ is a monomorphism of simplicial sets. In fact, we claim that for each $\beta \leqslant \alpha$, the map $f_{0,\gamma} : S(0)_\bullet \to S(\gamma)_\bullet$ is a monomorphism. The proof proceeds by transfinite induction on γ. In the case $\gamma = 0$, the map $f_{0,\gamma} = \mathrm{id}_{S(0)_\bullet}$ is an isomorphism. If γ is a nonzero limit ordinal, then the desired result follows from our inductive hypothesis, since the collection of monomorphisms in Set_Δ is closed under filtered colimits. If $\gamma = \beta + 1$ is a successor ordinal, then we can identify $f_{0,\gamma}$ with the composition
$$S(0)_\bullet \xrightarrow{f_{0,\beta}} S(\beta)_\bullet \xrightarrow{f_{\beta,\gamma}} S(\gamma)_\bullet,$$
where $f_{\beta,\gamma}$ is a monomorphism by assumption and $f_{0,\beta}$ is a monomorphism by virtue of our inductive hypothesis.

We now prove (b). Let T' be a collection of morphisms in Set_Δ which is weakly saturated and contains each of the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$; we wish to show that every monomorphism $i : A_\bullet \to B_\bullet$ belongs to T'. For each $k \geqslant -1$, let $B(k)_\bullet \subseteq B_\bullet$ denote the simplicial subset given by the union of the skeleton $\mathrm{sk}_k(B_\bullet)$ (Construction 1.1.3.5) with the image of i. Then the inclusion i can be written as a transfinite composition

$$A_\bullet \simeq B(-1)_\bullet \hookrightarrow B(0)_\bullet \hookrightarrow B(1)_\bullet \hookrightarrow B(2)_\bullet \hookrightarrow \cdots$$

Since T' is closed under transfinite composition, it will suffice to show that each of the inclusion maps $B(k-1)_\bullet \hookrightarrow B(k)_\bullet$ belongs to T'. Applying Proposition 1.1.3.11 to both $A_\bullet$ and $B_\bullet$, we obtain a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma\in Q} \partial\Delta^k & \longrightarrow & \coprod_{\sigma\in Q} \Delta^k \\ \downarrow & & \downarrow \\ B(k-1)_\bullet & \longrightarrow & B(k)_\bullet \end{array}$$

where Q denotes the collection of all nondegenerate k-simplices of $B_\bullet$ which do not belong to the image of i. Since T' is closed under pushouts, we are reduced to showing that the inclusion map

$$j : \coprod_{\sigma\in Q} \partial\Delta^k \hookrightarrow \coprod_{\sigma\in Q} \Delta^k$$

belongs to T'. Choosing a well-ordering of Q, we see that j can be written as a transfinite composition of morphisms

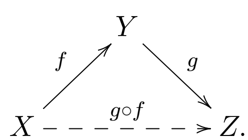
$$j_\sigma : (\coprod_{\tau \leqslant \sigma} \partial \Delta^k) \amalg (\coprod_{\tau > \sigma} \Delta^k) \hookrightarrow (\coprod_{\tau < \sigma} \partial \Delta^k) \amalg (\coprod_{\tau \geqslant \sigma} \Delta^k),$$

each of which is a pushout of the inclusion $\partial \Delta^k \hookrightarrow \Delta^k$. □

Proof of Proposition 1.4.5.3. Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration of simplicial sets and let T be the collection of all morphisms in Set_Δ which have the left lifting property with respect to p. Then T contains each of the inclusions $\partial \Delta^n \hookrightarrow \Delta^n$ (by virtue of our assumption that p is a trivial Kan fibration) and is weakly saturated (Proposition 1.4.4.16). It follows from Proposition 1.4.5.12 that every monomorphism of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$ belongs to T (and therefore has the left lifting property with respect to p). □

1.4.6 Uniqueness of Composition

0078 Let $\mathcal{C}$ be an ∞-category. Given a composable pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, one can form a composition $g \circ f$ by choosing a 2-simplex σ with $d_0(\sigma) = g$ and $d_2(\sigma) = f$, as indicated in the diagram



In general, neither the 2-simplex σ nor the resulting morphism $g \circ f = d_1(\sigma)$ is uniquely determined. However, we saw in §1.3.4 that the composition $g \circ f$ is unique up to homotopy (Proposition 1.3.4.2). We now prove a stronger result, which asserts that the 2-simplex σ (hence also the composite morphism $g \circ f = d_1(\sigma)$) is unique up to a contractible space of choices.

0079 **Theorem 1.4.6.1** (Joyal)**.** *Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set $S_\bullet$ is an ∞-category.*

(2) *The inclusion of simplicial sets $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial Kan fibration*

$$\mathrm{Fun}(\Delta^2, S_\bullet) \to \mathrm{Fun}(\Lambda^2_1, S_\bullet).$$

007A **Corollary 1.4.6.2.** *Let $f : X \to Y$ and $g : Y \to Z$ be a composable pair of morphisms in an ∞-category $\mathcal{C}$, so that the tuple $(g, \bullet, f)$ determines a map of simplicial sets $\Lambda^2_1 \to \mathcal{C}$ (see Exercise 1.1.2.14). Then the fiber product*

$$\mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{C})} \{(g, \bullet, f)\}$$

is a contractible Kan complex.

Proof. Combine Theorem 1.4.6.1 with Remark 1.4.5.9. □

Remark 1.4.6.3. In the situation of Corollary 1.4.6.2, one can think of the simplicial set 007B

$$Z_\bullet = \mathrm{Fun}(\Delta^2, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{C})} \{(g, \bullet, f)\}$$

as a "parameter space" for all choices of 2-simplex σ satisfying $d_0(\sigma) = g$ and $d_2(\sigma) = f$ (note that such 2-simplices can be identified with the vertices of $Z_\bullet$). Consequently, we can summarize Corollary 1.4.6.2 informally by saying that this parameter space is contractible.

We will give the proof of Theorem 1.4.6.1 at the end of this section. First, let us study its consequences.

Proof of Theorem 1.4.3.7. Let $S_\bullet$ be a simplicial set and let $\mathcal{D}$ be an ∞-category. We wish to show that the simplicial set $\mathrm{Fun}(S_\bullet, \mathcal{D})$ is an ∞-category. By virtue of Theorem 1.4.6.1, it will suffice to show that the restriction map

$$r : \mathrm{Fun}(\Delta^2, \mathrm{Fun}(S_\bullet, \mathcal{D})) \to \mathrm{Fun}(\Lambda^2_1, \mathrm{Fun}(S_\bullet, \mathcal{D}))$$

is a trivial Kan fibration. Note that we can identify r with the canonical map

$$\mathrm{Fun}(S_\bullet, \mathrm{Fun}(\Delta^2, \mathcal{D})) \to \mathrm{Fun}(S_\bullet, \mathrm{Fun}(\Lambda^2_1, \mathcal{D})),$$

which is a trivial Kan fibration by virtue of Corollary 1.4.5.6 and Theorem 1.4.6.1. □

We now introduce some terminology which will be useful for the proof of Theorem 1.4.6.1.

Definition 1.4.6.4. Let $f : A_\bullet \to B_\bullet$ be a morphism of simplicial sets. We will say that f is *inner anodyne* if it belongs to the weakly saturated class of morphisms generated by the collection of all inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ (so that $0 < i < n$). 007C

Remark 1.4.6.5. Let $f : A_\bullet \to B_\bullet$ be an inner anodyne map of simplicial sets. Then f is a monomorphism. This follows from the observation that the collection of monomorphisms is weakly saturated (Proposition 1.4.5.12), since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a monomorphism. 007D

Proposition 1.4.6.6. *Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:* 007E

(1) *The simplicial set $S_\bullet$ is an ∞-category.*

(2) *For every inner anodyne map of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$ and every map $f_0 : A_\bullet \to S_\bullet$, there exists a map $f : B_\bullet \to S_\bullet$ such that $f_0 = f \circ i$.*

Proof. The implication (2) ⇒ (1) is immediate (since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is inner anodyne). Conversely, if (1) is satisfied, then every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ has the left lifting property with respect to the projection map $p : S_\bullet \to \Delta^0$. It then follows from Remark 1.4.4.17 that every inner anodyne map has the left lifting property with respect to p. □

We will deduce Theorem 1.4.6.1 from the following technical result:

007F **Lemma 1.4.6.7** (Joyal)**.**

(*a*) *For every monomorphism of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$, the induced map*

$$(B_\bullet \times \Lambda^2_1) \coprod_{A_\bullet \times \Lambda^2_1} (A_\bullet \times \Delta^2) \subseteq A_\bullet \times \Delta^2$$

is inner anodyne.

(*b*) *The collection of inner anodyne morphisms is generated (as a weakly saturated class) by the inclusion maps*

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\,\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

for $m \geqslant 0$.

Proof. Let T be the weakly saturated class of morphisms generated by all inclusions of the form

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\,\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2,$$

and let S be the collection of all morphisms of simplicial sets $A_\bullet \to B_\bullet$ for which the map

$$(B_\bullet \times \Lambda^2_1) \coprod_{A_\bullet \times \Lambda^2_1} (A_\bullet \times \Delta^2) \subseteq A_\bullet \times \Delta^2$$

belongs to T. By construction, S contains all inclusions of the form $\partial\Delta^m \hookrightarrow \Delta^m$. Moreover, since T is weakly saturated, the class S is also weakly saturated. It follows that every monomorphism of simplicial sets belongs to S (Proposition 1.4.5.12). Consequently, to prove Lemma 1.4.6.7, it will suffice to show that T coincides with the class of inner anodyne morphisms of Set_Δ. We first show that every inner anodyne morphism belongs to T. Since T is weakly saturated, we are reduced to showing that every inner horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$ belongs to T. Since f belongs to S, the monomorphism

$$\overline{f} : (\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) \subseteq \Delta^n \times \Delta^2.$$

belongs to T. We conclude by observing that the morphism f is a retract of $\overline{f}$. More precisely, we have a commutative diagram of simplicial sets

$$\begin{array}{ccccc} \Lambda^n_i & \longrightarrow & (\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) & \longrightarrow & \Lambda^n_i \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow{s} & \Delta^n \times \Delta^2 & \xrightarrow{r} & \Delta^n, \end{array}$$

where the maps s and r are given on vertices by the formulae

$$s(j) = \begin{cases} (j,0) & \text{if } j < i \\ (j,1) & \text{if } j = i \\ (j,2) & \text{if } j > i \end{cases}$$

$$r(j,k) = \begin{cases} j & \text{if } j < i, k = 0 \\ j & \text{if } j > i, k = 2 \\ i & \text{otherwise.} \end{cases}$$

We now show that every morphism of T is inner anodyne. Since the collection of inner anodyne morphisms is weakly saturated, it will suffice to show that the inclusion map

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

is inner anodyne for each $m \geqslant 0$. For each $0 \leqslant i \leqslant j < m$, we let σ_{ij} denote the $(m+1)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$f_{ij} : [m+1] \to [m] \times [2]$$

$$f_{ij}(k) = \begin{cases} (k,0) & \text{if } 0 \leqslant k \leqslant i \\ (k-1,1) & \text{if } i+1 \leqslant k \leqslant j+1 \\ (k-1,2) & \text{if } j+2 \leqslant k \leqslant m+1. \end{cases}$$

For each $0 \leqslant i \leqslant j \leqslant m$, we let τ_{ij} denote the $(m+2)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$g_{ij} : [m+2] \to [m] \times [2]$$

$$g_{ij}(k) = \begin{cases} (k,0) & \text{if } 0 \leqslant k \leqslant i \\ (k-1,1) & \text{if } i+1 \leqslant k \leqslant j+1 \\ (k-2,2) & \text{if } j+2 \leqslant k \leqslant m+2. \end{cases}$$

We will regard each σ_{ij} and τ_{ij} as a simplicial subset of $\Delta^m \times \Delta^2$.

Set $X(0) = (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\,\Delta^m \times \Delta^2)$. For $0 \leqslant j < m$, we let

$$X(j+1) = X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj}.$$

We have a chain of inclusions

$$X(j) \subseteq X(j) \cup \sigma_{0j} \subseteq \cdots \subset X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj} = X(j+1).$$

Each of these inclusions fits into a pushout diagram

$$\begin{array}{ccc} \Lambda^{m+1}_{i+1} & \longrightarrow & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{(i-1)j} \\ \downarrow & & \downarrow \\ \sigma_{ij} & \longrightarrow & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{ij}, \end{array}$$

and is therefore inner anodyne. Set $Y(0) = X(m)$, so that the inclusion $X(0) \subseteq Y(0)$ is inner anodyne. We now set $Y(j+1) = Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{jj}$ for $0 \leqslant j \leqslant m$. As before, we have a chain of inclusions

$$Y(j) \subseteq Y(j) \cup \tau_{0j} \subseteq \cdots \subseteq Y_j \cup \tau_{0j} \cup \cdots \cup \tau_{jj} = Y(j+1),$$

each of which fits into a pushout diagram

$$\begin{array}{ccc} \Lambda^{m+2}_{i+1} & \longrightarrow & Y_j \cup \tau_{0j} \cup \cdots \cup \tau_{(i-1)j} \\ \downarrow & & \downarrow \\ \tau_{ij} & \longrightarrow & Y_j \cup \tau_{0j} \cup \cdots \cup \tau_{ij}, \end{array}$$

and is therefore inner andoyne. It follows that each inclusion $Y(j) \subseteq Y(j+1)$ is inner anodyne. Since the collection of inner anodyne morphisms is closed under composition, we conclude that the inclusion map $X(0) \hookrightarrow Y(0) \hookrightarrow Y(1) \hookrightarrow \cdots Y(m+2) = \Delta^m \times \Delta^2$ is inner anodyne, as desired. □

Proof of Theorem 1.4.6.1. Let $S_\bullet$ be a simplicial set and let $p : \mathrm{Fun}(\Delta^2, S_\bullet) \to \mathrm{Fun}(\Lambda^2_1, S_\bullet)$ denote the restriction map. Then p is a trivial Kan fibration if and only if every lifting problem

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & \mathrm{Fun}(\Delta^2, S_\bullet) \\ \downarrow & \nearrow & \downarrow p \\ \Delta^m & \longrightarrow & \mathrm{Fun}(\Lambda^2_1, S_\bullet) \end{array}$$

admits a solution. Unwinding the definitions, we see that this is equivalent to the requirement that every lifting problem of the form

$$\begin{array}{ccc} (\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial\,\Delta^m \times \Delta^2) & \longrightarrow & S_\bullet \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow \\ \Delta^m \times \Delta^2 & \longrightarrow & \Delta^0, \end{array}$$

admits a solution. Let T be the collection of all morphisms of simplicial sets which have the left lifting property with respect to the projection $S_\bullet \to \Delta^0$. Then p is a trivial Kan fibration if and only if T contains each of the inclusion maps

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial\,\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2.$$

Since T is weakly saturated (Proposition 1.4.4.16), this is equivalent to the requirement that T contains all inner anodyne morphisms (Lemma 1.4.6.7), which is in turn equivalent to the requirement that $S_\bullet$ is an ∞-category (Proposition 1.4.6.6). $\square$

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