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Part I

Higher Category Theory
Chapter 1

The Language of ∞-Categories

A principal goal of algebraic topology is to understand topological spaces by means of algebraic and combinatorial invariants. Let us consider some elementary examples.

- To any topological space $X$, one can associate the set $\pi_0(X)$ of path components of $X$. This is the quotient of $X$ by an equivalence relation $\simeq$, where $x \simeq y$ if there exists a continuous path $p : [0, 1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.

- To any topological space $X$ equipped with a base point $x \in X$, one can associate the fundamental group $\pi_1(X, x)$. This is a group whose elements are homotopy classes of continuous paths $p : [0, 1] \to X$ satisfying $p(0) = x = p(1)$.

For many purposes, it is useful to combine the set $\pi_0(X)$ and the fundamental groups $\{\pi_1(X, x)\}_{x \in X}$ into a single mathematical object. To any topological space $X$, one can associate an invariant $\pi_{\leq 1}(X)$ called the fundamental groupoid of $X$. The fundamental groupoid $\pi_{\leq 1}(X)$ is a category whose objects are the points of $X$, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a homotopy class of continuous paths $p : [0, 1] \to X$ satisfying $p(0) = x$ and $p(1) = y$. The set of path components $\pi_0(X)$ can then be recovered as the set of isomorphism classes of objects of the category $\pi_{\leq 1}(X)$, and each fundamental group $\pi_1(X, x)$ can be identified with the automorphism group of the point $x$ as an object of the category $\pi_{\leq 1}(X)$. The formalism of category theory allows us to assemble information about path components and fundamental groups into a single convenient package.

The fundamental groupoid $\pi_{\leq 1}(X)$ is a very important invariant of a topological space $X$, but is far from being a complete invariant. In particular, it does not contain any information about the higher homotopy groups $\{\pi_n(X, x)\}_{n \geq 2}$. We therefore ask the following:

**Question 1.0.0.1.** Let $X$ be a topological space. Can one devise a “category-theoretic” invariant of $X$, in the spirit of the fundamental groupoid $\pi_{\leq 1}(X)$, which contains information about all the homotopy groups of $X$?
We begin to address Question 1.0.0.1 in §1.1 by introducing the theory of simplicial sets. A simplicial set $S_\bullet$ is a collection of sets $\{S_n\}_{n\geq 0}$, which are related by face maps $\{d_i: S_n \to S_{n-1}\}_{0 \leq i \leq n}$ and degeneracy maps $\{s_i: S_n \to S_{n+1}\}_{0 \leq i \leq n}$ satisfying suitable identities (see Definition 1.1.12 and Exercise 1.1.11). Every topological space $X$ determines a simplicial set $\text{Sing}_\bullet(X)$, called the singular simplicial set of $X$, with the property that each $\text{Sing}_n(X)$ is the collection of continuous maps from the topological $n$-simplex into $X$ (Construction 1.1.7.1). Moreover, the homotopy groups of $X$ can be reconstructed from the simplicial set $\text{Sing}_\bullet(X)$ by a simple combinatorial procedure (see §3.2). Kan observed that this procedure can be applied more generally to any simplicial set $S_\bullet$ satisfying the following Kan extension condition:

(*) For $0 \leq i \leq n$, every map $\sigma_0: \Lambda^n_i \to S_\bullet$ admits an extension $\sigma: \Delta^n \to S_\bullet$.

Here $\Delta^n$ denotes a certain simplicial set called the standard $n$-simplex (Construction 1.1.2.1), and $\Lambda^n_i$ denotes a certain simplicial subset of $\Delta^n$ called the $i$th horn (Construction 1.1.2.9). Simplicial sets satisfying condition (*) are called Kan complexes. Every simplicial set of the form $\text{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.1.9.8), and the converse is true up to homotopy. More precisely, Milnor proved in [32] that the construction $X \mapsto \text{Sing}_\bullet(X)$ induces an equivalence from the (geometrically defined) homotopy theory of CW complexes to the (combinatorially defined) homotopy theory of Kan complexes; we will discuss this point in Chapter 3 (see Theorem 3.5.0.1).

The singular simplicial set $\text{Sing}_\bullet(X)$ is a natural candidate for the sort of invariant requested in Question 1.0.0.1: it is a mathematical object of a purely combinatorial nature which contains complete information about the homotopy groups of $X$ and their interrelationship (from which we can even reconstruct $X$ up to homotopy equivalence, provided that $X$ has the homotopy type of a CW complex). But in order to see that it qualifies as a complete answer, we must address the following:

**Question 1.0.0.2.** Let $X$ be a topological space. To what extent does the simplicial set $\text{Sing}_\bullet(X)$ behave like a category? What is the relationship between $\text{Sing}_\bullet(X)$ with the fundamental groupoid of $X$?

Our answer to Question 1.0.0.2 begins with the observation that the theory of simplicial sets is closely related to category theory. To every category $\mathcal{C}$, one can associate a simplicial set $N_\bullet(\mathcal{C})$, called the nerve of $\mathcal{C}$ (we will review the construction of $N_\bullet(\mathcal{C})$ in §1.2, see Construction 1.2.1.1). The construction $\mathcal{C} \mapsto N_\bullet(\mathcal{C})$ is fully faithful (Proposition 1.2.2.1); in particular, a category $\mathcal{C}$ is determined (up to canonical isomorphism) by the simplicial set $N_\bullet(\mathcal{C})$. Throughout much of this book, we will abuse notation by not distinguishing between a category $\mathcal{C}$ and its nerve $N_\bullet(\mathcal{C})$: that is, we will view a category as a special kind of simplicial set. These simplicial sets have a simple characterization: according to
Proposition 1.2.3.1, a simplicial set \( S_\bullet \) has the form \( N_{\bullet}(C) \) (for some category \( C \)) if and only if it satisfies the following variant of the Kan extension condition (Proposition 1.2.3.1):

\((\ast')\) For \( 0 < i < n \), every map \( \sigma_0 : \Lambda^n_i \to S_\bullet \) admits a unique extension \( \sigma : \Delta^n \to S_\bullet \).

The extension conditions \((\ast)\) and \((\ast')\) are closely related, but differ in two important respects. The Kan extension condition requires that every map of simplicial sets \( \sigma_0 : \Lambda^n_i \to S_\bullet \) admits an extension \( \sigma : \Delta^n \to S_\bullet \). Condition \((\ast')\) requires the existence of an extension only in the case \( 0 < i < n \), but demands that the extension is unique. Neither of these conditions implies the other: a simplicial set of the form \( N_{\bullet}(C) \) satisfies condition \((\ast)\) if and only if the category \( C \) is a groupoid (Proposition 1.2.4.2), and a simplicial set of the form \( \text{Sing}_{\bullet}(X) \) satisfies condition \((\ast')\) if and only if every continuous path \([0, 1] \to X\) is constant. However, conditions \((\ast)\) and \((\ast')\) admit a common generalization. We will say that a simplicial set \( S_\bullet \) is an \( \infty \)-category if it satisfies the following variant of \((\ast)\) and \((\ast')\), known as the weak Kan extension condition:

\((\ast'')\) For \( 0 < i < n \), every map \( \sigma_0 : \Lambda^n_i \to S_\bullet \) admits an extension \( \sigma : \Delta^n \to S_\bullet \).

The theory of \( \infty \)-categories can be viewed as a simultaneous generalization of homotopy theory and category theory. Every Kan complex is an \( \infty \)-category, and every category \( C \) determines an \( \infty \)-category (given by the nerve \( N_{\bullet}(C) \)). In particular, the notion of \( \infty \)-category answers the first part of Question 1.0.0.2: simplicial sets of the form \( \text{Sing}_{\bullet}(X) \) are almost never (the nerves of) categories, but are always \( \infty \)-categories. At this point, the reader might reasonably object that this is terminological legerdemain: to address the spirit of Question 1.0.0.2, we must demonstrate that simplicial sets of the form \( \text{Sing}_{\bullet}(X) \) (or, more generally, all simplicial sets satisfying condition \((\ast'')\)) really behave like categories. We begin in \( \S \) 1.3 by explaining how to extend various elementary category-theoretic ideas to the setting of \( \infty \)-categories. In particular, we can associate to each \( \infty \)-category \( S_\bullet \) a collection of objects (these are the elements of \( S_0 \)), a collection of morphisms (these are the elements of \( S_1 \)), and a composition law on morphisms. In particular, we show that any \( \infty \)-category \( S_\bullet \) determines an ordinary category \( hS_\bullet \), called the homotopy category of \( S_\bullet \) (Proposition 1.3.5.2). The construction of the homotopy category allows us to answer the second part of Question 1.0.0.2 for every topological space \( X \), the singular simplicial set \( \text{Sing}_{\bullet}(X) \) is an \( \infty \)-category, whose homotopy category \( h\text{Sing}_{\bullet}(X) \) is the fundamental groupoid \( \pi_{\leq 1}(X) \) (see Example 1.3.5.5).

Roughly speaking, the difference between an \( \infty \)-category \( S_\bullet \) and its homotopy category \( hS_\bullet \) is that the former can contain nontrivial homotopy-theoretic information (encoded by simplices of dimension \( n \geq 2 \), which can be loosely understood as “\( n \)-morphisms”) which is lost upon passage to the homotopy category \( hS_\bullet \). We can summarize the situation informally with the heuristic equation

\[ \{ \text{Categories} \} + \{ \text{Homotopy Theory} \} = \{ \text{\( \infty \)-Categories} \}, \]
or more precisely with the diagram

\[
\begin{array}{c}
\{\text{Categories}\} \xrightarrow{N_*} \{\infty\text{-Categories}\} \\
\cap \\
\{\text{Kan Complexes}\} \\
\downarrow^{\text{Sing}_*} \\
\{\text{Simplicial Sets}\} \quad \{\text{Topological Spaces}\}
\end{array}
\]

### 1.1 Simplicial Sets

For each integer \( n \geq 0 \), we let \(|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}\) denote the topological simplex of dimension \( n \). For any topological space \( X \), we will refer to a continuous map \( \sigma : |\Delta^n| \to X \) as a singular \( n \)-simplex in \( X \). Every singular \( n \)-simplex \( \sigma \) determines a finite collection of singular \((n-1)\)-simplices \( \{d_i \sigma\}_{0 \leq i \leq n} \), called the faces of \( \sigma \), which are given explicitly by the formula

\[
(d_i \sigma)(t_0, t_1, \ldots, t_{n-1}) = \sigma(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).
\]

Let \( \text{Sing}_n(X) = \text{Hom}_\text{Top}(|\Delta^n|, X) \) denote the set of singular \( n \)-simplices of \( X \). Many important algebraic invariants of \( X \) can be directly extracted from the sets \( \{\text{Sing}_n(X)\}_{n \geq 0} \) and the face maps \( \{d_i : \text{Sing}_n(X) \to \text{Sing}_{n-1}(X)\}_{0 \leq i \leq n} \).

#### Example 1.1.0.1 (Singular Homology)

For any topological space \( X \), the singular homology groups \( H_*(X; \mathbb{Z}) \) are defined as the homology groups of a chain complex

\[
\cdots \xrightarrow{\partial} \mathbb{Z}[\text{Sing}_2(X)] \xrightarrow{\partial} \mathbb{Z}[\text{Sing}_1(X)] \xrightarrow{\partial} \mathbb{Z}[\text{Sing}_0(X)],
\]

where \( \mathbb{Z}[\text{Sing}_n(X)] \) denotes the free abelian group generated by the set \( \text{Sing}_n(X) \) and the differential is given on generators by the formula

\[
\partial(\sigma) = \sum_{i=0}^{n} (-1)^i d_i \sigma.
\]

For some other algebraic invariants, it is convenient to keep track of a bit more structure. A singular \( n \)-simplex \( \sigma : |\Delta^n| \to X \) also determines a collection of singular \((n+1)\)-simplices \( \{s_i \sigma\}_{0 \leq i \leq n} \), given by the formula

\[
(s_i \sigma)(t_0, \ldots, t_{n+1}) = \sigma(t_0, t_1, \ldots, t_i, t_{i+1}, t_{i+2}, \ldots, t_{n+1}).
\]

The resulting constructions \( s_i : \text{Sing}_n(X) \to \text{Sing}_{n+1}(X) \) are called degeneracy maps, because singular \((n+1)\)-simplices of the form \( s_i \sigma \) factor through the linear projection \(|\Delta^{n+1}| \to |\Delta^n|\).

For example, the map \( s_0 : \text{Sing}_0(X) \to \text{Sing}_1(X) \) carries each point \( x \in X \simeq \text{Sing}_0(X) \) to the constant map \( x : |\Delta^1| \to X \) taking the value \( x \).
Example 1.1.0.2 (The Fundamental Group). Let $X$ be a topological space equipped with a base point $x \in X \simeq \text{Sing}_0(X)$. Then continuous paths $p : [0, 1] \to X$ satisfying $p(0) = x = p(1)$ can be identified with elements of the set $\{ \sigma \in \text{Sing}_1(X) : d_0(\sigma) = x = d_1(\sigma) \}$. The fundamental group $\pi_1(X, x)$ can then be described as the quotient $\{ \sigma \in \text{Sing}_1(X) : d_0(\sigma) = x = d_1(\sigma) \} / \simeq$, where $\simeq$ is the equivalence relation on $\text{Sing}_1(X)$ described by $(\sigma \simeq \sigma') \Leftrightarrow (\exists \tau \in \text{Sing}_2(X))[d_0(\tau) = s_0(x) \text{ and } d_1(\tau) = \sigma \text{ and } d_2(\tau) = \sigma']$.

The datum of a 2-simplex $\tau$ satisfying these conditions is equivalent to the datum of a continuous map $|\Delta^2| \to X$ with boundary behavior as indicated in the diagram

Such a map can be identified with a homotopy between the paths determined by $\sigma$ and $\sigma'$.

Motivated by the preceding examples, we can ask the following:

Question 1.1.0.3. Given a topological space $X$, what can we say about the collection of sets $\{ \text{Sing}_n(X) \}_{n \geq 0}$, together with the face and degeneracy maps

$$d_i : \text{Sing}_n(X) \to \text{Sing}_{n-1}(X) \quad s_i : \text{Sing}_n(X) \to \text{Sing}_{n+1}(X)?$$

What sort of mathematical structure do they form?

In [13], Eilenberg and Zilber supplied an answer to Question 1.1.0.3 by introducing what they called complete semi-simplicial complexes, which are now more commonly known as simplicial sets. Roughly speaking, a simplicial set $S_\bullet$ is a collection of sets $\{ S_n \}_{n \geq 0}$ indexed by the nonnegative integers, equipped with face and degeneracy operators $\{ d_i : S_n \to S_{n-1}, s_i : S_n \to S_{n+1} \}_{0 \leq i \leq n}$ satisfying a short list of identities. These identities can be summarized conveniently by saying that a simplicial set is a presheaf on the simplex category $\Delta$, whose definition we review in §1.1.1.

Simplicial sets are connected to algebraic topology by two closely related constructions:

- For every topological space $X$, the face and degeneracy operators defined above endow the collection $\{ \text{Sing}_n(X) \}_{n \geq 0}$ with the structure of a simplicial set. We denote this simplicial set by $\text{Sing}_\bullet(X)$ and refer to it as the singular simplicial set of $X$ (see Construction 1.1.7.1). These simplicial sets tend to be quite large: in any nontrivial example, the sets $\text{Sing}_n(X)$ will be uncountable for every nonnegative integer $n$. 

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- Any simplicial set \( S \) can be regarded as a “blueprint” for constructing a topological space \(|S|\) called the geometric realization of \( S \), which can be obtained as a quotient of the disjoint union \( \coprod_{n \geq 0} S_n \times |\Delta^n| \) by an equivalence relation determined by the face and degeneracy operators on \( S \). Many topological spaces of interest (for example, any space which admits a finite triangulation) can be realized as a geometric realization of a simplicial set \( S \) having only finitely many nondegenerate simplices; we will discuss some elementary examples in \( \S 1.1.2 \).

These constructions determine adjoint functors

\[
\text{Set}_\Delta \xrightarrow{\text{||}} \text{Top}
\]

relating the category \( \text{Set}_\Delta \) of simplicial sets to the category \( \text{Top} \) of topological spaces. We review the constructions of these functors in \( \S 1.1.7 \) and \( \S 1.1.8 \), viewing them as instances of a general paradigm (Variant \( 1.1.7.6 \) and Proposition \( 1.1.8.22 \)) which will appear repeatedly in Chapter 2.

For any (pointed) topological space \( X \), Examples \( 1.1.0.1 \) and \( 1.1.0.2 \) show that the singular homology and fundamental group of \( X \) can be recovered from the simplicial set \( \text{Sing}_\bullet(X) \). In fact, one can say more: under mild assumptions, the entire homotopy type of \( X \) can be recovered from \( \text{Sing}_\bullet(X) \). More precisely, there is always a canonical map \(|\text{Sing}_\bullet(X)| \to X\) (given by the counit of the adjunction described above), and Giever proved that it is always a weak homotopy equivalence (hence a homotopy equivalence when \( X \) has the homotopy type of a CW complex; see Proposition \( 3.5.3.8 \)). Consequently, for the purpose of studying homotopy theory, nothing is lost by replacing \( X \) by \( \text{Sing}_\bullet(X) \) and working in the setting of simplicial sets, rather than topological spaces. In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. However, not every simplicial set \( S \) behaves like the singular complex of a space; it is therefore necessary to single out a class of “good” simplicial sets to work with. In \( \S 1.1.9 \) we introduce a special class of simplicial sets, called Kan complexes (Definition \( 1.1.9.1 \)). By a theorem of Milnor (\cite{milnor}), the homotopy theory of Kan complexes is equivalent to the classical homotopy theory of CW complexes; we will return to this point in Chapter 3.

1.1.1 Simplicial and Cosimplicial Objects

We begin with some preliminaries.

**Notation 1.1.1.1.** For every nonnegative integer \( n \), we let \([n]\) denote the linearly ordered set \( \{0 < 1 < 2 < \cdots < n - 1 < n\} \).

**Definition 1.1.1.2 (The Simplex Category).** We define a category \( \Delta \) as follows:
• The objects of $\Delta$ are linearly ordered sets of the form $[n]$ for $n \geq 0$.

• A morphism from $[m]$ to $[n]$ in the category $\Delta$ is a function $\alpha : [m] \to [n]$ which is nondecreasing: that is, for each $0 \leq i \leq j \leq m$, we have $0 \leq \alpha(i) \leq \alpha(j) \leq n$.

We will refer to $\Delta$ as the simplex category.

Remark 1.1.1.3. The category $\Delta$ is equivalent to the category of all nonempty finite linearly ordered sets, with morphisms given by nondecreasing maps. In fact, we can say something even better: for every nonempty finite linearly ordered set $I$, there is a unique order-preserving bijection $I \simeq [n]$, for some $n \geq 0$.

Definition 1.1.1.4. Let $\mathcal{C}$ be any category. A simplicial object of $\mathcal{C}$ is a functor $\Delta^{\text{op}} \to \mathcal{C}$. Dually, a cosimplicial object of $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$.

Notation 1.1.1.5. We will often use the expression $\mathcal{C}^\bullet$ to denote a simplicial object of a category $\mathcal{C}$. In this case, we write $\mathcal{C}_n$ for the value of the functor $\mathcal{C}^\bullet$ on the object $[n] \in \Delta$. Similarly, we use the notation $\mathcal{C}^\bullet_n$ to indicate a cosimplicial object of $\mathcal{C}$, and $\mathcal{C}^n$ for its value on $[n] \in \Delta$.

Variant 1.1.1.6. Let $\Delta_{\text{inj}}$ denote the category whose objects are sets of the form $[n]$ (where $n$ is a nonnegative integer) and whose morphisms are strictly increasing functions $\alpha : [m] \to [n]$. If $\mathcal{C}$ is any category, we will refer to a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathcal{C}$ as a semisimplicial object of $\mathcal{C}$. We typically use the notation $\mathcal{C}^\bullet$ to indicate a semisimplicial object of $\mathcal{C}$, whose value on an object $[n] \in \Delta_{\text{inj}}^{\text{op}}$ we denote by $\mathcal{C}_n$.

Remark 1.1.1.7. The category $\Delta_{\text{inj}}$ of Variant 1.1.1.6 can be regarded as a (non-full) subcategory of the category $\Delta$ of Definition 1.1.1.2. Consequently, any simplicial object $\mathcal{C}^\bullet$ of a category $\mathcal{C}$ determines a semisimplicial object of $\mathcal{C}$, given by the composition

$$\Delta_{\text{inj}}^{\text{op}} \hookrightarrow \Delta^{\text{op}} \xrightarrow{\mathcal{C}^\bullet} \mathcal{C}.$$ 

We will often abuse notation by identifying a simplicial object $\mathcal{C}^\bullet$ of $\mathcal{C}$ with the underlying semisimplicial object of $\mathcal{C}$.

To a first degree of approximation, a simplicial object $\mathcal{C}^\bullet$ of a category $\mathcal{C}$ can be identified with the collection of objects $\{\mathcal{C}_n\}_{n \geq 0}$. However, these objects are equipped with additional structure, arising from the morphisms in the simplex category $\Delta$. We now spell this out more concretely.

Notation 1.1.1.8. Let $n$ be a positive integer. For $0 \leq i \leq n$, we let $\delta^i : [n-1] \to [n]$ denote the unique strictly increasing function whose image does not contain the element $i$, given concretely by the formula

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j + 1 & \text{if } j \geq i. \end{cases}$$
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If \( C \) is a (semi)simplicial object of a category \( C \), then we can evaluate \( C \) on the morphism \( \delta^i \) to obtain a morphism from \( C_n \) to \( C_{n-1} \). We will denote this map by \( d_i : C_n \rightarrow C_{n-1} \) and refer to it as the \( i \)th face map.

Dually, if \( C^\bullet \) is a cosimplicial object of a category \( C \), then the evaluation of \( C^\bullet \) on the morphism \( \delta_i \) determines a map \( d^i : C^{n-1} \rightarrow C^n \), which we refer to as the \( i \)th coface map.

**Notation 1.1.1.9.** For every pair of integers \( 0 \leq i \leq n \) we let \( \sigma^i : [n+1] \rightarrow [n] \) denote the function given by the formula

\[
\sigma^i(j) = \begin{cases} 
  j & \text{if } j \leq i \\
  j - 1 & \text{if } j > i.
\end{cases}
\]

If \( C \) is a simplicial object of a category \( C \), then we can evaluate \( C \) on the morphism \( \sigma^i \) to obtain a morphism from \( C_n \) to \( C_{n+1} \). We will denote this map by \( s_i : C_n \rightarrow C_{n+1} \) and refer to it as the \( i \)th degeneracy map.

Dually, if \( C^\bullet \) is a cosimplicial object of a category \( C \), then the evaluation on \( C^\bullet \) on the morphism \( \sigma_i \) determines a map \( s^i : C^{n+1} \rightarrow C^n \), which we refer to as the \( i \)th codegeneracy map.

**Exercise 1.1.1.10.** Let \( C \) be a semisimplicial object of a category \( C \). Show that the face maps of Notation 1.1.1.8 satisfy the following condition:

\( \ast \) For \( n \geq 2 \) and \( 0 \leq i < j \leq n \), we have \( d_i \circ d_j = d_{j-1} \circ d_i \) (as a map from \( C_n \) to \( C_{n-2} \)).

Conversely, show that any collection of objects \( \{C_n\}_{n \geq 0} \) and morphisms \( \{d_i : C_n \rightarrow C_{n-1}\}_{0 \leq i \leq n} \), satisfying \( \ast \) determines a unique semisimplicial object of \( C \).

**Exercise 1.1.1.11.** Let \( C \) be a simplicial object of a category \( C \). Show that the face and degeneracy maps of Notations 1.1.1.8 and 1.1.1.9 satisfy the simplicial identities

1. For \( n \geq 2 \) and \( 0 \leq i < j \leq n \), we have \( d_i \circ d_j = d_{j-1} \circ d_i \) (as a map from \( C_n \) to \( C_{n-2} \)).
2. For \( 0 \leq i \leq j \leq n \), we have \( s_i \circ s_j = s_{j+1} \circ s_i \) (as a map from \( C_n \) to \( C_{n+2} \)).
3. For \( 0 \leq i, j \leq n \), we have

\[
d_i \circ s_j = \begin{cases} 
  s_{j-1} \circ d_i & \text{if } i < j \\
  \text{id}_{C_n} & \text{if } i = j \text{ or } i = j + 1 \\
  s_j \circ d_{i-1} & \text{if } i > j + 1.
\end{cases}
\]

Conversely, show that any collection of objects \( \{C_n\}_{n \geq 0} \) and morphisms \( \{d_i : C_n \rightarrow C_{n-1}\}_{0 \leq i \leq n} \) and \( \{s_i : C_n \rightarrow C_{n+1}\}_{0 \leq i \leq n} \) satisfying (1), (2), and (3) determines a (unique) simplicial object of \( C \).
We will be primarily interested in the following special case of Definition 1.1.1.4:

**Definition 1.1.1.12.** Let Set denote the category of sets. A simplicial set is a simplicial object of Set: that is, a functor $\Delta^{\text{op}} \to \text{Set}$. A semisimplicial set is a semisimplicial object of Set: that is, a functor $\Delta^{\text{op}}_{\text{inj}} \to \text{Set}$. If $S_\bullet$ is a (semi)simplicial set, then we will refer to elements of $S_n$ as $n$-simplices of $S_\bullet$. We will also refer to the elements of $S_0$ as vertices of $S_\bullet$, and to the elements of $S_1$ as edges of $S_\bullet$.

We let $\text{Set}_\Delta = \text{Fun}(\Delta^{\text{op}}, \text{Set})$ denote the category of functors from $\Delta^{\text{op}}$ to Set. We refer to $\text{Set}_\Delta$ as the category of simplicial sets.

**Remark 1.1.1.13.** Since the category of sets has all (small) limits and colimits, the category of (semi)simplicial sets also has all (small) limits and colimits. Moreover, these limits and colimits are computed levelwise: for any functor $S_\bullet : C \to \text{Set}$ $\Delta^{\text{op}}(C) \mapsto S_\bullet(C)$, and any nonnegative integer $n$, we have canonical bijections

$$(\lim_{\overleftarrow{C} \in C} S(C))_n \simeq \lim_{\overleftarrow{C} \in C} (S_n(C)) \quad (\lim_{\overrightarrow{C} \in C} S(C))_n \simeq \lim_{\overrightarrow{C} \in C} (S_n(C)).$$

### 1.1.2 Simplices and Horns

We now consider some elementary examples of simplicial sets.

**Construction 1.1.2.1 (The Standard Simplex).** Let $n \geq 0$ be an integer. We let $\Delta^n$ denote the simplicial set given by the construction

$$([m] \in \Delta) \mapsto \text{Hom}_\Delta([m], [n]).$$

We will refer to $\Delta^n$ as the standard $n$-simplex. By convention, we extend this construction to the case $n = -1$ by setting $\Delta^{-1} = \emptyset$.

**Example 1.1.2.2.** The standard 0-simplex $\Delta^0$ is a final object of the category of simplicial sets: that is, it carries each $[n] \in \Delta^{\text{op}}$ to a set having a single element.

**Remark 1.1.2.3.** For each $n \geq 0$, the standard $n$-simplex $\Delta^n$ is characterized by the following universal property: for every simplicial set $X_\bullet$, Yonedo's lemma supplies a bijection

$$\text{Hom}_{\text{Set}_\Delta}(\Delta^n, X_\bullet) \simeq X_n.$$
Remark 1.1.2.4. Let $S_\bullet$ be a simplicial set. Suppose that, for every integer $n \geq 0$, we are given a subset $T_n \subseteq S_n$, and that the face and degeneracy maps

$$d_i : S_n \to S_{n-1} \quad s_i : S_n \to S_{n+1}$$

carry $T_n$ into $T_{n-1}$ and $T_{n+1}$, respectively. Then the collection $\{T_n\}_{n \geq 0}$ inherits the structure of a simplicial set $T_\bullet$. In this case, we will say that $T_\bullet$ is a \textit{simplicial subset} of $S_\bullet$ and write $T_\bullet \subseteq S_\bullet$.

Example 1.1.2.5. Let $S_\bullet$ be a simplicial set and let $v$ be a vertex of $S_\bullet$. Then $v$ can be identified with a map of simplicial sets $\Delta^0 \to S_\bullet$. This map is automatically a monomorphism (note that $\Delta^0$ has only a single $n$-simplex for every $n \geq 0$), whose image is a simplicial subset of $S_\bullet$. It will often be convenient to denote this simplicial subset by $\{v\}$. For example, we can identify vertices of the standard $n$-simplex $\Delta^n$ with integers $i$ satisfying $0 \leq i \leq n$; every such integer $i$ determines a simplicial subset $\{i\} \subseteq \Delta^n$ (whose $k$-simplices are the constant maps $[k] \to [n]$ taking the value $i$).

It will be useful to consider some other simplicial subsets of the standard $n$-simplex.

Construction 1.1.2.6 (The Boundary of $\Delta^n$). Let $n \geq 0$ be an integer. We define a simplicial set $(\partial \Delta^n) : \Delta^\text{op} \to \text{Set}$ by the formula

$$(\partial \Delta^n)([m]) = \{\alpha \in \text{Hom}_\Delta([m],[n]) : \alpha \text{ is not surjective}\}.$$ 

Note that we can regard $\partial \Delta^n$ as a simplicial subset of the standard $n$-simplex $\Delta^n$ of Construction 1.1.2.1. We will refer to $\partial \Delta^n$ as the \textit{boundary} of $\Delta^n$.

Example 1.1.2.7. The simplicial set $\partial \Delta^0$ is empty.

Exercise 1.1.2.8. Let $n \geq 0$ be an integer. For $0 \leq j \leq n$, the map $\delta^j : [n-1] \to [n]$ of Notation 1.1.1.8 determines a map of simplicial sets $\Delta^{n-1} \to \Delta^n$ which factors through the simplicial subset $\partial \Delta^n \subseteq \Delta^n$. We therefore obtain a map of simplicial sets $\Delta^{n-1} \to \partial \Delta^n$, which we will also denote by $\delta^j$. Show that, for any simplicial set $S_\bullet$, the construction

$$(f : \partial \Delta^n \to S_\bullet) \mapsto \{f \circ \delta^j\}_{0 \leq j \leq n}$$

determines an injective map

$$\text{Hom}_{\text{Set}}(\partial \Delta^n, S_\bullet) \to \prod_{j \in [n]} S_{n-1},$$

whose image is the collection of sequences of $(n-1)$-simplices $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ satisfying the identities $d_j(\sigma_k) = d_{k-1}(\sigma_j)$ for $0 \leq j < k \leq n$. 

Construction 1.1.2.9 (The Horn $\Lambda^n_i$). Suppose we are given a pair of integers $0 \leq i \leq n$. We define a simplicial set $\Lambda^n_i : \Delta^{op} \to \text{Set}$ by the formula

$$(\Lambda^n_i)([m]) = \{ \alpha \in \text{Hom}_\Delta([m], [n]) : [n] \not\subseteq \alpha([m]) \cup \{i\} \}.$$ 

We regard $\Lambda^n_i$ as a simplicial subset of the boundary $\partial \Delta^n \subseteq \Delta^n$. We will refer to $\Lambda^n_i$ as the $i$th horn in $\Delta^n$. We will say that $\Lambda^n_i$ is an inner horn if $0 < i < n$, and an outer horn if $i = 0$ or $i = n$.

Remark 1.1.2.10. Roughly speaking, one can think of the horn $\Lambda^n_i$ as obtained from the $n$-simplex $\Delta^n$ by removing its interior together with the face opposite its $i$th vertex (see Example 1.1.8.13).

Example 1.1.2.11. The horns contained in $\Delta^2$ are depicted in the following diagram:

Here the dotted arrows indicate edges of $\Delta^2$ which are not contained in the corresponding horn.

Example 1.1.2.12. The horns $\Lambda^0_0$ and $\Lambda^1_1$ are both isomorphic to $\Delta^0$, and the inclusion maps $\Lambda^0_0 \hookrightarrow \partial \Delta^1 \hookrightarrow \Lambda^1_1$ induce an isomorphism $\Delta^0 \amalg \Delta^0 \simeq \partial \Delta^1$.

Example 1.1.2.13. The horn $\Lambda^0_0$ is the empty simplicial set (and therefore coincides with the boundary $\partial \Delta^0$).

Exercise 1.1.2.14. Let $0 \leq i \leq n$ be integers. For $j \in [n] \setminus \{i\}$, we can regard the map $\delta^j$ of Exercise 1.1.2.8 as a map of simplicial sets from $\Delta^{n-1}$ to the horn $\Lambda^n_i \subseteq \Delta^n$. Show that, for any simplicial set $S_\bullet$, the construction

$$(f : \Lambda^n_i \to S_\bullet) \mapsto \{ f \circ \delta^j \}_{j \in [n] \setminus \{i\}}$$

determines an injection $\text{Hom}_{\text{Set}}(\Lambda^n_i, S_\bullet) \to \prod_{j \in [n] \setminus \{i\}} S_{n-1}$, whose image consists of "incomplete" sequences $(\sigma_0, \ldots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \ldots, \sigma_n)$ satisfying $d_j(\sigma_k) = d_{k-1}(\sigma_j)$ for $j, k \in [n] \setminus \{i\}$ with $j < k$. 
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1.1.3 The Skeletal Filtration

Roughly speaking, one can think of the simplicial sets $\Delta^n$ of Construction 1.1.2.1 as elementary building blocks out of which more complicated simplicial sets can be constructed. In this section, we make this idea more precise by introducing the skeletal filtration of a simplicial set. This filtration allows us to write every simplicial set $S_\bullet$ as the union of an increasing sequence of simplicial subsets

$$
{\rm sk}_0(S_\bullet) \subseteq {\rm sk}_1(S_\bullet) \subseteq {\rm sk}_2(S_\bullet) \subseteq {\rm sk}_3(S_\bullet) \subseteq \cdots ,
$$

where each $sk_n(S_\bullet)$ is obtained from $sk_{n-1}(S_\bullet)$ by attaching copies of $\Delta^n$ (see Proposition 1.1.3.13 below for a precise statement). We will need some preliminaries.

**Proposition 1.1.3.1.** Let $S_\bullet$ be a simplicial set and let $\sigma \in S_n$ be an $n$-simplex of $S_\bullet$ for some $n > 0$, which we will identify with a map of simplicial sets $\sigma : \Delta^n \to S_\bullet$. The following conditions are equivalent:

1. The simplex $\sigma$ belongs to the image of the degeneracy map $s_i : S_{n-1} \to S_n$ for some $0 \leq i \leq n - 1$ (see Notation 1.1.1.9).
2. The map $\sigma$ factors as a composition $\Delta^n \overset{f}{\to} \Delta^{n-1} \to S_\bullet$, where $f$ corresponds to a surjective map of linearly ordered sets $[n] \to [n-1]$.
3. The map $\sigma$ factors as a composition $\Delta^n \overset{f}{\to} \Delta^m \to S_\bullet$, where $m < n$ and $f$ corresponds to a surjective map of linearly ordered sets $[n] \to [m]$.
4. The map $\sigma$ factors as a composition $\Delta^n \to \Delta^m \to S_\bullet$, where $m < n$.

*Proof.* The implications $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are immediate. We will complete the proof by showing that $(4)$ implies $(1)$. Assume that $\sigma$ factors as a composition $\Delta^n \overset{f}{\to} \Delta^m \overset{\sigma'}{\to} S_\bullet$, where $m < n$. Let us abuse notation by identifying $f$ with a map of linearly ordered sets $[n] \to [m]$. Since $m < n$, this map cannot be injective. It follows that we can find some $i < n$ such that $f(i) = f(i+1)$. It follows that $f$ factors through the map $\sigma^i : [n] \to [n-1]$ of Notation 1.1.1.9, so that $\sigma$ belongs to the image of the degeneracy map $s_i$. $\square$

**Definition 1.1.3.2.** Let $S_\bullet$ be a simplicial set and let $\sigma : \Delta^n \to S_\bullet$ be an $n$-simplex of $S_\bullet$. We will say that $\sigma$ is degenerate if $n > 0$ and $\sigma$ satisfies the equivalent conditions of Proposition [1.1.3.1]. We say that $\sigma$ is nondegenerate if it is not degenerate (in particular, every 0-simplex of $S_\bullet$ is nondegenerate).

**Remark 1.1.3.3.** Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets. If $\sigma$ is a degenerate $n$-simplex of $S_\bullet$, then $f(\sigma)$ is a degenerate $n$-simplex of $T_\bullet$. The converse holds if $f$ is a monomorphism of simplicial sets (for example, if $S_\bullet$ is a simplicial subset of $T_\bullet$).
Proposition 1.1.3.4. Let $\sigma : \Delta^n \to S_\bullet$ be a map of simplicial sets. Then $\sigma$ can be factored as a composition

$$\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet,$$

where $\alpha$ corresponds to a surjective map of linearly ordered sets $[n] \to [m]$ and $\tau$ is a nondegenerate $m$-simplex of $S_\bullet$. Moreover, this factorization is unique.

Proof. Let $m$ be the smallest nonnegative integer for which $\sigma$ can be factored as a composition $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet$. It follows from the minimality of $m$ that $\alpha$ must induce a surjection of linearly ordered sets $[n] \to [m]$ (otherwise, we could replace $[m]$ by the image of $\alpha$) and that the $m$-simplex $\tau$ is nondegenerate. This proves the existence of the desired factorization.

To establish uniqueness, let us suppose we are given another factorization of $\sigma$ as a composition $\Delta^n \xrightarrow{\alpha'} \Delta^m' \xrightarrow{\tau'} S_\bullet$. By assumption, $\alpha$ and $\alpha'$ determine surjections of linearly ordered sets $[n] \to [m]$ and $[n] \to [m']$, and therefore admit sections which we will denote by $\beta$ and $\beta'$, respectively. The equality $\sigma = \tau \circ \alpha$ then gives

$$\tau = \sigma \circ \beta = \tau' \circ \alpha' \circ \beta.$$

Our assumption that $\tau$ is nondegenerate then guarantees that the map $\alpha' \circ \beta : [m] \to [m']$ is injective, so that $m \leq m'$. The same argument shows that $m' \leq m$, so we must have $m = m'$. Since the only nondecreasing injection from $[m]$ to itself is the identity map, we conclude that $\alpha' \circ \beta = \text{id}_{[m]}$. The desired uniqueness now follows from the calculations

$$\tau = \tau' \circ \alpha' \circ \beta = \tau' \quad \alpha = \alpha' \circ \beta \circ \alpha = \alpha'.$$

\[\square\]

Construction 1.1.3.5. Let $S_\bullet$ be a simplicial set, let $k \geq -1$ be an integer, and let $\sigma : \Delta^n \to S_\bullet$ be an $n$-simplex of $S_\bullet$. The proof of Proposition 1.1.3.4 shows that the following conditions are equivalent:

(a) Let $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S_\bullet$ be the factorization of Proposition 1.1.3.4 (so that $\alpha$ induces a surjection $[n] \to [m]$, the map $\tau$ is nondegenerate, and $\sigma = \tau \circ \alpha$). Then $m \leq k$.

(b) There exists a factorization $\Delta^n \to \Delta^{m'} \to S_\bullet$ of $\sigma$ for which $m' \leq k$.

For each $n \geq 0$, we let $sk_k(S_n)$ denote the subset of $S_n$ consisting of those $n$-simplices which satisfy conditions (a) and (b). From characterization (b), we see that the collection of subsets $\{sk_k(S_n) \subseteq S_n\}_{n \geq 0}$ is stable under the face and degeneracy operators of $S_\bullet$, and therefore determine a simplicial subset of $S_\bullet$ (Remark 1.1.2.4). We will denote this simplicial subset by $sk_k(S_\bullet)$ and refer to it as the $k$-skeleton of $S_\bullet$.

Remark 1.1.3.6. Let $S_\bullet$ be a simplicial set and let $k \geq -1$. If $n \leq k$, then $sk_k(S_\bullet)$ contains every $n$-simplex of $S_\bullet$. In particular, the union $\bigcup_{k \geq -1} sk_k(S_\bullet)$ is equal to $S_\bullet$. 
Remark 1.1.3.7. Let $S_\bullet$ be a simplicial set and let $\sigma$ be a nondegenerate $n$-simplex of $S_\bullet$. Then $\sigma$ is contained in $\text{sk}_k(S_\bullet)$ if and only if $n \leq k$.

Example 1.1.3.8. For any simplicial set $S_\bullet$, the $(−1)$-skeleton $\text{sk}_{−1}(S_\bullet)$ is empty.

We now show that the $k$-skeleton of a simplicial set $S_\bullet$ can be characterized by a universal property.

Definition 1.1.3.9. Let $S_\bullet$ be a simplicial set and let $k \geq −1$ be an integer. We will say that $S_\bullet$ has dimension $\leq k$ if, for $n > k$, every $n$-simplex of $S_\bullet$ is degenerate. If $k \geq 0$, we say that $S_\bullet$ has dimension $k$ if it has dimension $\leq k$ but does not have dimension $\leq k − 1$. We say that $S_\bullet$ is finite-dimensional if it has dimension $\leq k$ for some $k \gg 0$.

Proposition 1.1.3.10. Let $S_\bullet$ be a simplicial set and let $k \geq −1$ be an integer. Then:

(a) The simplicial set $\text{sk}_k(S_\bullet)$ has dimension $\leq k$.

(b) For every simplicial set $T_\bullet$ of dimension $\leq k$, composition with the inclusion map $\text{sk}_k(S_\bullet) \hookrightarrow S_\bullet$ induces a bijection

$$\text{Hom}_{\Delta\text{-}\text{Set}}(T_\bullet, \text{sk}_k(S_\bullet)) \rightarrow \text{Hom}_{\Delta\text{-}\text{Set}}(T_\bullet, S_\bullet).$$

In other words, the image of any map $T_\bullet \rightarrow S_\bullet$ is contained in $\text{sk}_k(S_\bullet)$.

Proof. Assertion (a) follows from Remark 1.1.3.7. To prove (b), suppose that $f : T_\bullet \rightarrow S_\bullet$ is a map of simplicial sets, where $T_\bullet$ has dimension $\leq k$. We wish to show that $f$ carries every $n$-simplex $\sigma$ of $T_\bullet$ to an $n$-simplex of $\text{sk}_k(S_\bullet)$. Using Proposition 1.1.3.4, we can reduce to the case where $\sigma$ is a nondegenerate $n$-simplex of $T_\bullet$. In this case, our assumption that $T_\bullet$ has dimension $\leq k$ guarantees that $n \leq k$, so that $f(\sigma)$ belongs to $\text{sk}_k(S_\bullet)$ by virtue of Remark 1.1.3.6.

Proposition 1.1.3.11. Let $S^-_\bullet$ and $S^+_\bullet$ be simplicial sets having dimensions $\leq k_−$ and $\leq k_+$, respectively. Then the product $S^-_\bullet \times S^+_\bullet$ has dimension $\leq k_− + k_+$.

Proof. Let $\sigma = (\sigma_-, \sigma_+)$ be a nondegenerate $n$-simplex of the product $S^-_\bullet \times S^+_\bullet$. Using Proposition 1.1.3.4, we see that $\sigma_-$ and $\sigma_+$ admit factorizations

$$\Delta^n \xrightarrow{\alpha_-} \Delta^{n−} \xrightarrow{\tau_-} S^-_\bullet \quad \Delta^n \xrightarrow{\alpha_+} \Delta^{n+} \xrightarrow{\tau_+} S^+_\bullet,$$

where $\tau_-$ and $\tau_+$ are nondegenerate, so that $n_- \leq k_−$ and $n_+ \leq k_+$. It follows that $\sigma$ factors as a composition

$$\Delta^n \xrightarrow{(\alpha_-, \alpha_+)} \Delta^{n−} \times \Delta^{n+} \xrightarrow{\tau_− \times \tau_+} S^-_\bullet \times S^+_\bullet.$$

The nondegeneracy of $\sigma$ guarantees that the map of partially ordered sets $[n] \xrightarrow{(\alpha_-, \alpha_+)} [n_−] \times [n_+]$ is a monomorphism, so that $n \leq n_- + n_+ \leq k_− + k_+$.
Exercise 1.1.3.12. Show that the inequality of Proposition 1.1.3.11 is sharp. That is, if $S^-$ and $S^+$ are nonempty simplicial sets of dimensions $k_-$ and $k_+$, respectively, then the product $S^- \times S^+$ has dimension $k_- + k_+$.

Let $S_\bullet$ be a simplicial set. For each $k \geq 0$, we let $S^\text{nd}_k$ denote the collection of all nondegenerate $k$-simplices of $S_\bullet$. Every element $\sigma \in S^\text{nd}_k$ determines a map of simplicial sets $\Delta^k \to \sk_k(S_\bullet)$. Since the boundary $\partial \Delta^k \subseteq \Delta^k$ has dimension $\leq k - 1$, this map carries $\partial \Delta^k$ into the $(k - 1)$-skeleton $\sk_{k-1}(S_\bullet)$.

**Proposition 1.1.3.13.** Let $S_\bullet$ be a simplicial set and let $k \geq 0$. Then the construction outlined above determines a pushout square

$$
\begin{array}{ccc}
\coprod_{\sigma \in S^\text{nd}_k} \partial \Delta^k & \to & \coprod_{\sigma \in S^\text{nd}_k} \Delta^k \\
\downarrow & & \downarrow \\
\sk_{k-1}(S_\bullet) & \to & \sk_k(S_\bullet)
\end{array}
$$

in the category $\Set_{\Delta}$ of simplicial sets.

**Proof.** Unwinding the definitions, we must prove the following:

(*) Let $\tau$ be an $n$-simplex of $\sk_k(S_\bullet)$ which is not contained in $\sk_{k-1}(S_\bullet)$. Then $\tau$ factors uniquely as a composition

$$
\Delta^n \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} S_\bullet,
$$

where $\sigma$ is a nondegenerate simplex of $S_\bullet$ and $\alpha$ does not factor through the boundary $\partial \Delta^k$ (in other words, $\alpha$ induces a surjection of linearly ordered sets $[n] \to [k]$).

Proposition 1.1.3.4 implies that any $n$-simplex of $S_\bullet$ admits a unique factorization $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\sigma} S_\bullet$, where $\alpha$ is surjective and $\sigma$ is nondegenerate. Our assumption that $\tau$ belongs to the $\sk_k(S_\bullet)$ guarantees that $m \leq k$, and our assumption that $\tau$ does not belong to $\sk_{k-1}(S_\bullet)$ guarantees that $m \geq k$.

\[\square\]

### 1.1.4 Discrete Simplicial Sets

Simplicial sets of dimension $\leq 0$ admit a simple classification:

**Proposition 1.1.4.1.** The evaluation functor

$$
ev_0 : \Set_\Delta \to \Set \quad X_\bullet \mapsto X_0$$

restricts to an equivalence of categories

$$\{\text{Simplicial sets of dimension } \leq 0\} \simeq \Set.$$

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We will give a proof of Proposition 1.1.4.1 at the end of this section. First, we make some general remarks which apply to simplicial objects of any category \( C \).

**Construction 1.1.4.2.** Let \( C \) be a category. For each object \( C \in C \), we let \( C_\bullet \) denote the constant functor \( \Delta^{\operatorname{op}} \to \{C\} \hookrightarrow C \) taking the value \( C \). We regard \( C_\bullet \) as a simplicial object of \( C \), which we will refer to as the constant simplicial object with value \( C \).

**Remark 1.1.4.3.** Let \( C \) be an object of the category \( C \). The constant simplicial object \( C_\bullet \) can be described concretely as follows:

- For each \( n \geq 0 \), we have \( C_n = C \).
- The face and degeneracy operators \( d_i : C_n \to C_{n-1} \quad s_i : C_n \to C_{n+1} \) are the identity maps from \( C \) to itself.

**Example 1.1.4.4.** Let \( S = \{s\} \) be a set containing a single element. Then \( S_\bullet \) is a final object of the category of simplicial sets: that is, it is isomorphic to the standard simplex \( \Delta^0 \).

The constant simplicial object \( C_\bullet \) of Construction 1.1.4.2 can be characterized by a universal mapping property:

**Proposition 1.1.4.5.** Let \( C \) be a category and let \( C_0 \) be an object of \( C \). For any simplicial object \( X_\bullet \) of \( C \), the canonical map

\[
\operatorname{Hom}_{\operatorname{Fun}(\Delta^{\operatorname{op}}, C)}(C_\bullet, X_\bullet) \to \operatorname{Hom}_C(C_0, X_0) = \operatorname{Hom}_C(C, X_0)
\]

is a bijection.

**Proof.** Let \( f : C \to X_0 \) be a morphism in \( C \); we wish to show that \( f \) can be promoted uniquely to a map of simplicial objects \( f_\bullet : C_\bullet \to X_\bullet \). The uniqueness of \( f_\bullet \) is clear. For existence, we define \( f_\bullet \) to be the natural transformation whose value on an object \([n] \in \Delta^{\operatorname{op}}\) is given by the composite map

\[
C_n = C \xrightarrow{f} X_0 \xrightarrow{X_{\alpha(n)}} X_n,
\]

where \( \alpha(n) \) denotes the unique morphism in \( \Delta \) from \([n]\) to \([0]\). To prove the naturality of \( f_\bullet \), we observe that for any nondecreasing map \( \beta : [m] \to [n] \) we have a commutative diagram

\[
\begin{array}{ccc}
C_n & \xrightarrow{f} & X_0 \\
\downarrow{C_{\alpha}} & & \downarrow{X_{\alpha(n)}} \\
C_m & \xrightarrow{f} & X_0 \\
\end{array}
\]

where the commutativity of the square on the right follows from the observation that \( \alpha(m) \) is equal to the composition \([m] \xrightarrow{\beta} [n] \xrightarrow{\alpha(n)} [0] \).
Remark 1.1.4.6. Let $\mathcal{C}$ be a category. Proposition 1.1.4.5 can be rephrased as follows:

- For any simplicial object $X_\bullet$ of $\mathcal{C}$, the limit $\lim_{\leftarrow [n] \in \Delta^{op}} X_n$ exists in the category $\mathcal{C}$.
- The canonical map $\lim_{\leftarrow [n] \in \Delta^{op}} X_n \to X_0$ is an isomorphism.

These assertions follow formally from the observation that $[0]$ is a final object of the category $\Delta$ (and therefore an initial object of the category $\Delta^{op}$).

Corollary 1.1.4.7. Let $\mathcal{C}$ be a category. Then the evaluation functor

$$evo : \text{Fun}(\Delta^{op}, \mathcal{C}) \to \mathcal{C} \quad X_\bullet \mapsto X_0$$

admits a left adjoint, given on objects by the formation of constant simplicial objects $C \mapsto C_\bullet$, described in Construction 1.1.4.2.

Corollary 1.1.4.8. Let $\mathcal{C}$ be a category. Then the construction $C \mapsto C_\bullet$ determines a fully faithful embedding from $\mathcal{C}$ to the category $\text{Fun}(\Delta^{op}, \mathcal{C})$ of simplicial objects of $\mathcal{C}$.

Proof. Let $C$ and $D$ be objects of $\mathcal{C}$; we wish to show that the canonical map

$$\theta : \text{Hom}_\mathcal{C}(C, D) \to \text{Hom}_{\text{Fun}(\Delta^{op}, \mathcal{C})}(C_\bullet, D_\bullet)$$

is a bijection. This is clear, since $\theta$ is right inverse to the evaluation map

$$\text{Hom}_{\text{Fun}(\Delta^{op}, \mathcal{C})}(C_\bullet, D_\bullet) \to \text{Hom}_\mathcal{C}(C, D)$$

which is bijective by virtue of Proposition 1.1.4.5.

We now specialize to the case where $\mathcal{C} = \text{Set}$ is the category of sets.

Definition 1.1.4.9. Let $X_\bullet$ be a simplicial set. We will say that $X_\bullet$ is discrete if there exists a set $S$ and an isomorphism of simplicial sets $X_\bullet \simeq S_\bullet$; here $S_\bullet$ denotes the constant simplicial set of Construction 1.1.4.2.

Specializing Corollary 1.1.4.8 to the case $\mathcal{C} = \text{Set}$, we obtain the following:

Corollary 1.1.4.10. The construction $S \mapsto S_\bullet$ determines a fully faithful embedding $\text{Set} \hookrightarrow \text{Set}_\Delta$. The essential image of this embedding is the full subcategory of $\text{Set}_\Delta$ spanned by the discrete simplicial sets.

Notation 1.1.4.11. Let $S$ be a set. We will often abuse notation by identifying $S$ with the constant simplicial set $S_\bullet$ of Construction 1.1.4.2 (by virtue of Corollary 1.1.4.10, this is mostly harmless). This abuse will occur most frequently in the special case where $S = \{v\}$ consists of a single vertex $v$ of some other simplicial set $X_\bullet$; in this case, we view $\{v\}$ as a simplicial subset of $X_\bullet$ which is abstractly isomorphic to $\Delta^0$ (see Example 1.1.2.5).
Remark 1.1.4.12. The fully faithful embedding
\[
\text{Set} \hookrightarrow \text{Set}_\Delta \quad S \mapsto S_•
\]
preserves (small) limits and colimits (since limits and colimits of simplicial sets are computed levelwise; see Remark 1.1.1.13). It follows that the collection of discrete simplicial sets is closed under the formation of (small) limits and colimits in \( \text{Set}_\Delta \).

Proposition 1.1.4.13. Let \( X_• \) be a simplicial set. The following conditions are equivalent:

1. The simplicial set \( X_• \) is discrete (Definition 1.1.4.9). That is, \( X_• \) is isomorphic to a constant simplicial set \( S_• \).

2. For every morphism \( \alpha : [m] \to [n] \) in the category \( \Delta \), the induced map \( X_n \to X_m \) is a bijection.

3. For every positive integer \( n \), the 0th face map \( d_0 : X_n \to X_{n-1} \) is a bijection.

4. The simplicial set \( X_• \) has dimension \( \leq 0 \), in the sense of Definition 1.1.3.9. That is, \( X_• \) does not contain any nondegenerate \( n \)-simplices for \( n > 0 \).

Proof. The implication (1) \( \Rightarrow \) (2) follows from Remark 1.1.4.3 and the implication (2) \( \Rightarrow \) (3) is immediate. To prove that (3) \( \Rightarrow \) (4), we observe that if the face map \( d_0 : X_n \to X_{n-1} \) is bijective, then the degeneracy operator \( s_0 : X_{n-1} \to X_n \) is also bijective (since it is a right inverse of \( d_0 \)). In particular, \( s_0 \) is surjective, so every \( n \)-simplex of \( X_• \) is degenerate.

We complete the proof by showing that (4) \( \Rightarrow \) (1). If \( X_• \) is a simplicial set of dimension \( \leq 0 \) and \( S = X_0 \) is the set of vertices of \( X_• \), then Proposition 1.1.3.13 supplies an isomorphism of simplicial sets \( \bigsqcup_{v \in S} \Delta^0 \simeq X_• \), whose domain can be identified with the constant simplicial set \( S_• \) (by virtue of Remark 1.1.4.12 and Example 1.1.4.4).

Proof of Proposition 1.1.4.1. By virtue of Proposition 1.1.4.13, it will suffice to show that the construction \( X_• \mapsto X_0 \) induces an equivalence of categories
\[
\{\text{Discrete simplicial sets}\} \to \text{Set}.
\]
This follows immediately from Corollary 1.1.4.10.

1.1.5 Directed Graphs as Simplicial Sets

We now generalize Proposition 1.1.4.13 to obtain a concrete description of simplicial sets having dimension \( \leq 1 \) (Proposition 1.1.5.9).

Definition 1.1.5.1. A directed graph \( G \) consists of the following data:
A set $\text{Vert}(G)$, whose elements we refer to as vertices of $G$.

A set $\text{Edge}(G)$, whose elements we refer to as edges of $G$.

A pair of functions $s, t : \text{Edge}(G) \to \text{Vert}(G)$ which assign to each edge $e \in \text{Edge}(G)$ a pair of vertices $s(e), t(e) \in \text{Vert}(G)$ that we refer to as the source and target of $e$, respectively.

**Warning 1.1.5.2.** The terminology of Definition 1.1.5.1 is not standard. Note that a directed graph $G$ can have distinct edges $e \neq e'$ having the same source $s(e) = s(e')$ and target $t(e) = t(e')$ (for this reason, directed graphs in the sense of Definition 1.1.5.1 are sometimes called multigraphs). Definition 1.1.5.1 also allows graphs which contain loops: that is, edges $e$ satisfying $s(e) = t(e)$.

**Remark 1.1.5.3.** It will sometimes be convenient to represent a directed graph $G$ by a diagram, having a node for each vertex $v$ of $G$ and an arrow for each edge $e$ of $G$, directed from the source of $e$ to the target of $e$. For example, the diagram

\[
\begin{array}{ccc}
\bullet & \downarrow & \downarrow \\
\uparrow & \downarrow & \downarrow \\
\bullet & \downarrow & \downarrow \\
\end{array}
\]

represents a directed graph with three vertices and five edges.

**Example 1.1.5.4.** To every simplicial set $X_\bullet$, we can associate a directed graph $\text{Gr}(X_\bullet)$ as follows:

- The vertex set $\text{Vert}(\text{Gr}(X_\bullet))$ is the set of 0-simplices of the simplicial set $X_\bullet$.
- The edge set $\text{Edge}(\text{Gr}(X_\bullet))$ is the set of nondegenerate 1-simplices of the simplicial set $X_\bullet$.
- For every edge $e \in \text{Edge}(\text{Gr}(X_\bullet)) \subseteq X_1$, the source $s(e)$ is the vertex $d_1(e)$, and the target $t(e)$ is the vertex $d_0(e)$ (here $d_0, d_1 : X_1 \to X_0$ are the face maps of Notation 1.1.1.8).

It will be convenient to construe Example 1.1.5.4 as providing a functor from the category of simplicial sets to the category of directed graphs. First, we need an appropriate definition for the latter category.

**Definition 1.1.5.5.** Let $G$ and $G'$ be directed graphs (in the sense of Definition 1.1.5.1). A **morphism** from $G$ to $G'$ is a function $f : \text{Vert}(G) \amalg \text{Edge}(G) \to \text{Vert}(G') \amalg \text{Edge}(G')$ which satisfies the following conditions:
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(a) For each vertex $v \in \text{Vert}(G)$, the image $f(v)$ belongs to $\text{Vert}(G')$.

(b) Let $e \in \text{Edge}(G)$ be an edge of $G$ with source $v = s(e)$ and target $w = t(e)$. Then exactly one of the following conditions holds:

- The image $f(e)$ is an edge of $G'$ having source $s(f(e)) = f(v)$ and target $t(f(e)) = f(w)$.
- The image $f(e)$ is a vertex of $G'$ satisfying $f(v) = f(e) = f(w)$.

We let Graph denote the category whose objects are directed graphs and whose morphisms are morphisms of directed graphs (with composition defined in the evident way).

**Warning 1.1.5.6.** Note that part (b) of Definition 1.1.5.5 allows the possibility that a morphism of directed graphs $G \to G'$ can “collapse” edges of $G$ to vertices of $G'$. Many other notions of morphism between (directed) graphs appear in the literature; we single out Definition 1.1.5.5 because of its close connection with the theory of simplicial sets (see Proposition 1.1.5.7 below).

Let $X_\bullet$ be a simplicial set and let $\text{Gr}(X_\bullet)$ be the directed graph of Example 1.1.5.4. Then the disjoint union $\text{Vert}(\text{Gr}(X_\bullet)) \amalg \text{Edge}(\text{Gr}(X_\bullet))$ can be identified with the set $X_1$ of all 1-simplices of $X_\bullet$ (where we identify $\text{Vert}(\text{Gr}(X_\bullet))$ with the collection of degenerate 1-simplices via the degeneracy map $s_0 : X_0 \to X_1$).

**Proposition 1.1.5.7.** Let $f : X_\bullet \to Y_\bullet$ be a map of simplicial sets. Then the induced map

$$\text{Vert}(\text{Gr}(X_\bullet)) \amalg \text{Edge}(\text{Gr}(X_\bullet)) \cong X_1 \xrightarrow{f} Y_1 \cong \text{Vert}(\text{Gr}(Y_\bullet)) \amalg \text{Edge}(\text{Gr}(Y_\bullet))$$

is a morphism of directed graphs from $\text{Gr}(X_\bullet)$ to $\text{Gr}(Y_\bullet)$, in the sense of Definition 1.1.5.5.

**Proof.** Since $f$ commutes with the degeneracy operator $s_0$, it carries degenerate 1-simplices of $X_\bullet$ to degenerate 1-simplices of $Y_\bullet$, and therefore satisfies requirement (a) of Definition 1.1.5.5. Requirement (b) follows from the fact that $f$ commutes with the face operators $d_0$ and $d_1$.

It follows from Proposition 1.1.5.7 that we can regard the construction $X_\bullet \mapsto \text{Gr}(X_\bullet)$ as a functor from the category $\text{Set}_\Delta$ of simplicial sets to the category Graph of directed graphs.

**Proposition 1.1.5.8.** Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. If $X_\bullet$ has dimension $\leq 1$, then the canonical map

$$\text{Hom}_{\text{Set}_\Delta}(X_\bullet, Y_\bullet) \to \text{Hom}_{\text{Graph}}(\text{Gr}(X_\bullet), \text{Gr}(Y_\bullet))$$

is bijective.
Proof. If $X\bullet$ has dimension $\leq 1$, then Proposition 1.1.3.13 provides a pushout diagram

\[
\begin{array}{ccc}
\prod_{e \in \text{Edge}(\text{Gr}(X\bullet))} \Delta^1 & \to & \prod_{e \in \text{Edge}(\text{Gr}(X\bullet))} \Delta^1 \\
\downarrow & & \downarrow \\
\prod_{v \in \text{Vert}(\text{Gr}(X\bullet))} \Delta^0 & \to & X\bullet.
\end{array}
\]

It follows that, for any simplicial set $Y\bullet$, we can identify $\text{Hom}_{\Delta}(X\bullet, Y\bullet)$ with the fiber product

\[
(\prod_{e \in \text{Edge}(\text{Gr}(X\bullet))} Y_1) \times_{\prod_{e \in \text{Edge}(\text{Gr}(X\bullet))} (Y_0 \times Y_0)} (\prod_{v \in \text{Vert}(\text{Gr}(X\bullet))} Y_0),
\]

which is precisely the set of morphisms of directed graphs from $\text{Gr}(X\bullet)$ to $\text{Gr}(Y\bullet)$.

It follows from Proposition 1.1.5.8 that the theory of simplicial sets of dimension $\leq 1$ is essentially equivalent to the theory of directed graphs.

Proposition 1.1.5.9. Let $\text{Set}_\Delta$ denote the category of simplicial sets and let $\text{Set}_{\Delta}^{\leq 1} \subseteq \text{Set}_\Delta$ denote the full subcategory spanned by the simplicial sets of dimension $\leq 1$. Then the construction $X\bullet \mapsto \text{Gr}(X\bullet)$ induces an equivalence of categories $\text{Set}_{\Delta}^{\leq 1} \to \text{Graph}$.

Proof. It follows from Proposition 1.1.5.8 that the functor $X\bullet \mapsto \text{Gr}(X\bullet)$ is fully faithful when restricted to simplicial sets of dimension $\leq 1$. It will therefore suffice to show that is essentially surjective. Let $G$ be any directed graph, and form a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\prod_{e \in \text{Edge}(G)} \Delta^1 & \to & \prod_{e \in \text{Edge}(G)} \Delta^1 \\
\downarrow_{(s,t)} & & \downarrow \\
\prod_{v \in \text{Vert}(G)} \Delta^0 & \to & X\bullet.
\end{array}
\]

Then $X\bullet$ is a simplicial set of dimension $\leq 1$, and the directed graph $\text{Gr}(X\bullet)$ is isomorphic to $G$. 

Remark 1.1.5.10. The proof of Proposition 1.1.5.9 gives an explicit description of the inverse equivalence $\text{Graph} \simeq C \to \text{Set}_\Delta$: it carries a directed graph $G$ to the 1-dimensional simplicial set $G\bullet$ given by the pushout

\[
(\prod_{v \in \text{Vert}(G)} \Delta^0) \amalg (\prod_{e \in \text{Edge}(G)} \Delta^1).
\]
Example 1.1.5.11. Let \( G \) be a directed graph and let \( G_\bullet \) denote the associated simplicial set of dimension \( \leq 1 \) (Remark 1.1.5.10). Then \( G_\bullet \) has dimension \( \leq 0 \) if and only if the edge set \( \text{Edge}(G) \) is empty. In this case, \( G_\bullet \) can be identified with the constant simplicial set associated to the vertex set \( \text{Vert}(G) \).

### 1.1.6 Connected Components of Simplicial Sets

In this section, we introduce the notion of a connected simplicial set (Definition 1.1.6.6) and show that every simplicial set \( S_\bullet \) admits an (essentially unique) decomposition as a disjoint union of connected subsets (Proposition 1.1.6.13), indexed by a set \( \pi_0(S_\bullet) \) which we call the set of connected components of \( S_\bullet \). Moreover, we characterize the construction \( S_\bullet \mapsto \pi_0(S_\bullet) \) as a left adjoint to the functor \( I \mapsto I_\bullet \) of Construction 1.1.4.2 (Corollary 1.1.6.21).

Definition 1.1.6.1. Let \( S_\bullet \) be a simplicial set and let \( S'_\bullet \subseteq S_\bullet \) be a simplicial subset of \( S_\bullet \) (Remark 1.1.2.4). We will say that \( S'_\bullet \) is a summand of \( S_\bullet \) if the simplicial set \( S_\bullet \) decomposes as a coproduct \( S'_\bullet \bigsqcup S''_\bullet \), for some other simplicial subset \( S''_\bullet \subseteq S_\bullet \).

Remark 1.1.6.2. In the situation of Definition 1.1.6.1, if \( S'_\bullet \subseteq S_\bullet \) is a summand, then the complementary summand \( S''_\bullet \) is uniquely determined: for each \( n \geq 0 \), we must have \( S''_n = S_n \setminus S'_n \). Consequently, the condition that \( S'_\bullet \) is a summand of \( S_\bullet \) is equivalent to the condition that the construction \( (|n| \in \Delta^{\text{op}}) \mapsto S_n \setminus S'_n \) is functorial: that is, that the face and degeneracy operators for the simplicial set \( S_\bullet \) preserve the subsets \( S_n \setminus S'_n \).

Remark 1.1.6.3. Let \( S_\bullet \) be a simplicial set. Then the collection of all summands of \( S_\bullet \) is closed under the formation of unions and intersections (this follows immediately from the criterion of Remark 1.1.6.2).

Remark 1.1.6.4 (Transitivity). Let \( S_\bullet \) be a simplicial set. If \( S'_\bullet \subseteq S_\bullet \) is a summand of \( S_\bullet \) and \( S''_\bullet \subseteq S'_\bullet \) is a summand of \( S'_\bullet \), then \( S''_\bullet \) is a summand of \( S_\bullet \).

Remark 1.1.6.5. Let \( f : S_\bullet \to T_\bullet \) be a map of simplicial sets and let \( T'_\bullet \subseteq T_\bullet \) be a summand. Then the inverse image \( f^{-1}(T'_\bullet) \simeq S_\bullet \times_{T_\bullet} T'_\bullet \) is a summand of \( S_\bullet \).

Definition 1.1.6.6. Let \( S_\bullet \) be a simplicial set. We will say that \( S_\bullet \) is connected if it is nonempty and every summand \( S'_\bullet \subseteq S_\bullet \) is either empty or coincides with \( S_\bullet \).

Example 1.1.6.7. For each \( n \geq 0 \), the standard \( n \)-simplex \( \Delta^n \) is connected.
Definition 1.1.6.8 (Connected Components). Let $S_\bullet$ be a simplicial set. We will say that a simplicial subset $S'_\bullet \subseteq S_\bullet$ is a connected component of $S_\bullet$ if $S'_\bullet$ is a summand of $S_\bullet$ (Definition 1.1.6.1) and $S'_\bullet$ is connected (Definition 1.1.6.6). We let $\pi_0(S_\bullet)$ denote the set of all connected components of $S_\bullet$.

Warning 1.1.6.9. Let $S_\bullet$ be a simplicial set. As we will soon see, the set $\pi_0(S_\bullet)$ admits many different descriptions:

- We can identify $\pi_0(S_\bullet)$ with the set of connected components of $S_\bullet$ (Definition 1.1.6.8).
- We can identify $\pi_0(S_\bullet)$ with a colimit of the diagram $\Delta^{op} \to \text{Set}_\Delta$ given by the simplicial set $S_\bullet$ (Remark 1.1.6.20).
- We can identify $\pi_0(S_\bullet)$ with the quotient of the set of vertices $S_0$ by an equivalence relation $\sim$ generated by the set of edges $S_1$ (Remark 1.1.6.23).
- We can identify $\pi_0(S_\bullet)$ with the set of connected components of the directed graph $\text{Gr}(S_\bullet)$ (Variant 1.1.6.24).
- When $S_\bullet$ is a Kan complex, we can identify $\pi_0(S_\bullet)$ as the set of isomorphism classes of objects in the fundamental groupoid $\pi_{\leq 1}(S_\bullet)$ (Remark 1.3.6.14).

Because of this abundance of perspectives, it often will be convenient to view $I = \pi_0(S_\bullet)$ as an abstract index set which is equipped with a bijection

$$I \simeq \{\text{Connected components of } S_\bullet\} \quad (i \in I) \mapsto (S'_\bullet \subseteq S_\bullet),$$

rather than as the set of connected components itself.

Example 1.1.6.10. Let $I$ be a set and let $I_\bullet$ be the constant simplicial set associated to $I$ (Construction 1.1.4.2). Then the connected components of $I_\bullet$ are exactly the simplicial subsets of the form $\{i\} = \{i\}$ for $i \in I$. In particular, we have a canonical bijection $I \simeq \pi_0(I_\bullet)$.

Proposition 1.1.6.11. Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets, and suppose that $S_\bullet$ is connected. Then there is a unique connected component $T'_\bullet \subseteq T_\bullet$ such that $f(S_\bullet) \subseteq T'_\bullet$.

Proof. Let $T'_\bullet$ be the smallest summand of $T_\bullet$ which contains the image of $f$ (the existence of $T'_\bullet$ follows from Remark 1.1.6.3). We can take $T'_\bullet$ to be the intersection of all those summands of $T_\bullet$ which contain the image of $f$. We will complete the proof by showing that $T'_\bullet$ is connected. Since $S_\bullet$ is nonempty, $T'_\bullet$ must be nonempty. Let $T''_\bullet \subseteq T'_\bullet$ be a summand; we wish to show that $T''_\bullet = T'_\bullet$ or $T''_\bullet = \emptyset$. Note that $f^{-1}(T''_\bullet)$ is a summand of $S_\bullet$ (Remark 1.1.6.5). Since $S_\bullet$ is connected, we must have $f^{-1}(T''_\bullet) = S_\bullet$ or $f^{-1}(T''_\bullet) = \emptyset$. Replacing $T''_\bullet$ by its complement if necessary, we may assume that $f^{-1}(T''_\bullet) = S_\bullet$, so that $f$ factors through $T''_\bullet$. Since $T''_\bullet$ is a summand of $T_\bullet$ (Remark 1.1.6.4), the minimality of $T'_\bullet$ guarantees that $T''_\bullet = T'_\bullet$, as desired. \qed
Corollary 1.1.6.12. Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:

(a) The simplicial set $S_\bullet$ is connected.

(b) For every set $I$, the canonical map

$$I \simeq \text{Hom}_{\text{Set}_\Delta}(\Delta^0, I_\bullet) \to \text{Hom}_{\text{Set}_\Delta}(S_\bullet, I_\bullet)$$

is bijective.

Proof. The implication $(a) \Rightarrow (b)$ follows from Proposition 1.1.6.11 and Example 1.1.6.10. Conversely, suppose that $(b)$ is satisfied. Applying $(b)$ in the case $I = \emptyset$, we conclude that there are no maps from $S_\bullet$ to the empty simplicial set, so that $S_\bullet$ is nonempty. If $S_\bullet$ is a disjoint union of simplicial subsets $S'_\bullet, S''_\bullet \subseteq S_\bullet$, then we obtain a map of simplicial sets

$$S_\bullet \simeq S'_\bullet \coprod S''_\bullet \to \Delta^0 \coprod \Delta^0$$

and assumption $(b)$ guarantees that this map factors through one of the summands on the right hand side; it follows that either $S'_\bullet$ or $S''_\bullet$ is empty.

Proposition 1.1.6.13. Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is the disjoint union of its connected components.

Proof. Let $\sigma$ be an $n$-simplex of $S_\bullet$; we wish to show that there is a unique connected component of $S_\bullet$ which contains $\sigma$. This follows from Proposition 1.1.6.11 applied to the map $\Delta^n \to S_\bullet$ classified by $\sigma$ (since the standard $n$-simplex $\Delta^n$ is connected; see Example 1.1.6.7).

Corollary 1.1.6.14. Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is empty if and only if $\pi_0(S_\bullet)$ is empty.

Corollary 1.1.6.15. Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is connected if and only if $\pi_0(S_\bullet)$ has exactly one element.

Exercise 1.1.6.16 (Classification of Summands). Let $S_\bullet$ be a simplicial set. Show that a simplicial subset $S'_\bullet \subseteq S_\bullet$ is a summand if and only if it can be written as a union of connected components of $S_\bullet$. Consequently, we have a canonical bijection

$$\{\text{Subsets of } \pi_0(S_\bullet)\} \simeq \{\text{Summands of } S_\bullet\}.$$ 

Remark 1.1.6.17 (Functoriality of $\pi_0$). Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets. It follows from Proposition 1.1.6.11 that for each connected component $S'_\bullet \subseteq S_\bullet$, there is a unique connected component $T'_\bullet \subseteq T_\bullet$ such that $f(S'_\bullet) \subseteq T'_\bullet$. The construction $S'_\bullet \mapsto T'_\bullet$ then determines a map of sets $\pi_0(f) : \pi_0(S_\bullet) \to \pi_0(T_\bullet)$. This construction is compatible with composition, and therefore allows us to view the construction $S_\bullet \mapsto \pi_0(S_\bullet)$ as a functor $\pi_0 : \text{Set}_\Delta \to \text{Set}$ from the category of simplicial sets to the category of sets.
CHAPTER 1. THE LANGUAGE OF $\infty$-CATEGORIES

We now show that the connected component functor $\pi_0 : \text{Set}_\Delta \to \text{Set}$ can be characterized by a universal property.

**Construction 1.1.6.18 (The Component Map).** Let $S_\bullet$ be a simplicial set. For every $n$-simplex $\sigma$ of $S_\bullet$, Proposition 1.1.6.13 implies that there is a unique connected component $S'_\bullet \subseteq S_\bullet$ which contains $\sigma$. The construction $\sigma \mapsto S'_\bullet$ then determines a map of simplicial sets

$$u : S_\bullet \to \pi_0(S_\bullet),$$

where $\pi_0(S_\bullet)$ denotes the constant simplicial set associated to $\pi_0(S_\bullet)$ (Construction 1.1.4.2). We will refer to $u$ as the component map.

**Proposition 1.1.6.19.** Let $S_\bullet$ be a simplicial set and let $u : S_\bullet \to \pi_0(S_\bullet)$ be the component map of Construction 1.1.6.18. For every set $J$, composition with $u$ induces a bijection

$$\text{Hom}_{\text{Set}}(\pi_0(S_\bullet), J) \to \text{Hom}_{\text{Set}}(S_\bullet, J_\bullet).$$

**Proof.** Decomposing $S_\bullet$ as the union of its connected components, we can reduce to the case where $S_\bullet$ is connected, in which case the desired result is a reformulation of Corollary 1.1.6.12. $\square$

**Remark 1.1.6.20 ($\pi_0$ as a Colimit).** Let $S_\bullet$ be a simplicial set. It follows from Proposition 1.1.6.19 that the component map $u : S_\bullet \to \pi_0(S_\bullet)$ exhibits $\pi_0(S_\bullet)$ as the colimit of the diagram $\Delta^{\text{op}} \to \text{Set}$ determined by $S_\bullet$.

**Corollary 1.1.6.21.** The connected component functor

$$\pi_0 : \text{Set}_\Delta \to \text{Set} \quad S_\bullet \mapsto \pi_0(S_\bullet)$$

of Remark 1.1.6.17 is left adjoint to the constant simplicial set functor

$$\text{Set} \to \text{Set}_\Delta \quad I \mapsto I_\bullet$$

of Construction 1.1.4.2. More precisely, the construction $S_\bullet \mapsto (u : S_\bullet \to \pi_0(S_\bullet))$ is the unit of an adjunction.

We now make Remark 1.1.6.20 more concrete.

**Proposition 1.1.6.22.** Let $S_\bullet$ be a simplicial set, and let $u_0 : S_0 \to \pi_0(S_\bullet)$ be the map of sets given by the component map of Construction 1.1.6.18. Then $u_0$ exhibits $\pi_0(S_\bullet)$ as the coequalizer of the face maps $d_0, d_1 : S_1 \rightrightarrows S_0$. 

Remark 1.1.6.23. Let $S_\bullet$ be a simplicial set. Proposition 1.1.6.22 supplies a coequalizer diagram of sets

$$S_1 \xrightarrow{d_0, d_1} S_0 \twoheadrightarrow \pi_0(S_\bullet).$$

In other words, it allows us to identify $\pi_0(S_\bullet)$ with the quotient of $S_0/\sim$, where $\sim$ is the equivalence relation generated by the set of edges of $S_\bullet$ (that is, the smallest equivalence relation with the property that $d_0(e) \sim d_1(e)$, for every edge $e \in S_1$). In particular, the set $\pi_0(S_\bullet)$ depends only on the 1-skeleton of $S_\bullet$.

Variant 1.1.6.24. Let $S_\bullet$ be a simplicial set. Then the set of connected components $\pi_0(S_\bullet)$ can also be described as the coequalizer of the pair of maps $d_0, d_1 : S_1^{nd} \to S_0$, where $S_1^{nd} \subseteq S_1$ denotes the set of nondegenerate edges of $S_\bullet$ (since every degenerate edge $e \in S_1$ automatically satisfies $d_0(e) = d_1(e)$). We therefore have a coequalizer diagram of sets

$$\text{Edge}(G) \xrightarrow{a, b} \text{Vert}(G) \twoheadrightarrow \pi_0(S_\bullet),$$

where $G = \text{Gr}(S_\bullet)$ is the directed graph of Example 1.1.5.4. In other words, we can identify $\pi_0(S_\bullet)$ with the set of connected components of $G$, in the usual graph-theoretic sense.

Proof of Proposition 1.1.6.22. Let $I$ be a set and let $f : S_0 \to I$ be a function satisfying $f \circ d_0 = f \circ d_1$ (as functions from $S_1$ to $I$). We wish to show that $f$ factors uniquely as a composition

$$S_0 \xrightarrow{u_0} \pi_0(S_\bullet) \to I.$$

By virtue of Proposition 1.1.6.19, this is equivalent to the assertion that there is a unique map of simplicial sets $F : S_\bullet \to I_\bullet$ which coincides with $f$ on simplices of degree zero. Let $\sigma$ be an $n$-simplex of $S_\bullet$, which we identify with a map of simplicial sets $\sigma : \Delta^n \to S_\bullet$. For $0 \leq i \leq n$, we regard $\sigma(i)$ as a vertex of $S_\bullet$. Note that if $0 \leq i \leq j \leq n$, then we have $f(\sigma(i)) = f(\sigma(j))$: to prove this, we can assume without loss of generality that $i = 0$ and $j = n = 1$, in which case it follows from our hypothesis that $f \circ d_0 = f \circ d_1$. It follows that there is a unique element $F(\sigma) \in I$ such that $F(\sigma) = f(\sigma(i))$ for each $0 \leq i \leq n$. The construction $\sigma \mapsto F(\sigma)$ defines a map of simplicial sets $F : S_\bullet \to I_\bullet$ with the desired properties.

Proposition 1.1.6.25. The collection of connected simplicial sets is closed under finite products.

Proof. Since the final object $\Delta^0 \in \text{Set}_\Delta$ is connected (Example 1.1.6.7), it will suffice to show that the collection of connected simplicial sets is closed under pairwise products. Let $S_\bullet$ and $T_\bullet$ be connected simplicial sets; we wish to show that $S_\bullet \times T_\bullet$ is connected. Equivalently, we
wish to show that \( \pi_0(S_\bullet \times T_\bullet) \) consists of a single element (Corollary 1.1.6.15). By virtue of Proposition 1.1.6.22, the component map supplies a surjection

\[ u_0 : S_0 \times T_0 \to \pi_0(S_\bullet \times T_\bullet). \]

It will therefore suffice to show that for every pair of vertices \((s, t), (s', t') \in S_0 \times T_0\) belong to the same connected component of \(S_\bullet \times T_\bullet\). Let \(K_\bullet \subseteq S_\bullet \times T_\bullet\) be the connected component which contains the vertex \((s', t)\). Since \(S_\bullet\) is connected, the map

\[ S_\bullet \cong S_\bullet \times \{t\} \hookrightarrow S_\bullet \times T_\bullet \]

factors through a unique connected component of \(S_\bullet \times T_\bullet\), which must be equal to \(K_\bullet\). It follows that \(K_\bullet\) contains the vertex \((s, t)\). A similar argument (with the roles of \(S_\bullet\) and \(T_\bullet\) reversed) shows that \(K_\bullet\) contains \((s', t')\).

\[ \square \]

**Corollary 1.1.6.26.** The functor \(\pi_0 : \text{Set}_\Delta \to \text{Set}\) preserves finite products.

**Proof.** Since \(\pi_0(\Delta^0)\) is a singleton (Example 1.1.6.7), it will suffice to show that for every pair of simplicial sets \(S_\bullet\) and \(T_\bullet\), the canonical map

\[ \pi_0(S_\bullet \times T_\bullet) \to \pi_0(S_\bullet) \times \pi_0(T_\bullet) \]

is bijective. Writing \(S_\bullet\) and \(T_\bullet\) as a disjoint union of connected components (Proposition 1.1.6.13), we can reduce to the case where \(S_\bullet\) and \(T_\bullet\) are connected, in which case the desired result follows from Proposition 1.1.6.25.

\[ \square \]

**Warning 1.1.6.27.** The collection of connected simplicial sets is not closed under infinite products (so the functor \(\pi_0 : \text{Set}_\Delta \to \text{Set}\) does not commute with infinite products). For example, let \(G\) be the directed graph with vertex set \(\text{Vert}(G) = \mathbb{Z}_{\geq 0} = \text{Edge}(G)\), with source and target maps

\[ s, t : \text{Edge}(G) \to \text{Vert}(G) \quad s(n) = n \quad t(n) = n + 1. \]

More informally, \(G\) is the directed graph depicted in the diagram

```
0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots
```

The associated 1-dimensional simplicial set \(G_\bullet\) is connected. However, the infinite product \(S_\bullet = \prod_{n \in \mathbb{Z}_{\geq 0}} G_\bullet\) is not connected. By definition, the vertices of \(S_\bullet\) can be identified with functions \(f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}\). It is not difficult to see that two such functions \(f, g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}\) belong to the same connected component of \(S_\bullet\) if and only if the function \(n \mapsto |f(n) - g(n)|\) is bounded. In particular, the identity function \(n \mapsto n\) and the zero function \(n \mapsto 0\) do not belong to the same connected component of \(S_\bullet\).
1.1. SIMPLICIAL SETS

1.1.7 The Singular Simplicial Set of a Topological Space

Topology provides an abundant supply of examples of simplicial sets.

**Construction 1.1.7.1.** Let $X$ be a topological space. We define a simplicial set $\text{Sing}_\bullet(X)$ as follows:

- To each object $[n] \in \Delta$, we assign the set $\text{Sing}_n(X) = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$ of singular $n$-simplices in $X$.

- To each non-decreasing map $\alpha : [m] \to [n]$, we assign the map $\text{Sing}_n(X) \to \text{Sing}_m(X)$ given by precomposition with the continuous map $|\Delta^m| \to |\Delta^n|$

$$(t_0, t_1, \ldots, t_m) \mapsto (\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, \ldots, \sum_{\alpha(i)=n} t_i).$$

We will refer to $\text{Sing}_\bullet(X)$ as the *singular simplicial set of* $X$. We view the construction $X \mapsto \text{Sing}_\bullet(X)$ as a functor from the category of topological spaces to the category of simplicial sets, which we will denote by $\text{Sing}_\bullet : \text{Top} \to \text{Set}_\Delta$.

**Example 1.1.7.2.** Let $X$ be a topological space and let $\text{Sing}_\bullet(X)$ be its singular simplicial set. The vertices of $\text{Sing}_\bullet(X)$ can be identified with points of $X$. The edges of $\text{Sing}_\bullet(X)$ can be identified with continuous paths $p : [0, 1] \to X$.

**Remark 1.1.7.3** (Connected Components of $\text{Sing}_\bullet(X)$). Let $X$ be a topological space. We let $\pi_0(X)$ denote the set of *path components* of $X$: that is, the quotient of $X$ by the equivalence relation

$$(x \sim y) \iff (\exists p : [0, 1] \to X) [p(0) = x \text{ and } p(1) = y].$$

It follows from Remark [1.1.6.23] that we have a canonical bijection $\pi_0(\text{Sing}_\bullet(X)) \simeq \pi_0(X)$. That is, we can identify connected components of the simplicial set $\text{Sing}_\bullet(X)$ (in the sense of Definition [1.1.6.8]) with path components of the topological space $X$.

**Remark 1.1.7.4** (Connectedness of $\text{Sing}_\bullet(X)$). Let $X$ be a topological space. Then the simplicial set $\text{Sing}_\bullet(X)$ is connected if and only if $X$ is path connected (this follows from Remark [1.1.7.3]).

**Warning 1.1.7.5.** Let $X$ be a topological space. If the simplicial set $\text{Sing}_\bullet(X)$ is connected, then the topological space $X$ is path connected and therefore connected. Beware that the converse is not necessarily true: there exist topological spaces $X$ which are connected but not path connected, in which case the singular simplicial set $\text{Sing}_\bullet(X)$ will not be connected.
It will be convenient to consider a generalization of Construction 1.1.7.1.

**Variant 1.1.7.6.** Let $C$ be any category and let $Q^\bullet$ be a cosimplicial object of $C$, which we view as a functor $Q : \Delta \to C$. For every object $X \in C$, the construction $([n] \in \Delta) \mapsto \text{Hom}_C(Q([n]), X)$ determines a functor from $\Delta^{\text{op}}$ to the category of sets, which we can view as a simplicial set. We will denote this simplicial set by $\text{Sing}_C^Q(X)$, so that we have canonical bijections $\text{Sing}_C^Q([n])(X) \simeq \text{Hom}_C(Q^n, X)$. We view the construction $X \mapsto \text{Sing}_C^Q(X)$ as a functor from the category $C$ to the category of simplicial sets, which we denote by $\text{Sing}_C^Q : C \to \text{Set}_\Delta$.

**Example 1.1.7.7.** The construction $[n] \mapsto |\Delta^n|$ determines a functor from the simplex category $\Delta$ to the category $\text{Top}$ of topological spaces, which assigns to each morphism $\alpha : [m] \to [n]$ the continuous map

$$|\Delta^m| \to |\Delta^n| \quad (t_0, \ldots, t_m) \mapsto \left( \sum_{\alpha(i)=0} t_i, \ldots, \sum_{\alpha(i)=n} t_i \right).$$

We regard this functor as a cosimplicial topological space, which we denote by $|\Delta^\bullet|$. Applying Variant 1.1.7.6 to this cosimplicial space yields a functor $\text{Sing}_\text{Top}^{|\Delta^\bullet|} : \text{Top} \to \text{Set}_\Delta$, which coincides with the singular simplicial set functor $\text{Sing}_\bullet$ of Construction 1.1.7.1.

**Example 1.1.7.8.** The construction $[n] \mapsto \Delta^n$ determines a functor from the simplex category $\Delta$ to the category $\text{Set}_\Delta = \text{Fun}(\Delta^{\text{op}}, \text{Set})$ of simplicial sets (this is the Yoneda embedding for the simplex category $\Delta$). We regard this functor as a cosimplicial object of $\text{Set}_\Delta$, which we denote by $\Delta^\bullet$. Applying Variant 1.1.7.6 to this cosimplicial object, we obtain a functor from the category of simplicial sets to itself, which is canonically isomorphic to the identity functor $\text{id}_{\text{Set}_\Delta} : \text{Set}_\Delta \to \text{Set}_\Delta$ (see Remark 1.1.2.3).

**Remark 1.1.7.9.** The cosimplicial space $|\Delta^\bullet|$ of Example 1.1.7.7 can be described more informally as follows:

- To each nonempty finite linearly ordered set $I$, it assigns a topological simplex $|\Delta^I|$ whose vertices are the elements of $I$: that is, the convex hull of the set $I$ inside the real vector space $\mathbb{R}[I]$ generated by $I$.

- To every nondecreasing map $\alpha : I \to J$, the induced map $|\Delta^I| \to |\Delta^J|$ is given by the restriction of the $\mathbb{R}$-linear map $\mathbb{R}[I] \to \mathbb{R}[J]$ determined by $\alpha$. Equivalently, it is the unique affine map which coincides with $\alpha$ on the vertices of the simplex $|\Delta^I|$.

### 1.1.8 The Geometric Realization of a Simplicial Set

Let $X$ be a topological space. By definition, $n$-simplices of the simplicial set $\text{Sing}_\bullet(X)$ are continuous maps $|\Delta^n| \to X$. This observation determines a bijection

$$\text{Hom}_{\text{Top}}(|\Delta^n|, X) \simeq \text{Hom}_{\text{Set}_\Delta}(\Delta^n, \text{Sing}_\bullet(X)).$$
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We now consider a generalization of this construction, which can be applied to simplicial sets other than $\Delta^n$.

001Y **Definition 1.1.8.1.** Let $S_\bullet$ be a simplicial set and let $Y$ be a topological space. We will say that a map of simplicial sets $u : S_\bullet \to \text{Sing}_\bullet(Y)$ exhibits $Y$ as a geometric realization of $S_\bullet$ if, for every topological space $X$, the composite map

$$\text{Hom}_{\text{Top}}(Y, X) \to \text{Hom}_{\text{Set}_\Delta}(\text{Sing}_\bullet(Y), \text{Sing}_\bullet(X)) \xrightarrow{\text{out}} \text{Hom}_{\text{Set}_\Delta}(S_\bullet, \text{Sing}_\bullet(X))$$

is bijective.

001Z **Example 1.1.8.2.** For each $n \geq 0$, the identity map $1 : |\Delta^n| \to |\Delta^n|$ determines an $n$-simplex of the simplicial set $\text{Sing}_\bullet(|\Delta^n|)$, which we can identify with a map of simplicial sets $\Delta^n \to \text{Sing}_\bullet(|\Delta^n|)$ which exhibits $|\Delta^n|$ as a geometric realization of $\Delta^n$.

0020 **Notation 1.1.8.3.** Let $S_\bullet$ be a simplicial set. It follows immediately from the definitions that if there exists a map $u : S_\bullet \to \text{Sing}_\bullet(Y)$ which exhibits $Y$ as a geometric realization of $S_\bullet$, then the topological space $Y$ is determined up to homeomorphism and depends functorially on $S_\bullet$. We will emphasize this dependence by writing $|S_\bullet|$ to denote a geometric realization of $S_\bullet$ (by virtue of Example 1.1.8.2, this is compatible with our existing notation in the case where $S_\bullet$ is the standard $n$-simplex).

Every simplicial set admits a geometric realization:

0021 **Proposition 1.1.8.4.** For every simplicial set $S_\bullet$, there exists a topological space $Y$ and a map $u : S_\bullet \to \text{Sing}_\bullet(Y)$ which exhibits $Y$ as a geometric realization of $S_\bullet$.

0022 **Corollary 1.1.8.5.** The singular simplicial set functor $\text{Sing}_\bullet : \text{Top} \to \text{Set}_\Delta$ admits a left adjoint, given by the geometric realization construction $S_\bullet \mapsto |S_\bullet|$.

Our starting point is the following formal observation:

0023 **Lemma 1.1.8.6.** Let $\mathcal{J}$ be a small category equipped with a functor $F : \mathcal{J} \to \text{Set}_\Delta$, which we will denote by $(J \in \mathcal{J}) \mapsto (F(J)_\bullet \in \text{Set}_\Delta)$. Let $S_\bullet = \lim_{J \in \mathcal{J}} F(J)_\bullet$ be a colimit of $F$. If each of the simplicial sets $F(J)_\bullet$ admits a geometric realization $|F(J)_\bullet|$, then $S_\bullet$ also admits a geometric realization, given by the colimit $Y = \lim_{J \in \mathcal{J}} |F(J)_\bullet|$.

**Proof.** For each $J \in \mathcal{J}$, choose a map $u_J : F(J)_\bullet \to \text{Sing}_\bullet(|F(J)_\bullet|)$ which exhibits $|F(J)_\bullet|$ as a geometric realization of $F(J)_\bullet$. We can then amalgamate the composite maps

$$F(I)_\bullet \xrightarrow{u_I} \text{Sing}_\bullet(|F(I)_\bullet|) \to \text{Sing}_\bullet(Y)$$

to a single map of simplicial sets $u : S_\bullet \to \text{Sing}_\bullet(Y)$. We claim that $u$ exhibits $Y$ as a geometric realization of the simplicial set $S_\bullet$. Let $X$ be any topological space; we wish to show that the composite map

$$\text{Hom}_{\text{Top}}(Y, X) \to \text{Hom}_{\text{Set}_\Delta}(\text{Sing}_\bullet(Y), \text{Sing}_\bullet(X)) \xrightarrow{\text{out}} \text{Hom}_{\text{Set}_\Delta}(S_\bullet, \text{Sing}_\bullet(X))$$
is bijective. This is clear, since this composite map can be written as an inverse limit of the bijections \( \text{Hom}_{\text{Top}}(\mathcal{F}(J)_\bullet, X) \simeq \text{Hom}_{\text{Set}_\Delta}(\mathcal{F}(J)_\bullet, \text{Sing}_\bullet(X)) \) determined by \( u_J \).

It is possible to deduce Proposition 1.1.8.4 and Corollary 1.1.8.5 in a completely formal way from Lemma 1.1.8.6, since every simplicial set can be presented as a colimit of simplices (see Proposition 1.1.8.22 below). However, we will instead give a less direct argument which yields some additional information about the structure of the topological spaces \( |S_\bullet| \). We begin by studying simplicial subsets of the standard simplex \( \Delta^n \).

**Notation 1.1.8.7.** Let \( n \geq 0 \) be an integer and let \( \mathcal{U} \) be a collection of nonempty subsets of \([n] = \{0, 1, \ldots, n\}\). We will say that \( \mathcal{U} \) is downward closed if \( \emptyset \neq I \subseteq J \in \mathcal{U} \) implies that \( I \in \mathcal{U} \). If this condition is satisfied, we let \( \Delta^n_\mathcal{U} \) denote the simplicial subset of \( \Delta^n \) whose \( m \)-simplices are nondecreasing maps \( \alpha : [m] \to [n] \) for which the image of \( \alpha \) is an element of \( \mathcal{U} \). Similarly, we set \( |\Delta^n_\mathcal{U}| = \{ (t_0, \ldots, t_n) \in |\Delta^n| : \{ i \in [n] : t_i \neq 0 \} \in \mathcal{U} \} \).

**Example 1.1.8.8.** For each \( n \geq 0 \), the boundary \( \partial \Delta^n \) of Construction 1.1.2.6 is given by \( \Delta^n_\mathcal{U} \), where \( \mathcal{U} \) is the collection of all nonempty proper subsets of \([n]\).

**Example 1.1.8.9.** For \( 0 \leq i \leq n \), the horn \( \Lambda^n_i \) of Construction 1.1.2.9 is given by \( \Delta^n_\mathcal{U} \), where \( \mathcal{U} \) is the collection of all nonempty subsets of \([n]\) which are distinct from \([n]\) and \([n] \setminus \{i\}\).

**Exercise 1.1.8.10.** Show that every simplicial subset of the standard \( n \)-simplex \( \Delta^n \) has the form \( \Delta^n_\mathcal{U} \), where \( \mathcal{U} \) is some (uniquely determined) downward closed collection of nonempty subsets of \([n]\).

**Proposition 1.1.8.11.** Let \( n \) be a nonnegative integer and let \( \mathcal{U} \) be a downward closed collection of nonempty subsets of \([n]\). Then the canonical map \( \Delta^n \to \text{Sing}_\bullet(|\Delta^n|) \) restricts to a map of simplicial sets \( f_\mathcal{U} : \Delta^n_\mathcal{U} \to \text{Sing}_\bullet(|\Delta^n|_\mathcal{U}) \), which exhibits the topological space \( |\Delta^n|_\mathcal{U} \) as a geometric realization of \( \Delta^n_\mathcal{U} \).

**Proof.** We proceed by induction on the cardinality of \( \mathcal{U} \). If \( \mathcal{U} \) is empty, then the simplicial set \( \Delta^n_\mathcal{U} \) and the topological space \( |\Delta^n|_\mathcal{U} \) are both empty, in which case there is nothing to prove. We may therefore assume that \( \mathcal{U} \) is nonempty. Choose some \( S \in \mathcal{U} \) whose cardinality is as large as possible. Set

\[ \mathcal{U}_0 = \mathcal{U} \setminus \{S\} \quad \mathcal{U}_1 = \{T \subseteq S : T \neq \emptyset\} \quad \mathcal{U}_{01} = \mathcal{U}_0 \cap \mathcal{U}_1. \]

Our inductive hypothesis implies that the maps \( f_{\mathcal{U}_0} \) and \( f_{\mathcal{U}_{01}} \) exhibit \( |\Delta^n|_{\mathcal{U}_0} \) and \( |\Delta^n|_{\mathcal{U}_{01}} \) as geometric realizations of \( \Delta^n_{\mathcal{U}_0} \) and \( \Delta^n_{\mathcal{U}_{01}} \), respectively. Moreover, if \( S = \{ i_0 < i_1 < \cdots < i_m \} \subseteq [n] \), then we can identify \( f_{\mathcal{U}_1} \) with the tautological map \( \Delta^m \to \text{Sing}_\bullet(|\Delta^m|) \), so that
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$f_{t_1}$ exhibits $|\Delta^n|_{t_1}$ as a geometric realization of $\Delta^n_{t_1}$ by virtue of Example 1.1.8.2. It follows immediately from the definitions that the diagram of simplicial sets

$$
\begin{array}{ccc}
\Delta^n_{t_01} & \longrightarrow & \Delta^n_{t_0} \\
\downarrow & & \downarrow \\
\Delta^n_{t_1} & \longrightarrow & \Delta^n_t
\end{array}
$$

is a pushout square. By virtue of Lemma 1.1.8.6, we are reduced to proving that the diagram of topological spaces

$$
\begin{array}{ccc}
|\Delta^n|_{t_01} & \longrightarrow & |\Delta^n|_{t_0} \\
\downarrow & & \downarrow \\
|\Delta^n|_{t_1} & \longrightarrow & |\Delta^n|_t
\end{array}
$$

is also a pushout square. This is clear, since $|\Delta^n|_{t_0}$ and $|\Delta^n|_{t_1}$ are closed subsets of $|\Delta^n|$ whose union is $|\Delta^n|_t$ and whose intersection is $|\Delta^n|_{t_01}$.

Example 1.1.8.12. Let $n$ be a nonnegative integer. Combining Example 1.1.8.8 with Proposition 1.1.8.11, we see that the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ induces a homeomorphism from $|\partial \Delta^n|$ to the boundary of the topological $n$-simplex $|\Delta^n|$, given by

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j\}.$$

Example 1.1.8.13. Let $0 \leq i \leq n$. Combining Example 1.1.8.9 with Proposition 1.1.8.11, we see that the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ induces a homeomorphism from $|\Lambda^n_i|$ to the subset of $|\Delta^n|$ given by

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\}.$$

Proof of Proposition 1.1.8.4. Let $S_\bullet$ be a simplicial set. We first show that for each $n \geq -1$, the skeleton $\text{sk}_n(S_\bullet)$ admits a geometric realization. The proof proceeds by induction on $n$, the case $n = -1$ being trivial (since $\text{sk}_{-1}(S_\bullet)$ is empty). Let $S_n^{nd}$ denote the collection of nondegenerate $n$-simplices of $S_\bullet$. We note that Proposition 1.1.3.13 provides a pushout diagram

$$
\begin{array}{ccc}
\coprod_{\sigma \in S_n^{nd}} \partial \Delta^n & \longrightarrow & \coprod_{\sigma \in S_n^{nd}} \Delta^n \\
\downarrow & & \downarrow \\
\text{sk}_{n-1}(S_\bullet) & \longrightarrow & \text{sk}_n(S_\bullet).
\end{array}
$$

Combining our inductive hypothesis, Example 1.1.8.2, Example 1.1.8.12, and Lemma 1.1.8.6, we deduce that $\text{sk}_n(S_\bullet)$ admits a geometric realization $|\text{sk}_n(S_\bullet)|$ which fits into a pushout.
Combining the equality $S_n = \bigcup_n \text{sk}_n(S_\bullet)$ of Remark 1.1.3.6 with Lemma 1.1.8.6, we deduce that the simplicial set $S_\bullet$ also admits a geometric realization, given by the direct limit $\lim_{\rightarrow n} |\text{sk}_n(S_\bullet)|$.

\[ \bigl( \bigcup_{\sigma \in S_n} |\partial \Delta^n| \bigr) \longrightarrow \bigl( \bigcup_{\sigma \in S_n} |\Delta^n| \bigr) \]

\[ \downarrow \quad \downarrow \]

\[ |\text{sk}_{n-1}(S_\bullet)| \longrightarrow |\text{sk}_n(S_\bullet)|. \]

**Remark 1.1.8.14.** The proof of Proposition 1.1.8.4 shows that the geometric realization $|S_\bullet|$ of a simplicial set $S_\bullet$ has a canonical realization as a CW complex, having one cell of dimension $n$ for each nondegenerate $n$-simplex $\sigma$ of $S_\bullet$; this cell can be described explicitly as the image of the map

\[ |\Delta^n| \setminus |\partial \Delta^n| \hookrightarrow |\Delta^n| \xrightarrow{\sigma} |S_\bullet|. \]

The proof of Proposition 1.1.8.4 also yields the following fact, which we will use repeatedly throughout this book:

**Lemma 1.1.8.15.** Let $\mathcal{U}$ be a full subcategory of the category $\text{Set}_\Delta$ of simplicial sets. Suppose $\mathcal{U}$ satisfies the following three conditions:

1. Suppose we are given a pushout diagram of simplicial sets

\[ \begin{array}{ccc}
X_\bullet & \xrightarrow{f} & Y_\bullet \\
\downarrow & & \downarrow \\
X'_\bullet & \longrightarrow & Y'_\bullet
\end{array} \]

where $f$ is a monomorphism. If $X_\bullet$, $Y_\bullet$, and $X'_\bullet$ belong to $\mathcal{U}$, then $Y'_\bullet$ belongs to $\mathcal{U}$.

2. Suppose we are given a sequence of monomorphisms of simplicial sets

\[ X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow X(3) \hookrightarrow \cdots \]

If each $X(m)\bullet$ belongs to $\mathcal{U}$, then the sequential colimit $\lim_{\rightarrow m} X(m)\bullet$ belongs to $\mathcal{U}$.

3. For each $n \geq 0$ and every set $I$, the coproduct $\coprod_{i \in I} \Delta^n$ belongs to $\mathcal{U}$.

Then every simplicial set belongs to $\mathcal{U}$. 
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Proof. Set $S_\bullet$ be a simplicial set; we wish to show that $S_\bullet$ belongs to $\mathcal{U}$. By virtue of Remark 1.1.3.6, we can identify $S_\bullet$ with the colimit $\lim_{\to n} sk_n(S_\bullet)$. By virtue of (2), it will suffice to show that each skeleton $sk_n(S_\bullet)$ belongs to $\mathcal{U}$. We may therefore assume without loss of generality that $S_\bullet$ has dimension $\leq n$, for some integer $n$. We proceed by induction on $n$. In the case $n = -1$, the simplicial set $S_\bullet$ is empty, and the desired result is a special case of (3). To carry out the inductive step, we invoke Proposition 1.1.3.13 to choose a pushout diagram

$$
\begin{array}{ccc}
\coprod_{\sigma \in S_n} \partial \Delta^n & \longrightarrow & \coprod_{\sigma \in S_n} \Delta^n \\
\downarrow & & \downarrow \\
\text{sk}_{n-1}(S_\bullet) & \longrightarrow & S_\bullet.
\end{array}
$$

By virtue of assumption (1), it will suffice to show that the simplicial sets $\text{sk}_{n-1}(S_\bullet)$, $\coprod_{\sigma \in S_n} \partial \Delta^n$, and $\coprod_{\sigma \in S_n} \Delta^n$ belong to $\mathcal{U}$. In the first two cases, this follows from our inductive hypothesis. In the third, it follows from assumption (3). \hfill \square

Remark 1.1.8.16. In the statement of Lemma 1.1.8.15, we can replace (3) by the following pair of conditions:

(3') For each $n \geq 0$, the standard $n$-simplex $\Delta^n$ belongs to $\mathcal{U}$.

(3'') The subcategory $\mathcal{U} \subseteq \text{Set}_\Delta$ is closed under the formation of coproducts.

Corollary 1.1.8.17. Let $\mathcal{U}$ be a full subcategory of the category $\text{Set}_\Delta$ of simplicial sets. If $\mathcal{U}$ is closed under small colimits and contains the standard $n$-simplex $\Delta^n$ for each $n \geq 0$, then $\mathcal{U} = \text{Set}_\Delta$.

Proof. If $\mathcal{U}$ is closed under small colimits, then it satisfies conditions (1) and (2) of Lemma 1.1.8.15 along with condition (3'') of Remark 1.1.8.16. Consequently, if it contains each of the standard simplices $\Delta^n$, then $\mathcal{U} = \text{Set}_\Delta$. \hfill \square

Remark 1.1.8.18. We can state Corollary 1.1.8.17 more informally as follows: the category $\text{Set}_\Delta$ of simplicial sets is generated, under small colimits, by objects of the form $\Delta^n$. In fact, one can say more: it is freely generated (under small colimits) by the essential image of the Yoneda embedding

$$
\Delta \hookrightarrow \text{Set}_\Delta \quad [n] \mapsto \Delta^n.
$$

This is a general fact about presheaf categories, which we will return to in §[?].

Let us now sketch another proof of Corollary 1.1.8.17, which illustrates some ideas which will be useful later.
Construction 1.1.8.19 (The Category of Simplices of a Simplicial Set). Let \( S \) be a simplicial set. We define a category \( \Delta_S \) as follows:

- The objects of \( \Delta_S \) are pairs \(([n], \sigma)\), where \([n]\) is an object of \( \Delta \) and \( \sigma \) is an \( n \)-simplex of \( S \).

- A morphism from \(([n], \sigma)\) to \(([n'], \sigma')\) in the category \( \Delta_S \) is a nondecreasing function \( f : [n] \to [n'] \) with the property that the induced map \( S_{n'} \to S_n \) carries \( \sigma' \) to \( \sigma \).

We will refer to \( \Delta_S \) as the category of simplices of \( S \).

Remark 1.1.8.20. Passage from a simplicial set \( S \) to the category of simplices \( \Delta_S \) is a special case of the Grothendieck construction, which we will study in more detail in Chapter [?].

Alternative Proof of Corollary 1.1.8.17. Via the Yoneda embedding \( \Delta \hookrightarrow \text{Set}_\Delta \), we can identify \( \Delta_S \) with the category whose objects are simplicial sets of the form \( \Delta^n \) (for some \( n \geq 0 \)), which are equipped with a map of simplicial sets \( \Delta^n \to S \). In particular, we have a canonical map of simplicial sets \( \varinjlim_{([n], \sigma) \in \Delta_S} \Delta^n \to S \). To prove Corollary 1.1.8.17, it suffices to observe that this map is an isomorphism. This is an elementary calculation which we leave to the reader (see \$[?] \) for more details).

Remark 1.1.8.21. Each of our proofs of Corollary 1.1.8.17 gives additional information that the other does not. Our first proof shows that every simplicial set \( S \) can be built as a colimit of standard simplices in a very specific way: namely, by forming pushouts along boundary inclusions \( \partial \Delta^n \hookrightarrow \Delta^n \) (for a more precise assertion, see the proof of Proposition 1.4.5.12). This extra information was used in the proof of Proposition 1.1.8.4 to show that the geometric realization \( |S| \) is a CW complex (and not merely a topological space which is colimit of disks). On the other hand, our second proof shows that every simplicial set \( S \) can be built in a single step as the colimit of a diagram of standard simplices (which can be chosen in a specific, canonical way).

In Chapter 2 we will encounter a number of variants of the geometric realization construction \( S \mapsto |S| \), which arise as special cases of the following:

Proposition 1.1.8.22. Let \( C \) be a category, let \( Q \) be a cosimplicial object of \( C \), and let \( \text{Sing}^Q : C \to \text{Set}_\Delta \) be the functor of Variant 1.1.7.6. If the category \( C \) admits small colimits, then the functor \( \text{Sing}^Q \) admits a left adjoint \( \text{Set}_\Delta \to C \), which we will denote by \( S \mapsto |S|^Q \).

Proof. Let us say that a simplicial set \( S \) is good if the functor

\[(C \in C) \mapsto \text{Hom}_{\text{Set}_\Delta}(S, \text{Sing}^Q(C))\]
is corepresentable by an object of the category $\mathcal{C}$ (in which case we denote the corepresenting object by $|S_\bullet|^Q$). It follows from Yoneda’s lemma that the standard $n$-simplex $\Delta^n$ is good for each $n \geq 0$, with $|\Delta^n|^Q \simeq Q^n$. If $\mathcal{C}$ admits small colimits, then the proof of Lemma 1.1.8.6 shows that the collection of good simplicial sets is closed under small colimits. It now suffices to observe that every simplicial set $S_\bullet$ can be written as a small colimit of simplices (Lemma 1.1.8.17).

**Remark 1.1.8.23.** The functor $\pi_0 : \text{Set}_\Delta \to \text{Set}$ of Corollary 1.1.6.21 can be regarded as a special case of Proposition 1.1.8.22: it agrees with the functor $\bullet |^Q$, where $Q^\bullet : \Delta \to \text{Set}$ is a constant functor whose value is a singleton set $\ast \in \text{Set}_\Delta$.

**Proposition 1.1.8.24.** Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:

1. The geometric realization $|S_\bullet|$ is a path-connected topological space.
2. The geometric realization $|S_\bullet|$ is a connected topological space.
3. The simplicial set $S_\bullet$ is connected, in the sense of Definition 1.1.6.6.

**Proof.** The implication (1) $\Rightarrow$ (2) holds for any topological space. To prove that (2) $\Rightarrow$ (3), we observe that any decomposition $S_\bullet \simeq S'_\bullet \coprod S''_\bullet$ into disjoint nonempty simplicial subsets determines a homeomorphism $|S_\bullet| \simeq |S'_\bullet| \coprod |S''_\bullet|$. We will complete the proof by showing that (3) $\Rightarrow$ (1). Note that we have a commutative diagram of sets

\[
\begin{array}{ccc}
\lim_{\gamma : \Delta^n \to S_\bullet} |\Delta^n| & \xrightarrow{\sim} & |S_\bullet| \\
\downarrow & & \downarrow \\
\lim_{\gamma : \Delta^n \to S_\bullet} \pi_0(|\Delta^n|) & \longrightarrow & \pi_0(|S_\bullet|),
\end{array}
\]

where the upper horizontal map is bijective and the right vertical map is surjective. It follows that the lower horizontal map is also surjective. Since each of the topological spaces $|\Delta^n|$ is path connected, the colimit in the lower left can be identified with the set $\pi_0(S_\bullet)$ (Remark 1.1.8.23). If $S_\bullet$ is connected, the set $\pi_0(S_\bullet)$ consists of a single element, so that $\pi_0(|S_\bullet|)$ is also a singleton. \hfill $\Box$

**Corollary 1.1.8.25.** For every simplicial set $S_\bullet$, we have a canonical bijection

$\pi_0(S_\bullet) \simeq \pi_0(|S_\bullet|)$. 

**Proof.** Writing $S_\bullet$ as a disjoint union of connected components (Proposition 1.1.6.11) we can reduce to the case where $S_\bullet$ is connected, in which case both sets have a single element (Proposition 1.1.8.24). \hfill $\Box$
1.1.9 Kan Complexes

We now articulate an important property enjoyed by simplicial sets of the form $\operatorname{Sing}_\bullet(X)$.

**Definition 1.1.9.1.** Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is a Kan complex if it satisfies the following condition:

(*) For $n > 0$ and $0 \leq i \leq n$, any map of simplicial sets $\sigma_0 : \Lambda_i^n \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$. Here $\Lambda_i^n \subseteq \Delta^n$ denotes the $i$th horn (see Construction 1.1.2.9).

**Exercise 1.1.9.2.** Show that for $n > 0$, the standard simplex $\Delta^n$ is not a Kan complex (for a more general statement, see Proposition 1.2.4.2).

**Example 1.1.9.3.** Let $S_\bullet$ be a simplicial set of dimension exactly 1 (that is, a simplicial set $S_\bullet$ which arises from a directed graph with at least one edge). Then $S_\bullet$ is not Kan complex.

**Example 1.1.9.4** (Products of Kan Complexes). Let $\{S_\alpha_\bullet\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set $A$, and let $S_\bullet = \prod_{\alpha \in A} S_\alpha_\bullet$ be their product. If each $S_\alpha_\bullet$ is a Kan complex, then $S_\bullet$ is a Kan complex. The converse holds provided that each $S_\alpha_\bullet$ is nonempty.

**Example 1.1.9.5** (Coproducts of Kan Complexes). Let $\{S_\alpha_\bullet\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set $A$, and let $S_\bullet = \coprod_{\alpha \in A} S_\alpha_\bullet$ be their coproduct. For each $0 \leq i \leq n$, the restriction map

$$\theta : \operatorname{Hom}_{\text{Set}}(\Delta^n, S_\bullet) \to \operatorname{Hom}_{\text{Set}}(\Lambda_i^n, S_\bullet)$$

can be identified with the coproduct (formed in the arrow category $\text{Fun}([1], \text{Set})$) of restriction maps $\theta_\alpha : \operatorname{Hom}_{\text{Set}}(\Delta^n, S_\alpha_\bullet) \to \operatorname{Hom}_{\text{Set}}(\Lambda_i^n, S_\alpha_\bullet)$ (this follows from the observation that the simplicial sets $\Delta^n$ and $\Lambda_i^n$ are connected). It follows that $\theta$ is surjective if and only if each $\theta_\alpha$ is surjective. Allowing $n$ and $i$ to vary, we conclude that $S_\bullet$ is a Kan complex if and only if each summand $S_\alpha_\bullet$ is a Kan complex.

**Remark 1.1.9.6.** Let $S_\bullet$ be a simplicial set. Combining Example 1.1.9.5 with Proposition 1.1.6.13, we deduce that $S_\bullet$ is a Kan complex if and only if each connected component of $S_\bullet$ is a Kan complex.

**Example 1.1.9.7.** Let $S$ be a set and let $S_\bullet$ denote the associated constant simplicial set (Construction 1.1.4.2). Then $S_\bullet$ is a Kan complex (this follows from Remark 1.1.9.6, since each connected component of $S_\bullet$ is isomorphic to $\Delta^0$; see Example 1.1.6.10).

**Proposition 1.1.9.8.** Let $X$ be a topological space. Then the singular simplicial set $\operatorname{Sing}_\bullet(X)$ is a Kan complex.
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Proof. Let $\sigma_0 : \Lambda^0_i \to \text{Sing}_\bullet(X)$ be a map of simplicial sets for $n > 0$; we wish to show that $\sigma_0$ can be extended to an $n$-simplex of $X$. Using the geometric realization functor, we can identify $\sigma_0$ with a continuous map of topological spaces $f_0 : |\Lambda^0_i| \to X$; we wish to show that $f_0$ factors as a composition

$$|\Lambda^0_i| \to |\Delta^n| \xrightarrow{\tau} X.$$ 

Using Example 1.1.8.13 we can identify $|\Lambda^0_i|$ with the subset

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\} \subset |\Delta^n|.$$

In this case, we can take $f$ to be the composition $f_0 \circ r$, where $r$ is any continuous retraction of $|\Delta^n|$ onto the subset $|\Lambda^0_i|$. For example, we can take $r$ to be the map given by the formula

$$r(t_0, \ldots, t_n) = (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \ldots, t_n - c)$$

$$c = \min\{t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}.$$

Algebra furnishes another rich supply of examples:

**Proposition 1.1.9.9.** Let $G_\bullet$ be a simplicial group (that is, a simplicial object of the category of groups). Then (the underlying simplicial set of) $G_\bullet$ is a Kan complex.

Proof. Let $n$ be a positive integer and $\bar{\sigma} : \Lambda^n_i \to G_\bullet$ be a map of simplicial sets for some $0 \leq i \leq n$, which we will identify with a tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \bullet, \sigma'_{i+1}, \ldots, \sigma_n)$ of elements of the group $G_{n-1}$ (Exercise 1.1.2.14). We wish to prove that there exists an element $\tau \in G_n$ satisfying $d_j \tau = \sigma_j$ for $j \neq i$. Let $e$ denote the identity element of $G_{n-1}$. We first treat the special case where $\sigma_{i+1} = \cdots = \sigma_n = e$. If, in addition, we have $\sigma_0 = \sigma_1 = \cdots = \sigma_{i-1} = e$, then we can take $\tau$ to be the identity element of $G_n$. Otherwise, there exists some smallest integer $j < i$ such that $\sigma_j \neq e$. We proceed by descending induction on $j$. Set $\tau'' = s_j \sigma_j \in G_n$, and consider the map $\bar{\sigma}' : \Lambda^n_i \to G_\bullet$ given by the tuple $(\sigma_0', \sigma_1', \ldots, \sigma_{i-1}', \bullet, \sigma'_{i+1}, \ldots, \sigma_n')$ with $\sigma'_k = \sigma_k(d_k \tau'')^{-1}$. We then have $\sigma'_0 = \sigma'_1 = \cdots = \sigma'_j = e$ and $\sigma'_{i+1} = \cdots = \sigma'_n = e$. Invoking our inductive hypothesis we conclude that there exists an element $\tau' \in G_n$ satisfying $d_k \tau' = \sigma'_k$ for $k \neq i$. We can then complete the proof by taking $\tau$ to be the product $\tau' \tau''$.

If not all of the equalities $\sigma_{i+1} = \cdots = \sigma_n = e$ hold, then there exists some largest integer $j > i$ such that $\sigma_j \neq e$. We now proceed by ascending induction on $j$. Set $\tau'' = s_j \sigma_j$ and let $\bar{\sigma}' : \Lambda^n_i \to G_\bullet$ be the map given by the tuple $(\sigma_0', \sigma_1', \ldots, \sigma_{i-1}', \bullet, \sigma'_{i+1}, \ldots, \sigma_n')$ with $\sigma'_k = \sigma_k(d_k \tau'')^{-1}$, as above. We then have $\sigma_j = \sigma_{j+1} = \cdots = \sigma_n = e$, so the inductive hypothesis guarantees the existence of an element $\tau' \in G_n$ satisfying $d_k \tau' = \sigma'_k$ for $k \neq i$. As before, we complete the proof by setting $\tau = \tau' \tau''$. 

\qed
Let $S$ be a simplicial set. According to Remark 1.1.6.23, we can identify the set of connected components $\pi_0(S)$ with the quotient $S_0/\sim$, where $\sim$ is the equivalence relation generated by the image of the map $\left( d_0, d_1 \right): S_1 \to S_0 \times S_0$. In the special case where $S = \text{Sing}(X)$ is the singular simplicial set of a topological space $X$, this description simplifies: the image of the map $\left( d_0, d_1 \right): \text{Sing}_1(X) \to \text{Sing}_0(X) \times \text{Sing}_0(X) = X \times X$ is already an equivalence relation, and $\pi_0(S)$ can be identified with the set of path components $\pi_0(X)$ (Remark 1.1.7.3). A similar phenomenon occurs for any Kan complex:

**Proposition 1.1.9.10.** Let $S$ be a Kan complex containing a pair of vertices $x, y \in S_0$. Then $x$ and $y$ belong to the same connected component of $S$ if and only if there exists an edge $e \in S_1$ satisfying $d_0(e) = x$ and $d_1(e) = y$.

**Proof.** Let $R$ denote the image of the map $\left( d_0, d_1 \right): S_1 \to S_0 \times S_0$. According to Remark 1.1.6.23, we can identify $\pi_0(S)$ with the quotient of $S_0$ by the equivalence relation generated by $R$. It will therefore suffice to show that $R$ is already an equivalence relation on $S_0$. To prove this, we must verify three things:

- The relation $R$ is reflexive. This follows from the observation that for every vertex $x \in S_0$, the map $\left( d_0, d_1 \right)$ carries the degenerate edge $s_0(x)$ to the pair $(x, x) \in S_0 \times S_0$.

- The relation $R$ is symmetric. Suppose that $(x, y) \in R$: that is, there exists an edge $e \in S_1$ satisfying $d_0(e) = x$ and $d_1(e) = y$. Then the tuple $(e, s_0(x), \bullet)$ determines a map of simplicial sets $\sigma_0: \Lambda^2_2 \to S$ (see Exercise 1.1.2.14), which we depict as a diagram:

  \[
  \begin{array}{ccc}
  y & \downarrow e \\
  x & \Rightarrow s_0(x) \\
  \end{array}
  \]

  Since $S$ is a Kan complex, we can complete this diagram to a 2-simplex $\sigma: \Delta^2 \to S$. Then $e' = d_2(\sigma)$ is an edge of $S$ satisfying $d_0(e') = y$ and $d_1(e') = x$, which proves that the pair $(y, x)$ belongs to $R$.

- The relation $R$ is transitive. Suppose that we are given vertices $x, y, z \in S_0$ with $(x, y) \in R$ and $(y, z) \in R$; we wish to show that $(x, z) \in R$. Choose edges $e, e' \in S_1$ satisfying $d_0(e) = x$, $d_1(e) = y = d_0(e')$, and $d_1(e') = z$. Then the tuple $(e', \bullet, e)$ determines a map of simplicial sets $\tau_0: \Lambda^2_1 \to S$ (see Exercise 1.1.2.14), which we depict as a diagram:

  \[
  \begin{array}{ccc}
  e' & \downarrow y \\
  z & \Rightarrow x \\
  \end{array}
  \]
Our assumption that $S$ is a Kan complex guarantees that we can extend $\tau_0$ to a 2-simplex $\tau: \Delta^2 \to S$. Then $e'' = d_1(\tau)$ is an edge of $S$ satisfying $d_0(e'') = x$ and $d_1(e'') = z$, which proves that $(x, z) \in R$.

\[ \square \]

**Corollary 1.1.9.11.** Let $\{S(\alpha)\}_{\alpha \in A}$ be a collection of Kan complexes parametrized by a set $A$, and let $S = \prod_{\alpha \in A} S(\alpha)$ denote their product. Then the canonical map

$$\pi_0(S) \to \prod_{\alpha \in A} \pi_0(S(\alpha))$$

is bijective. In particular, $S$ is connected if and only if each factor $S(\alpha)$ is connected.

### 1.2 The Nerve of a Category

In §1.1, we introduced the theory of simplicial sets and discussed its relationship to the theory of topological spaces. Every topological space $X$ determines a simplicial set $\text{Sing}(X)$ (Construction 1.1.7.1), and simplicial sets of the form $\text{Sing}(X)$ have a special property: they are Kan complexes (Proposition 1.1.9.8). In this section, we will study a different class of simplicial sets, which arise instead from the theory of categories. In §1.2.1, we associate to every category $\mathcal{C}$ a simplicial set $N(\mathcal{C})$, called the **nerve of $\mathcal{C}$**. We show in §1.2.2 that the construction $\mathcal{C} \mapsto N(\mathcal{C})$ is fully faithful (Proposition 1.2.2.1). In §1.2.3, we show that a simplicial set $S$ belongs to the essential image of the functor $\mathcal{C} \mapsto N(\mathcal{C})$ if and only if it satisfies a certain lifting condition (Proposition 1.2.3.1). This lifting condition is similar to the Kan extension condition (Definition 1.1.9.1), but not identical to it: in §1.2.4, we show that a simplicial set of the form $N(\mathcal{C})$ is a Kan complex if and only if every morphism in $\mathcal{C}$ is invertible (Proposition 1.2.4.2).

In §1.2.5, we show that the construction $\mathcal{C} \mapsto N(\mathcal{C})$ has a left adjoint, which associates to each simplicial set $S$ a category $\text{hS}$ which we call the **homotopy category of $S$** (Definition 1.2.5.1). This category admits a particularly simple description in the case where the simplicial set $S$ has dimension $\leq 1$: in §1.2.6, we show that it can be identified with the **path category** of the directed graph $G$ corresponding to $S$ (under the equivalence of Proposition 1.1.5.9).

### 1.2.1 Construction of the Nerve

We begin with a few definitions.

**Construction 1.2.1.1.** For every integer $n \geq 0$, let us view the linearly ordered set $[n] = \{0 < 1 < \cdots < n - 1 < n\}$ as a category (where there is a unique morphism from $i$ to
CHAPTER 1. THE LANGUAGE OF $\infty$-CATEGORIES

$j$ when $i \leq j$). For any category $\mathcal{C}$, we let $N_n(\mathcal{C})$ denote the set of all functors from $[n]$ to $\mathcal{C}$. Note that for any nondecreasing map $\alpha : [m] \to [n]$, precomposition with $\alpha$ determines a map of sets $N_n(\mathcal{C}) \to N_m(\mathcal{C})$. We can therefore view the construction $[n] \to N_n(\mathcal{C})$ as a simplicial set. We will denote this simplicial set by $N_\bullet(\mathcal{C})$ and refer to it as the nerve of $\mathcal{C}$.

**Remark 1.2.1.2** (The Classifying Space of a Category). Let $\mathcal{C}$ be a category. Then the topological space $|N_\bullet(\mathcal{C})|$ is called the classifying space of the category $\mathcal{C}$.

**Remark 1.2.1.3.** Let $\mathcal{C}$ be a category and let $n \geq 1$. Elements of $N_n(\mathcal{C})$ can be identified with diagrams

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$ (see Remark 1.4.7.8). In other words, we can identify elements of $N_n(\mathcal{C})$ with $n$-tuples $(f_1, \ldots, f_n)$ of morphisms of $\mathcal{C}$ having the property that, for $0 < i < n$, the source of $f_{i+1}$ coincides with the target of $f_i$.

**Example 1.2.1.4.** Let $\mathcal{C}$ be a category. Then:

- Vertices of the simplicial set $N_\bullet(\mathcal{C})$ can be identified with objects of the category $\mathcal{C}$.
- Edges of the simplicial set $N_\bullet(\mathcal{C})$ can be identified with morphisms in the category $\mathcal{C}$.
- Let $f : X \to Y$ be a morphism in $\mathcal{C}$, regarded as an edge of the simplicial set $N_\bullet(\mathcal{C})$. Then the faces of $f$ are given by the target $d_0f = Y$ and the source $d_1f = X$, respectively.
- Let $X$ be an object of $\mathcal{C}$, which we regard as a vertex of the simplicial set $N_\bullet(\mathcal{C})$. Then the degenerate edge $s_0(X)$ is the identity morphism $\text{id}_X : X \to X$.

**Remark 1.2.1.5** (Face Operators on $N_\bullet(\mathcal{C})$). Let $\mathcal{C}$ be a category and suppose we are given an $n$-simplex $\sigma$ of the simplicial set $N_\bullet(\mathcal{C})$ for some $n > 0$, which we identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n.$$

Then:

- The 0th face $d_0\sigma \in N_{n-1}(\mathcal{C})$ can be identified with the diagram

$$C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3 \to \cdots \xrightarrow{f_n} C_n$$

obtained from $\sigma$ by “deleting” the object $C_0$ (and the morphism $f_1$ with source $C_0$).

- The $n$th face $d_n\sigma \in N_{n-1}(\mathcal{C})$ can be identified with the diagram

$$C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_{n-2} \xrightarrow{f_{n-1}} C_{n-1}$$

obtained from $\sigma$ by “deleting” the object $C_n$ (and the morphism $f_n$ with target $f_n$).
• For $0 < i < n$, the $i$th face $d_i \sigma \in N_{n-1}(\mathcal{C})$ can be identified with the diagram

$$C_0 \xrightarrow{f_1} C_1 \rightarrow \cdots \rightarrow C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \rightarrow \cdots \xrightarrow{f_n} C_n$$

obtained by “deleting” the object $C_i$ (and composing the morphisms $f_i$ and $f_{i+1}$).

**Remark 1.2.1.6** (Degeneracy Operators on $N_{\bullet}(\mathcal{C})$). Let $\mathcal{C}$ be a category and suppose we are given an $n$-simplex $\sigma$ of the simplicial set $N_{\bullet}(\mathcal{C})$ which we identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_n} C_n.$$

Then, for $0 \leq i \leq n$, we can identify $s_i \sigma \in N_{n+1}(\mathcal{C})$ with the diagram

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{id_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \rightarrow \cdots \xrightarrow{f_n} C_n$$

obtained from $\sigma$ by “inserting” the identity morphism $id_{C_i}$.

**Remark 1.2.1.7.** Let $\mathcal{C}$ be a category and let $\sigma$ be an $n$-simplex of $N_{\bullet}(\mathcal{C})$, corresponding to a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_n} C_n.$$

Then $\sigma$ is degenerate (Definition 1.1.3.2) if and only if some $f_i$ is an identity morphism of $\mathcal{C}$ (in which case we must have $C_{i-1} = C_i$).

**Remark 1.2.1.8.** Let $I$ be a set equipped with a partial ordering $\leq_I$. Then we can regard $I$ as a category whose objects are the elements of $I$, with morphisms given by

$$\text{Hom}_I(i,j) = \begin{cases} * & \text{if } i \leq_I j \\ \emptyset & \text{otherwise.} \end{cases}$$

We will denote the nerve of this category by $N_{\bullet}(I)$, and refer to it as the nerve of the partially ordered set $I$. For each $n \geq 0$, we can identify $n$-simplices of $N_{\bullet}(I)$ with monotone functions $[n] \to I$: that is, with nondecreasing sequences $(i_0 \leq_I i_1 \leq_I \cdots \leq_I i_n)$ of elements of $I$.

**Example 1.2.1.9.** For each $n \geq 0$, the nerve $N_{\bullet}([n])$ can be identified with the standard $n$-simplex $\Delta^n$ of Construction 1.1.2.1.

**Remark 1.2.1.10.** The construction $\mathcal{C} \mapsto N_{\bullet}(\mathcal{C})$ determines a functor $N_{\bullet} : \text{Cat} \to \text{Set}_{\Delta}$ from the category $\text{Cat}$ of (small) categories to the category $\text{Set}_{\Delta}$ of simplicial sets. This is a special case of the construction described in Variant 1.1.7.6. More precisely, we can identify $N_{\bullet}$ with the functor $\text{Sing}^Q$, where $Q : \Delta \to \text{Cat}$ is the functor which carries each object $[n] \in \Delta$ to itself, regarded as a category. It follows from Proposition 1.1.8.22 that this functor admits a left adjoint, which we will study in §1.2.5.
1.2.2 Recovering a Category from its Nerve

Passage from a category \( C \) to the nerve \( N_\bullet(C) \) does not lose any information:

**Proposition 1.2.2.1.** The nerve functor \( N_\bullet : \text{Cat} \to \text{Set}_\Delta \) is fully faithful.

Throughout this book, we will often abuse terminology by identifying a category \( C \) with its nerve \( N_\bullet(C) \). By virtue of Proposition 1.2.2.1, this is essentially harmless: the nerve construction allows us to identify categories with certain kinds of simplicial sets.

**Proof of Proposition 1.2.2.1.** Let \( C \) and \( C' \) be categories. We wish to show that the nerve functor \( N_\bullet \) induces a bijection

\[
\theta : \text{Hom}_{\text{Cat}}(C,C') \to \text{Hom}_{\text{Set}_\Delta}(N_\bullet(C),N_\bullet(C')).
\]

Here the source of \( \theta \) is the set of all functors from \( C \) to \( C' \). We first note that \( \theta \) is injective: a functor \( F : C \to C' \) is determined by its behavior on the objects and morphisms of \( C \), and therefore by the behavior of \( \theta(F) \) on the vertices and edges of the simplicial set \( N_\bullet(C) \) (see Example 1.2.1.4). Let us prove the surjectivity of \( \theta \). Let \( f : N_\bullet(C) \to N_\bullet(C') \) be a morphism of simplicial sets; we wish to show that there exists a functor \( F : C \to C' \) such that \( f = \theta(F) \).

For each \( n \geq 0 \), the morphism \( f \) determines a map of sets \( N_n(C) \to N_n(C') \), which we will also denote by \( f \). In the case \( n = 0 \), this map carries each object \( C \in C \) to an object of \( C' \), which we will denote by \( F(C) \). For every pair of objects \( C,D \in C \), the map \( f \) carries each morphism \( u : C \to D \) to a morphism \( f(u) \) in the category \( C' \). Since \( f \) commutes with face maps, the morphism \( f(u) \) has source \( F(C) \) and target \( F(D) \) (see Example 1.2.1.4), and can therefore be regarded as an element of \( \text{Hom}_{C'}(F(C),F(D)) \); we denote this element by \( f(u) \).

We will complete the proof by verifying the following:

(a) The preceding construction determines a functor \( F : C \to C' \).

(b) We have an equality \( f = \theta(F) \) of maps from \( N_\bullet(C) \) to \( N_\bullet(C') \).

To prove (a), we first note that the compatibility of \( f \) with degeneracy maps implies that we have \( F(\text{id}_C) = \text{id}_{F(C)} \) for each \( C \in C \) (see Example 1.2.1.4). It will therefore suffice to show that for every pair of composable morphisms \( u : C \to D \) and \( v : D \to E \) in the category \( C \), we have \( F(v) \circ F(u) = F(v \circ u) \) as elements of the set \( \text{Hom}_{C'}(F(C),F(E)) \). For this, we observe that the diagram \( C \xrightarrow{u} D \xrightarrow{v} E \) can be identified with a 2-simplex \( \sigma \) of \( N_\bullet(C) \). Using the equality \( d_i(f(\sigma)) = f(d_i(\sigma)) \) for \( i = 0, 2 \), we see that \( f(\sigma) \) corresponds to the diagram \( F(C) \xrightarrow{F(u)} F(D) \xrightarrow{F(v)} F(E) \) in \( C' \). We now compute

\[
F(v) \circ F(u) = d_1(f(\sigma)) = f(d_1(\sigma)) = F(v \circ u).
\]

This completes the proof of (a). To prove (b), we must show that \( f(\tau) = \theta(F)(\tau) \) for each \( n \)-simplex \( \tau \) of \( N_\bullet(C) \). This follows by construction in the case \( n \leq 1 \), and follows in
1.2. THE NERVE OF A CATEGORY

general since an \( n \)-simplex of \( N_\bullet(C') \) is determined by its 1-dimensional faces (see Remark 1.2.1.3).

1.2.3 Characterization of Nerves

We now describe the essential image of the functor \( N_\bullet : \text{Cat} \to \text{Set}_\Delta \).

Proposition 1.2.3.1. Let \( S_\bullet \) be a simplicial set. Then \( S_\bullet \) is isomorphic to the nerve of a category if and only if it satisfies the following condition:

\[
(*)' \quad \text{For every pair of integers } 0 < i < n \text{ and every map of simplicial sets } \sigma_0 : \Lambda^n_i \to S_\bullet, \text{ there exists a unique map } \sigma : \Delta^n \to S_\bullet \text{ such that } \sigma_0 = \sigma|_{\Lambda^n_i}. \]

The proof of Proposition 1.2.3.1 will require some preliminaries. We begin by establishing the necessity of condition \((*)'\).

Lemma 1.2.3.2. Let \( C \) be a category. Then the simplicial set \( N_\bullet(C) \) satisfies condition \((*)'\) of Proposition 1.2.3.1.

Proof. Choose integers \( 0 < i < n \) together with a map of simplicial sets \( \sigma_0 : \Lambda^n_i \to N_\bullet(C) \); we wish to show that \( \sigma_0 \) can be extended uniquely to a \( n \)-simplex of \( N_\bullet(C) \). For \( 0 \leq j \leq n \), let \( C_j \in C \) denote the image under \( \sigma_0 \) of the \( j \)th vertex of \( \Delta^n \) (which belongs to the horn \( \Lambda^n_i \)). We first consider the case where \( n \geq 3 \). In this case, \( \Lambda^n_i \) contains every edge of \( \Delta^n \). For \( 0 \leq j \leq k \leq n \), let \( f_{k,j} : C_j \to C_k \) denote the 1-simplex of \( N_\bullet(C) \) obtained by evaluating \( \sigma_0 \) on the edge of \( \Delta^n \) corresponding to the pair \((j,k)\). We claim that the construction

\[
j \mapsto C_j \quad (j \leq k) \mapsto f_{k,j}
\]

determines a functor \([n] \to C\), which we can then identify with an \( n \)-simplex of \( N_\bullet(C) \) having the desired properties. It is easy to see that \( f_{j,j} = \text{id}_{C_j} \) for each \( 0 \leq j \leq n \), so it will suffice to show that \( f_{\ell,k} \circ f_{k,j} = f_{\ell,j} \) for every triple \( 0 \leq j \leq k \leq \ell \leq n \). The triple \((j,k,\ell)\) determines a 2-simplex \( \tau \) of \( \Delta^n \). If \( \tau \) is contained in \( \Lambda^n_i \), then \( \tau' = \sigma_0(\tau) \) is a 2-simplex of \( N_\bullet(C) \) satisfying \( d_0(\tau') = f_{\ell,k}, d_1(\tau') = f_{\ell,j} \), and \( d_2(\tau') = f_{k,j} \), so that \( \tau' \) “witnesses” the identity \( f_{\ell,k} \circ f_{k,j} = f_{\ell,j} \). It will therefore suffice to treat the case where the simplex \( \tau \) does not belong to the \( \Lambda^n_i \). In this case, our assumption that \( n \geq 3 \) guarantees that we must have \( \{j,k,\ell\} = [n] \setminus \{i\} \). It follows that \( n = 3 \), so that either \( i = 1 \) or \( i = 2 \). We will treat the case \( i = 1 \) (the case \( i = 2 \) follows by a similar argument). Note that \( \Lambda^3_1 \) contains all of the nondegenerate 2-simplices of \( \Delta^3 \) other than \( \tau \); applying the map \( \sigma_0 \), we obtain 2-simplices of \( N_\bullet(C) \) which witness the identities

\[
f_{3,0} = f_{3,1} \circ f_{1,0} \quad f_{3,1} = f_{3,2} \circ f_{2,1} \quad f_{2,0} = f_{2,1} \circ f_{1,0}.
\]
We now compute
\[ f_{3,0} = f_{3,1} \circ f_{1,0} = (f_{3,2} \circ f_{2,1}) \circ f_{1,0} = f_{3,2} \circ (f_{2,1} \circ f_{1,0}) = f_{3,2} \circ f_{2,0} \]
so that \[ f_{\ell,j} = f_{\ell,k} \circ f_{k,j} \], as desired.

It remains to treat the case \( n = 2 \), so that we must also have \( i = 1 \). In this situation, the map \( \sigma_0 : \Lambda_1^n \to N_\bullet(C) \) determines a pair of composable morphisms \( f_{1,0} : C_0 \to C_1 \) and \( f_{2,1} : C_1 \to C_2 \). This data extends uniquely to a 2-simplex \( \sigma \) of \( C \) satisfying \( d_1(\sigma) = f_{2,1} \circ f_{1,0} \) (see Remark 1.2.1.3).

Lemma 1.2.3.3. Let \( f : S_\bullet \to T_\bullet \) be a map of simplicial sets. Assume that \( f \) induces bijections \( S_0 \to T_0 \) and \( S_1 \to T_1 \), and that both \( S_\bullet \) and \( T_\bullet \) satisfy condition \((\ast')\) of Proposition 1.2.3.1. Then \( f \) is an isomorphism.

Proof. We claim that, for every simplicial set \( K_\bullet \), composition with \( f \) induces a bijection
\[ \theta_{K_\bullet} : \text{Hom}_{\text{Set}}(K_\bullet, S_\bullet) \to \text{Hom}_{\text{Set}}(K_\bullet, T_\bullet). \]
Writing \( K_\bullet \) as a union of its skeleta \( \text{sk}_n K_\bullet \), we can reduce to the case where \( K \) has dimension \( \leq n \), for some integer \( n \geq -1 \) (see Definition 1.1.3.9). We now proceed by induction on \( n \). The case \( n = -1 \) is trivial (since a simplicial set of dimension \( \leq -1 \) is empty). Let us therefore assume that \( n \geq 0 \), so that Proposition 1.1.3.13 supplies a pushout diagram of simplicial sets
\[
\begin{array}{ccc}
\coprod \partial \Delta^n & \to & \coprod \Delta^n \\
\downarrow & & \downarrow \\
\text{sk}_{n-1} K_\bullet & \to & K_\bullet.
\end{array}
\]
It follows from our inductive hypothesis that the maps \( \theta_{\partial \Delta^n} \) and \( \theta_{\text{sk}_{n-1} K_\bullet} \) are bijective. Consequently, to show that \( \theta_{K_\bullet} \) is bijective, it will suffice to show that \( \theta_{\Delta^n} \) is bijective: that is, that \( f \) induces a bijection \( S_n \to T_n \). For \( n \leq 1 \), this follows from our hypothesis. To handle the case \( n \geq 2 \), we observe that there is a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{Set}_\Delta}(\Delta^n, S_\bullet) & \to & \text{Hom}_{\text{Set}_\Delta}(\Lambda_1^n, S_\bullet) \\
\downarrow \theta_{\Delta^n} & & \downarrow \theta_{\Lambda_1^n} \\
\text{Hom}_{\text{Set}_\Delta}(\Delta^n, T_\bullet) & \to & \text{Hom}_{\text{Set}_\Delta}(\Lambda_1^n, T_\bullet).
\end{array}
\]
Here the right vertical map is bijective by virtue of our inductive hypothesis, and the horizontal maps are bijective by virtue of our assumption that both \( S_\bullet \) and \( T_\bullet \) satisfy assumption \((\ast')\). It follows that the left vertical map is also bijective, as desired. \( \Box \)
1.2. THE NERVE OF A CATEGORY

Proof of Proposition 1.2.3.1

Let $S_•$ be a simplicial set satisfying condition $(\ast')$ of Proposition 1.2.3.1; we will show that there is a category $C$ and an isomorphism of simplicial sets $u : S_• \to N_•(C)$ (the converse assertion follows from Lemma 1.2.3.2). It follows from Proposition 1.2.2.1 that the category $C$ is uniquely determined (up to isomorphism), and from the proof of Proposition 1.2.2.1 we can extract an explicit construction of $C$:

- The objects of $C$ are the vertices of $S_•$.
- Given a pair of objects $C, D \in C$, we let $\text{Hom}_C(C, D)$ denote the collection of edges $e$ of $S_•$ satisfying $d_0(e) = D$ and $d_1(e) = C$.
- For each object $C \in C$, we define the identity morphism $\text{id}_C \in \text{Hom}_C(C, C)$ to be the degenerate edge $s_0(C)$.
- Given a triple of objects $C, D, E \in C$ and a pair of morphisms $f \in \text{Hom}_C(C, D)$ and $g \in \text{Hom}_C(D, E)$, we can apply hypothesis $(\ast')$ (in the special case $n = 2$ and $i = 1$) to conclude that there is a unique 2-simplex $\sigma$ of $S_•$ satisfying $d_2(\sigma) = f$ and $d_0(\sigma) = g$. We define the composition $g \circ f \in \text{Hom}_C(C, E)$ to be the edge $d_1(\sigma)$.

We claim that $C$ is a category. For this, we must check the following:

- The composition law on $C$ is unital: for every morphism $f : C \to D$ in $C$, we have equalities
  $$\text{id}_D \circ f = f = f \circ \text{id}_C.$$  
  Let us verify the identity on the left; the proof in the other case is similar. For this, we must construct a 2-simplex $\sigma$ of $S_•$ such that $d_0(\sigma) = \text{id}_D$ and $d_1(\sigma) = d_2(\sigma) = f$.
  The degenerate 2-simplex $s_1(f)$ has these properties.

- The composition law on $C$ is associative. That is, for every triple of composable morphisms
  $$f : W \to X, \quad g : X \to Y, \quad h : Y \to Z$$
  in $C$, we have an identity $h \circ (g \circ f) = (h \circ g) \circ f$ in the category $C$. Applying condition $(\ast')$ repeatedly, we deduce the following:

  - There is a unique 2-simplex $\sigma_0$ of $C$ satisfying $d_0(\sigma_0) = h$ and $d_2(\sigma_0) = g$ (it follows that $d_1(\sigma_0) = h \circ g$).
  - There is a unique 2-simplex $\sigma_3$ of $C$ satisfying $d_0(\sigma_3) = g$ and $d_2(\sigma_3) = f$ (it follows that $d_1(\sigma_3) = g \circ f$).
  - There is a unique 2-simplex $\sigma_2$ of $C$ satisfying $d_0(\sigma_2) = h \circ g$ and $d_2(\sigma_2) = f$ (it follows that $d_1(\sigma_2) = (h \circ g) \circ f$).
There is a unique 3-simplex $\tau$ of $\mathcal{C}$ satisfying $d_0(\tau) = \sigma_0$, $d_2(\tau) = \sigma_2$, and $d_3(\tau) = \sigma_3$ (this follows by applying $(\ast')$ to the horn inclusion $\Lambda^3_1 \to \Delta^3$).

The 3-simplex $\tau$ can be depicted in the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{h} \\
W & \xRightarrow{(h \circ g) \circ f} & Z, \\
\end{array}
\]

Set $\sigma_1 = d_1(\tau)$. Then $\sigma_1$ is a 2-simplex of $\mathcal{S}$ satisfying $d_0(\sigma_1) = h$, $d_1(\sigma_1) = (h \circ g) \circ f$, and $d_2(\sigma_1) = g \circ f$. It follows that $\sigma_1$ "witnesses" the identity $h \circ (g \circ f) = (h \circ g) \circ f$.

Note that every $n$-simplex $\sigma : \Delta^n \to \mathcal{S}$ determines a functor $[n] \to \mathcal{C}$, given on objects by the values of $\sigma$ on the vertices of $\Delta^n$ and on morphisms by the values of $\sigma$ on the edges of $\Delta^n$. This construction determines a map of simplicial sets $u : \mathcal{S} \to N_{\ast}(\mathcal{C})$, which is clearly bijective on simplices of dimension $\leq 1$. Since the simplicial sets $\mathcal{S}$ and $N_{\ast}(\mathcal{C})$ both satisfy condition $(\ast')$ (Lemma 1.2.3.2), it follows from Lemma 1.2.3.3 that $u$ is an isomorphism.

Remark 1.2.3.4. The characterization of Proposition 1.2.3.1 has many variants. For example, one can replace condition $(\ast')$ by the following $a$ priori weaker condition:

$(\ast'_0)$ For every $n \geq 2$ and every map of simplicial sets $\sigma_0 : \Lambda^n_1 \to \mathcal{S}$, there exists a unique map $\sigma : \Delta^n \to \mathcal{S}$ satisfying $\sigma_0 = \sigma |_{\Lambda^n_1}$.

1.2.4 The Nerve of a Groupoid

According to Proposition 1.2.2.1 every category $\mathcal{C}$ can be recovered, up to canonical isomorphism, from the nerve $N_{\ast}(\mathcal{C})$. In particular, any isomorphism-invariant condition on a category $\mathcal{C}$ can be reformulated as a condition on the simplicial set $N_{\ast}(\mathcal{C})$. We now illustrate this principle with a simple example.

Definition 1.2.4.1. Let $\mathcal{C}$ be a category. We say that a morphism $f : C \to D$ in $\mathcal{C}$ is an isomorphism if there exists a morphism $g : D \to C$ satisfying the identities

$$f \circ g = \text{id}_D \quad \text{and} \quad g \circ f = \text{id}_C.$$

In this case, the morphism $g$ is uniquely determined and we write $g = f^{-1}$. We say that $\mathcal{C}$ is a groupoid if every morphism in $\mathcal{C}$ is invertible.
Proposition 1.2.4.2. Let \( C \) be a category. Then \( C \) is a groupoid (Definition 1.2.4.1) if and only if the simplicial set \( N_\bullet(C) \) is a Kan complex (Definition 1.1.9.1).

Example 1.2.4.3 (The Milnor Construction). Let \( M \) be a monoid. We can then form a category \( BM \) having a single object \( X \), where \( \text{Hom}_{BM}(X, X) = M \) and the composition of morphisms in \( BM \) is given by multiplication in \( M \). We will denote the nerve of the category \( BM \) by \( B\bullet M \).

In the special case where \( M = G \) is a group, the geometric realization \( |B\bullet G| \) is a topological space called the classifying space of \( G \). It can be characterized (up to homotopy equivalence) by the fact that it is a CW complex with either of the following properties:

- The space \( |B\bullet G| \) is connected, and its homotopy groups (with respect to any choice of base point) are given by the formula
  \[
  \pi_\ast(|B\bullet G|) \simeq \begin{cases} 
  G & \text{if } \ast = 1 \\
  0 & \text{if } \ast > 1.
  \end{cases}
  \]

- For any paracompact topological space \( X \), there is a canonical bijection
  \[
  \{\text{Continuous maps } f : X \to |B\bullet G|}/\text{homotopy} \simeq \{G\text{-torsors } P \to X\}/\text{isomorphism}.
  \]

We refer the reader to [31] for a more detailed discussion (including an extension to the setting of topological groups).

Proof of Proposition 1.2.4.2. Suppose first that \( N_\bullet(C) \) is a Kan complex; we wish to show that \( C \) is a groupoid. Let \( f : C \to D \) be a morphism in \( C \). Using the surjectivity of the map \( \text{Hom}_{\text{Set}_\Delta}(\Delta^2, N_\bullet(C)) \to \text{Hom}_{\text{Set}_\Delta}(\Delta^2_\ast, N_\bullet(C)) \), we see that there exists a 2-simplex \( \sigma \) of \( N_\bullet(C) \) satisfying \( d_0(\sigma) = f \) and \( d_1(\sigma) = \text{id}_D \). Setting \( g = d_2(\sigma) \), we conclude that \( f \circ g = \text{id}_D \); that is, \( g \) is a left inverse to \( f \). Similarly, the surjectivity of the map \( \text{Hom}_{\text{Set}_\Delta}(\Delta^2, N_\bullet(C)) \to \text{Hom}_{\text{Set}_\Delta}(\Delta^2_\ast, N_\bullet(C)) \) allows us to construct a map \( h : D \to C \) satisfying \( h \circ f = \text{id}_C \). The calculation
  \[
  g = \text{id}_C \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ \text{id}_D = h
  \]

then shows that \( g = h \) is an inverse of \( f \), so that \( f \) is invertible as desired.

Now suppose that \( C \) is a groupoid. We wish to show that, for \( 0 \leq i \leq n \), every map \( \sigma_0 : \Delta^n_i \to N_\bullet(C) \) can be extended to an \( n \)-simplex \( \sigma : \Delta^n \to N_\bullet(C) \). For \( 0 < i < n \), this follows from Lemma 1.2.3.2 (and does not require the assumption that \( C \) is a groupoid). We will treat the case where \( i = 0 \); the case \( i = n \) follows by similar reasoning. We consider several cases:
• In the case $n = 1$, the map $\sigma_0 : \Lambda^0_n \to N_\bullet(C)$ can be identified with an object $C \in \mathcal{C}$. In this case, we can take $\sigma$ to be an edge of $N_\bullet(C)$ corresponding to any morphism with target $C$ (for example, we can take $\sigma$ to be the identity map $\text{id}_C$).

• In the case $n = 2$, we can identify $\sigma_0$ with a pair of morphisms in $\mathcal{C}$ having the same source, which we can depict as a diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & E.
\end{array}
\]

Our assumption that $\mathcal{C}$ is a groupoid guarantees that we can extend this diagram to a 2-simplex of $\mathcal{C}$, whose $0$th face is given by the morphism $g \circ f^{-1} : D \to E$.

• In the case $n \geq 3$, the map $\sigma_0$ determines a collection of objects $\{C_i\}_{0 \leq i \leq n}$ and morphisms $f_{j,i} : C_i \to C_j$ for $i \leq j$ (as in the proof of Lemma 1.2.3.2). We wish to show that these morphisms determine a functor $[n] \to \mathcal{C}$ (which we can then identify with an $n$-simplex $\sigma$ of $N_\bullet(C)$ satisfying $\sigma|_{\Lambda^0_n} = \sigma_0$). For this, we must verify the identity $f_{k,j} \circ f_{j,i} = f_{k,i}$ for $0 \leq i \leq j \leq k \leq n$. Note that this identity is satisfied whenever the triple $(i \leq j \leq k)$ determines a 2-simplex of $\Delta^n$ belonging to the horn $\Lambda^0_n$. This is automatic unless $n = 3$ and $(i, j, k) = (1, 2, 3)$. To handle this exceptional case, we compute

\[
(f_{3,2} \circ f_{2,1}) \circ f_{1,0} = f_{3,2} \circ (f_{2,1} \circ f_{1,0}) = f_{3,2} \circ f_{2,0} = f_{3,0} = f_{3,1} \circ f_{1,0}.
\]

Since $\mathcal{C}$ is a groupoid, composing with $f^{-1}_{1,0}$ on the right yields the desired identity $f_{3,2} \circ f_{2,1} = f_{3,1}$.

\[\square\]

We close this section with a general observation regarding the relationship between categories and groupoids.

**Construction 1.2.4.4.** Let $\mathcal{C}$ be a category. We define a subcategory $\mathcal{C}^\simeq \subseteq \mathcal{C}$ as follows:

• Every object of $\mathcal{C}$ belongs to $\mathcal{C}^\simeq$.

• A morphism $f : X \to Y$ of $\mathcal{C}$ belongs to $\mathcal{C}^\simeq$ if and only if $f$ is an isomorphism.

We will refer to $\mathcal{C}^\simeq$ as the *core* of $\mathcal{C}$.
Remark 1.2.4.5. Let $\mathcal{C}$ be a category. The core $\mathcal{C}^\sim$ is determined (up to isomorphism) by the following properties:

- The category $\mathcal{C}^\sim$ is a groupoid.
- If $\mathcal{D}$ is a groupoid, then every functor $F : \mathcal{D} \to \mathcal{C}$ factors (uniquely) through $\mathcal{C}^\sim$.

1.2.5 The Homotopy Category of a Simplicial Set

We now show that the functor $\mathcal{C} \mapsto N_\bullet(\mathcal{C})$ of Construction 1.2.1.1 admits a left adjoint (Corollary 1.2.5.5).

Definition 1.2.5.1. Let $\mathcal{C}$ be a category. We will say that a map of simplicial sets $u : S_\bullet \to N_\bullet(\mathcal{C})$ exhibits $\mathcal{C}$ as the homotopy category of $S_\bullet$ if, for every category $\mathcal{D}$, the composite map

$$\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \to \text{Hom}_{\text{Set}}(N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D})) \xrightarrow{\text{ev}_u} \text{Hom}_{\text{Set}}(S_\bullet, N_\bullet(\mathcal{D}))$$

is bijective (note that the map on the left is always bijective, by virtue of Proposition 1.2.2.1).

Exercise 1.2.5.2. Let $X$ be a topological space and let $\tau_{\leq 1}(X)$ denote its fundamental groupoid. Show that there is a unique map of simplicial sets $u : \text{Sing}_\bullet(X) \to N_\bullet(\pi_{\leq 1}(X))$ with the following properties:

- On 0-simplices, $u$ carries each point $x \in X$ (regarded as a vertex of $\text{Sing}_\bullet(X)$) to itself (regarded as an object of $\pi_{\leq 1}(X)$).
- On 1-simplices, $u$ carries each path $p : [0, 1] \to X$ (regarded as an edge of $\text{Sing}_\bullet(X)$) to its homotopy class $[p]$ (regarded as a morphism of the category $\pi_{\leq 1}(X)$).

Moreover, show that $u$ exhibits the fundamental groupoid $\pi_{\leq 1}(X)$ as a homotopy category of the singular simplicial set $\text{Sing}_\bullet(X)$. For a generalization, see Proposition 1.3.5.7.

Notation 1.2.5.3. Let $S_\bullet$ be a simplicial set. It follows immediately from the definition that if there exists a category $\mathcal{C}$ and a map $u : S_\bullet \to N_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of $S_\bullet$, then the category $\mathcal{C}$ is unique up to isomorphism and depends functorially on $S_\bullet$. To emphasize this dependence, we will refer to $\mathcal{C}$ as the homotopy category of $S_\bullet$ and denote it by $hS_\bullet$.

Proposition 1.2.5.4. Let $S_\bullet$ be a simplicial set. Then there exists a category $\mathcal{C}$ and a map of simplicial sets $u : S_\bullet \to N_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of $S_\bullet$. 
Proof. Let $Q^\bullet$ denote the cosimplicial object of $\text{Cat}$ given by the inclusion $\Delta \hookrightarrow \text{Cat}$. Unwinding the definitions, we see that a homotopy category of $S^\bullet$ can be identified with a realization $|S^\bullet|^Q$, whose existence is a special case of Proposition 1.1.8.22. Alternatively, we can give a direct construction of the homotopy category $hS^\bullet$:

- The objects of $hS^\bullet$ are the vertices of $S^\bullet$.
- Every edge $e$ of $S^\bullet$ determines a morphism $[e]$ in $hS^\bullet$, whose source is the vertex $d_1(e)$ and whose target is the vertex $d_0(e)$.
- The collection of morphisms in $hS^\bullet$ is generated under composition by morphisms of the form $[e]$, subject only to the relations

$$[s_0(x)] = \text{id}_x \text{ for } x \in S_0 \quad [d_1(\sigma)] = [d_0(\sigma)] \circ [d_2(\sigma)] \text{ for } \sigma \in S_2.$$ 

$\square$

**Corollary 1.2.5.5.** The nerve functor $N^\bullet : \text{Cat} \to \text{Set}_\Delta$ admits a left adjoint, given on objects by the construction $S^\bullet \mapsto hS^\bullet$.

**Remark 1.2.5.6.** Let $C$ be a category. Then the counit of the adjunction described in Corollary 1.2.5.5 induces an isomorphism of categories $hN^\bullet(C) \cong C$ (this is a restatement of Proposition 1.2.2.1). In other words, every category $C$ can be recovered as the homotopy category of its nerve $N^\bullet(C)$.

**Warning 1.2.5.7.** Let $S^\bullet$ be a simplicial set. Our proof of Proposition 1.2.5.4 gives a construction of the homotopy category $hS^\bullet$ by generators and relations. The result of this construction is not easy to describe. If $x$ and $y$ are vertices of $S^\bullet$, then every morphism from $x$ to $y$ in $hS^\bullet$ can be represented by a composition

$$[e_n] \circ [e_{n-1}] \circ \cdots \circ [e_1],$$

where $\{e_i\}_{0 \leq i \leq n}$ is a sequence of edges satisfying

$$d_1(e_1) = x \quad d_0(e_i) = d_1(e_{i+1}) \quad d_0(e_n) = y.$$ 

In general, it can be difficult to determine whether or not two such compositions represent the same morphism of $hS^\bullet$ (even for finite simplicial sets, this question is algorithmically undecidable). However, there are two situations in which the homotopy category $hS^\bullet$ admits a simpler description:

- Let $S^\bullet$ be a simplicial set of dimension $\leq 1$, which we can identify with a directed graph $G$ (Proposition 1.1.5.9). In this case, the homotopy category $hS^\bullet$ is generated freely by the vertices and edges of the graph $G$: that is, it can be identified with the path category of $G$ (Proposition 1.2.6.5) which we study in §1.2.6.
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• Let $S_\bullet$ be an $\infty$-category. In this case, every morphism in the homotopy category $\mathcal{C} = hS_\bullet$ can be represented by a single edge of $S_\bullet$, rather than a composition of edges (in other words, the canonical map $u : S_\bullet \to N_\bullet(\mathcal{C})$ is surjective on edges), and two edges of $S_\bullet$ represent the same morphism in $hS_\bullet$ if and only if they are homotopic (Definition 1.3.3.1). This leads to a more explicit description of the homotopy category $\mathcal{C}$ (generalizing Exercise 1.2.5.2) which we will discuss in §1.3.5 (see Proposition 1.3.5.7).

1.2.6 Example: The Path Category of a Directed Graph

Let $S_\bullet$ be a simplicial set of dimension $\leq 1$. In this section, we will show that the homotopy category $hS_\bullet$ of Notation 1.2.5.3 admits a concrete description, which can be conveniently described using the language of directed graphs.

Construction 1.2.6.1 (The Path Category). Let $G$ be a directed graph (Definition 1.1.5.1). For each edge $e \in \text{Edge}(G)$, we let $s(e), t(e) \in \text{Vert}(G)$ denote the source and target of $e$, respectively. If $x$ and $y$ are vertices of $\text{Vert}(G)$, then a path from $x$ to $y$ is a sequence of edges $(e_m, e_{m-1}, \ldots, e_1)$ satisfying

$$s(e_1) = x \quad t(e_i) = s(e_{i+1}) \quad t(e_m) = y,$$

By convention, we regard the empty sequence of edges as a path from each vertex $x \in \text{Vert}(G)$ to itself.

We define a category $\text{Path}[G]$ as follows:

• The objects of $\text{Path}[G]$ are the vertices of $G$.

• For every pair of vertices $x, y \in \text{Vert}(G)$, we let $\text{Hom}_{\text{Path}[G]}(x, y)$ denote the set of all paths $(e_m, \ldots, e_1)$ from $x$ to $y$.

• For every vertex $x \in \text{Vert}(G)$, the identity morphism $\text{id}_x$ in the category $\text{Path}[G]$ is the empty path from $x$ to itself.

• Let $x, y, z \in \text{Vert}(G)$. Then the composition law

$$\circ : \text{Hom}_{\text{Path}[G]}(y, z) \times \text{Hom}_{\text{Path}[G]}(x, y) \to \text{Hom}_{\text{Path}[G]}(x, z)$$

is described by the formula

$$(e_n, \ldots, e_1) \circ (e'_m, \ldots, e'_1) = (e_n, \ldots, e_1, e'_m, \ldots, e'_1).$$

In other words, composition in $\text{Path}[G]$ is given by concatenation of paths.

We will refer to $\text{Path}[G]$ as the path category of the directed graph $G$. 
Example 1.2.6.2. Fix an integer \( n \geq 0 \). Let \( G \) be the directed graph with vertex set \( \text{Vert}(G) = \{v_0, v_1, \ldots, v_n\} \), and edge set \( \text{Edge}(G) = \{e_1, \ldots, e_n\} \), where each edge \( e_i \) has source \( s(e_i) = v_{i-1} \) and target \( t(e_i) = v_i \); we can represent \( G \) graphically by the diagram

\[
\begin{array}{ccccccccc}
  & v_0 & \rightarrow & v_1 & \rightarrow & \cdots & \rightarrow & v_{n-1} & \rightarrow & v_n \\
  e_1 & & e_2 & & e_{n-1} & & e_n & & \\
\end{array}
\]

Let \( v_i \) and \( v_j \) be a pair of vertices of \( G \). Then:

- If \( i \leq j \), there is a unique path from \( v_i \) to \( v_j \), given by the sequence of edges \( (e_j, e_{j-1}, \ldots, e_{i+1}) \).
- If \( i > j \), then there are no paths from \( v_i \) to \( v_j \).

It follows that the path category \( \text{Path}[G] \) is isomorphic to the linearly ordered set \( [n] = \{0 < 1 < 2 < \cdots < n\} \) (regarded as a category).

Example 1.2.6.3. Let \( G \) be a directed graph having a single vertex \( \text{Vert}(G) = \{x\} \). Then the path category \( \text{Path}[G] \) has a single object \( x \), and can therefore be identified with the category \( BM \) associated to the monoid \( M = \text{End}_{\text{Path}[G]}(x) = \text{Hom}_{\text{Path}[G]}(x, x) \) (see Example 1.2.4.3). Note that the elements of \( M \) can be identified with (possibly empty) sequences of elements of the set \( \text{Edge}(G) \), and that the multiplication on \( M \) is given by concatenation of sequences. In other words, \( M \) can be identified with the free monoid generated by the set \( \text{Edge}(M) \) (this identification is not completely tautological: it can be regarded as a special case of Proposition 1.2.6.5 below).

Example 1.2.6.4. Let \( G \) be a directed graph having a single vertex \( \text{Vert}(G) = \{x\} \) and a single edge \( \text{Edge}(G) = \{e\} \) (necessarily satisfying \( s(e) = x = t(e) \)). Then the path category \( \text{Path}[G] \) has a single object \( x \) whose endomorphism monoid \( \text{End}_{\text{Path}[G]}(x) = \text{Hom}_{\text{Path}[G]}(x, x) \) can be identified with the set \( \mathbb{Z}_{\geq 0} \) of nonnegative integers (with monoid structure given by addition).

Let \( G \) be a directed graph, and let \( G_* \) denote the associated 1-dimensional simplicial set (see Proposition 1.1.5.9). Then there is an evident map of simplicial sets \( u : G_* \to N_* (\text{Path}[G]) \), which carries each vertex \( v \in \text{Vert}(G) \) to itself and each edge \( e \in \text{Edge}(G) \) to the path consisting of the single edge \( e \).

Proposition 1.2.6.5. Let \( G \) be a directed graph. Then the map of simplicial sets \( u : G_* \to N_* (\text{Path}[G]) \) exhibits \( \text{Path}[G] \) as the homotopy category of the simplicial set \( G_* \), in the sense of Definition 1.2.5.1. In other words, for every category \( C \), the composite map

\[
\text{Hom}_{\text{Cat}}(\text{Path}[G], C) \to \text{Hom}_{\text{Set}_\Delta} (N_* (\text{Path}[G]), N_* (C)) \xrightarrow{\circ u} \text{Hom}_{\text{Set}_\Delta} (G_*, N_* (C))
\]

is a bijection.
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Proof. Let \( f : G \to N(C) \) be a map of simplicial sets, in the sense of Definition 1.1.5.5. We wish to show that there is a unique functor \( F : \text{Path}[G] \to C \) for which the composite map

\[
G \xrightarrow{u} N(\text{Path}[G]) \xrightarrow{N(F)} N(C)
\]

agrees with \( F \). Unwinding the definitions, we see that this agreement imposes the following requirements on \( F \):

(a) For each vertex \( v \in \text{Vert}(G) \), we have \( F(x) = f(x) \) (as objects of \( C \)).

(b) For each edge \( e \in \text{Edge}(G) \) having \( x = s(e) \) and target \( y = t(e) \), the functor \( F \) carries the path \( (e) \) to the morphism \( f(e) : f(x) \to f(y) \) in \( C \).

The existence and uniqueness of the functor \( F \) is now clear: it is determined on objects by property (a), and on morphisms by the formula

\[
F(e_n, e_{n-1}, \ldots, e_1) = f(e_n) \circ f(e_{n-1}) \circ \cdots \circ f(e_1).
\]

\( \square \)

**Remark 1.2.6.6.** In the proof of Proposition 1.2.6.5, we have implicitly invoked the fact that every category \( C \) satisfies the generalized associative law: every sequence of composable morphisms

\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n
\]

has a well-defined composition \( f_n \circ f_{n-1} \circ \cdots \circ f_1 \), which can be computed in terms of the binary composition law by inserting parentheses arbitrarily. One might object that this logic is circular: the generalized associative law is essentially equivalent to Proposition 1.2.6.5 (applied to the directed graph \( G \) described in Example 1.2.6.2). In §1.4.7, we will establish an \( \infty \)-categorical generalization of Proposition 1.2.6.5 (Theorem 1.4.7.1), whose proof will avoid this sort of circular reasoning (see Remark 1.4.7.4).

**Definition 1.2.6.7.** A category \( C \) is free if it is isomorphic to \( \text{Path}[G] \), for some directed graph \( G \).

We close this section with a characterization of those categories which are free in the sense of Definition 1.2.6.7.

**Definition 1.2.6.8.** Let \( C \) be a category. We will say that a morphism \( f : X \to Y \) in \( C \) is indecomposable if \( f \) is not an identity morphism, and for every factorization \( f = g \circ h \) have either \( g = \text{id}_Y \) (so \( h = f \)) or \( h = \text{id}_X \) (so \( g = f \)).

**Example 1.2.6.9.** Let \( G \) be a directed graph and let \( \overline{e} \) be a morphism in the path category \( \text{Path}[G] \), given by a sequence of edges \( (e_n, e_{n-1}, \ldots, e_1) \) satisfying \( t(e_i) = s(e_{i+1}) \). Then \( \overline{e} \) is indecomposable if and only if \( n = 1 \).
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**Warning 1.2.6.10.** Definitions 1.2.6.7 and 1.2.6.8 are not invariant under equivalence of categories. If \( F : \mathcal{C} \to \mathcal{D} \) is an equivalence of categories and \( \mathcal{C} \) is free, then \( \mathcal{D} \) need not be free; if \( f \) is an indecomposable morphism in \( \mathcal{C} \), then \( F(f) \) need not be an indecomposable morphism of \( \mathcal{D} \).

Let \( \mathcal{C} \) be any category. We define a directed graph \( \text{Gr}_0(\mathcal{C}) \) as follows:

- The vertices of \( \text{Gr}_0(\mathcal{C}) \) are the objects of \( \mathcal{C} \).
- The edges of \( \text{Gr}_0(\mathcal{C}) \) are the indecomposable morphisms of \( \mathcal{C} \) (where an indecomposable morphism \( f : X \to Y \) is regarded as an edge with source \( s(f) = X \) and target \( t(f) = Y \)).

By construction, the graph \( \text{Gr}_0(\mathcal{C}) \) comes equipped with a canonical map \( \text{Gr}_0(\mathcal{C}) \to N_\bullet(\mathcal{C}) \), which we can identify (by means of Proposition 1.2.6.5) with a functor \( F : \text{Path}[\text{Gr}_0(\mathcal{C})] \to \mathcal{C} \).

**Proposition 1.2.6.11.** Let \( \mathcal{C} \) be a category. The following conditions on \( \mathcal{C} \) are equivalent:

(a) The category \( \mathcal{C} \) is free. That is, there exists a directed graph \( G \) and an isomorphism of categories \( \mathcal{C} \simeq \text{Path}[G] \).

(b) The functor \( F : \text{Path}[\text{Gr}_0(\mathcal{C})] \to \mathcal{C} \) an isomorphism of categories.

(c) The functor \( F : \text{Path}[\text{Gr}_0(\mathcal{C})] \to \mathcal{C} \) is an equivalence of categories.

(d) The functor \( F : \text{Path}[\text{Gr}_0(\mathcal{C})] \to \mathcal{C} \) is fully faithful.

(e) Every morphism \( f \) in \( \mathcal{C} \) admits a unique factorization \( f = f_n \circ f_{n-1} \circ \cdots \circ f_1 \), where each \( f_i \) is an indecomposable morphism of \( \mathcal{C} \).

**Proof.** The functor \( F \) is bijective on objects, which shows that (b) \( \iff \) (c) \( \iff \) (d). The equivalence of (d) and (e) follows from the definition of morphisms in the path category \( \text{Path}[\text{Gr}_0(\mathcal{C})] \). The implication (b) \( \Rightarrow \) (a) is immediate, and the converse follows from Example 1.2.6.9. \( \square \)

### 1.3 ∞-Categories

In 1.1 and 1.2 we considered two closely related conditions on a simplicial set \( S_\bullet \):

(∗) For \( n > 0 \) and \( 0 \leq i \leq n \), every map of simplicial sets \( \sigma_0 : \Lambda^n_i \to S_\bullet \) can be extended to a map \( \sigma : \Delta^n \to S_\bullet \).

(∗′) For \( 0 < i < n \), every map of simplicial sets \( \sigma_0 : \Lambda^n_i \to S_\bullet \) can be extended uniquely to a map \( \sigma : \Delta^n \to S_\bullet \).
Simplicial sets satisfying (\(\ast\)) are called Kan complexes and form the basis for a combinatorial approach to homotopy theory, while simplicial sets satisfying (\(\ast'\)) can be identified with categories (Propositions 1.2.2.1 and 1.2.3.1). These notions admit a common generalization:

**Definition 1.3.0.1.** An \(\infty\)-category is a simplicial set \(S\) which satisfies the following condition:

\((\ast'')\) For \(0 < i < n\), every map of simplicial sets \(\sigma_0 : \Lambda^n_i \to S\) can be extended to a map \(\sigma : \Delta^n \to S\).

**Remark 1.3.0.2.** Condition \((\ast'')\) is commonly known as the weak Kan extension condition. It was introduced by Boardman and Vogt in [2], who refer to \(\infty\)-categories as weak Kan complexes. The theory was developed further by Joyal ([21] and [20]), who refers to \(\infty\)-categories as quasicategories.

**Example 1.3.0.3.** Every Kan complex is an \(\infty\)-category. In particular, if \(X\) is a topological space, then the singular simplicial set \(\text{Sing}_\bullet(X)\) is an \(\infty\)-category.

**Example 1.3.0.4.** For every category \(C\), the nerve \(N_\bullet(C)\) is an \(\infty\)-category.

**Remark 1.3.0.5.** We will often abuse terminology by identifying a category \(C\) with its nerve \(N_\bullet(C)\) (this abuse is essentially harmless by virtue of Proposition 1.2.2.1). Adopting this convention, we can state Example 1.3.0.4 more simply: every category is an \(\infty\)-category.

To minimize the possibility of confusion, we will sometimes refer to categories as ordinary categories.

**Example 1.3.0.6** (Products of \(\infty\)-Categories). Let \(\{S_\alpha\}_{\alpha \in A}\) be a collection of simplicial sets parametrized by a set \(A\), and let \(S_\bullet = \prod_{\alpha \in A} S_\alpha\) denote their product. If each \(S_\alpha\) is an \(\infty\)-category, then \(S_\bullet\) is an \(\infty\)-category. The converse holds provided that each \(S_\alpha\) is nonempty.

**Example 1.3.0.7** (Coproducts of \(\infty\)-Categories). Let \(\{S_\alpha\}_{\alpha \in A}\) be a collection of simplicial sets parametrized by a set \(A\), and let \(S_\bullet = \coprod_{\alpha \in A} S_\alpha\) denote their coproduct. For each \(0 < i < n\), the restriction map

\[
\theta : \text{Hom}_{\Delta}(\Delta^n, S_\bullet) \to \text{Hom}_{\Delta}(\Lambda^n_i, S_\bullet)
\]

can be identified with the coproduct (formed in the arrow category \(\text{Fun}([1], \text{Set})\)) of restriction maps \(\theta_\alpha : \text{Hom}_{\Delta}(\Delta^n, S_\alpha) \to \text{Hom}_{\Delta}(\Lambda^n_i, S_\alpha)\). It follows that \(\theta\) is a surjection if and only if each \(\theta_\alpha\) is a surjection. Allowing \(n\) and \(i\) to vary, we conclude that \(S_\bullet\) is an \(\infty\)-category if and only if each connected component of \(S_\bullet\) is an \(\infty\)-category.

**Remark 1.3.0.8.** Let \(S_\bullet\) be a simplicial set. Combining Example 1.3.0.7 with Proposition 1.1.6.13, we deduce that \(S_\bullet\) is an \(\infty\)-category if and only if each connected component of \(S_\bullet\) is an \(\infty\)-category.
CHAPTER 1. THE LANGUAGE OF ∞-CATEGORIES

Remark 1.3.0.9. Suppose we are given a filtered diagram of simplicial sets \( \{ S(\alpha)_\bullet \} \) having colimit \( S_\bullet = \lim_{\longrightarrow} S(\alpha)_\bullet \). If each \( S(\alpha)_\bullet \) is an \( \infty \)-category, then \( S_\bullet \) is an \( \infty \)-category.

Throughout this book, we will generally use calligraphic letters (like \( C \), \( D \), and \( E \)) to denote \( \infty \)-categories, and we will generally describe them using terminology borrowed from category theory. For example, if \( C = S_\bullet \) is an \( \infty \)-category, then we will refer to vertices of the simplicial set \( S_\bullet \) as objects of the \( \infty \)-category \( C \) (see §1.3.1). One of the central themes of this book is that \( \infty \)-categories behave much like ordinary categories. In particular, for any \( \infty \)-category \( C \), there is a notion of composition for morphisms of \( C \), which we study in §1.3.4. Given a pair of morphisms \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) in \( C \) (corresponding to edges \( f, g \in S_1 \) satisfying \( d_0(f) = d_1(g) \)), the pair \( (f, g) \) defines a map of simplicial sets \( \sigma_0 : \Lambda^2_1 \rightarrow C \). Applying condition \((s'')\), we can extend \( \sigma_0 \) to a 2-simplex \( \sigma \) of \( C \), which we can think of heuristically as a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow \\
X & \xrightarrow{h} & Z.
\end{array}
\]

In this case, we will refer to the morphism \( h = d_1(\sigma) \) as a composition of \( f \) and \( g \). However, this comes with a caveat: the extension \( \sigma \) is usually not unique, so the morphism \( h \) is not completely determined by \( f \) and \( g \). However, we will show that it is unique up to a certain notion of homotopy which we study in §1.3.3. We apply this observation in §1.3.5 to give a concrete description of the homotopy category \( hC \) (in the sense of Definition 1.2.5.1) when \( C \) is an \( \infty \)-category (see Definition 1.3.5.3 and Proposition 1.3.5.7).

1.3.1 Objects and Morphisms

We begin by introducing some terminology.

Definition 1.3.1.1. Let \( C = S_\bullet \) be an \( \infty \)-category. An object of \( C \) is a vertex of the simplicial set \( S_\bullet \) (that is, an element of the set \( S_0 \)). A morphism of \( C \) is an edge of the simplicial set \( S_\bullet \) (that is, an element of \( S_1 \)). If \( f \in S_1 \) is a morphism of \( C \), we will refer to the object \( X = d_1(f) \) as the source of \( f \) and to the object \( Y = d_0(f) \) as the target of \( f \). In this case, we will say that \( f \) is a morphism from \( X \) to \( Y \). For any object \( X \) of \( C \), we can regard the degenerate edge \( s_0(X) \) as a morphism from \( X \) to itself; we will denote this morphism by \( \text{id}_X \) and refer to it as the identity morphism of \( X \).

Notation 1.3.1.2. Let \( C \) be an \( \infty \)-category. We will often write \( X \in C \) to indicate that \( X \) is an object of \( C \). We use the phrase “\( f : X \rightarrow Y \) is a morphism of \( C \)” to indicate that \( f \) is a morphism of \( C \) having source \( X \) and target \( Y \).
Example 1.3.1.3. Let $\mathcal{C}$ be an ordinary category, and regard the simplicial set $N_\bullet(\mathcal{C})$ as an $\infty$-category. Then:

- The objects of the $\infty$-category $N_\bullet(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The morphisms of the $\infty$-category $N_\bullet(\mathcal{C})$ are the morphisms of $\mathcal{C}$. Moreover, the source and target of a morphism of $\mathcal{C}$ coincide with the source and target of the corresponding morphism in $N_\bullet(\mathcal{C})$.
- For every object $X \in \mathcal{C}$, the identity morphism $\text{id}_X$ does not depend on whether we view $X$ as an object of the category $\mathcal{C}$ or the $\infty$-category $N_\bullet(\mathcal{C})$.

Example 1.3.1.4. Let $X$ be a topological space, and regard the simplicial set $\text{Sing}_\bullet(X)$ as an $\infty$-category. Then:

- The objects of $\text{Sing}_\bullet(X)$ are the points of $X$.
- The morphisms of $\text{Sing}_\bullet(X)$ are continuous paths $f : [0, 1] \to X$. The source of a morphism $f$ is the point $f(0)$, and the target is the point $f(1)$.
- For every point $x \in X$, the identity morphism $\text{id}_x$ is the constant path $[0, 1] \to X$ taking the value $x$.

1.3.2 The Opposite of an $\infty$-Category

Let $\mathcal{C}$ be an ordinary category. Then we can construct a new category $\mathcal{C}^{\text{op}}$, called the opposite category of $\mathcal{C}$, as follows:

- The objects of the opposite category $\mathcal{C}^{\text{op}}$ are the objects of $\mathcal{C}$.
- For every pair of objects $C, D \in \mathcal{C}$, we have $\text{Hom}_{\mathcal{C}^{\text{op}}}(C, D) = \text{Hom}_\mathcal{C}(D, C)$.
- Composition of morphisms in $\mathcal{C}^{\text{op}}$ is given by the composition of morphisms in $\mathcal{C}$, with the order reversed.

The construction $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$ admits a straightforward generalization to the setting of $\infty$-categories. In fact, it can be extended to arbitrary simplicial sets.

Notation 1.3.2.1. Let $\text{Lin}$ denote the category whose objects are finite linearly ordered sets and whose morphisms are nondecreasing functions. Let $I$ be an object of $\text{Lin}$, regarded as a set with a linear ordering $\leq_I$. We let $I^{\text{op}}$ denote the same set with the opposite ordering, so that

$$(i \leq_{I^{\text{op}}} j) \iff (j \leq_I i).$$

The construction $I \mapsto I^{\text{op}}$ determines an equivalence from the category $\text{Lin}$ to itself.
Recall that the simplex category $\Delta$ of Definition 1.1.1.2 is the full subcategory of $\text{Lin}$ spanned by objects of the form $[n] = \{0 < 1 < \cdots < n\}$, and is equivalent to the full subcategory of $\text{Lin}$ spanned by those linearly ordered sets which are finite and nonempty (Remark 1.1.1.3). There is a unique functor $\text{Op} : \Delta \to \Delta$ for which the diagram

$$
\begin{array}{ccc}
\Delta & \longrightarrow & \text{Lin} \\
\downarrow \text{Op} & & \downarrow I \mapsto I^{\text{op}} \\
\Delta & \longrightarrow & \text{Lin}
\end{array}
$$

commutes up to isomorphism, where the horizontal maps are given by the inclusion. The functor $\text{Op}$ can be described more concretely as follows:

- For each object $[n] \in \Delta$, we have $\text{Op}([n]) = [n]$ (note that the construction $i \mapsto n - i$ determines an isomorphism of $[n]$ with the opposite linear ordering $[n]^{\text{op}}$).
- For each morphism $\alpha : [m] \to [n]$ in $\Delta$, the morphism $\text{Op}(\alpha) : [m] \to [n]$ is given by the formula $\text{Op}(\alpha)(i) = n - \alpha(m - i)$.

**Construction 1.3.2.2.** Let $S_\bullet$ be a simplicial set, which we regard as a functor $\Delta^{\text{op}} \to \text{Set}$. We let $S_\bullet^{\text{op}}$ denote the simplicial set given by the composition

$$
\Delta^{\text{op}} \xrightarrow{\text{Op}} \Delta^{\text{op}} \xrightarrow{S_\bullet} \text{Set},
$$

where $\text{Op}$ is the functor described in Notation 1.3.2.1. We will refer to $S_\bullet^{\text{op}}$ as the opposite of the simplicial set $S_\bullet$.

**Remark 1.3.2.3.** Let $S_\bullet$ be a simplicial set. Then the opposite simplicial set $S_\bullet^{\text{op}}$ can be described more concretely as follows:

- For each $n \geq 0$, we have $S_n^{\text{op}} = S_n$.
- The face and degeneracy maps of $S_\bullet^{\text{op}}$ are given by

$$
(d_i : S_n^{\text{op}} \to S_{n-1}^{\text{op}}) = (d_{n-i} : S_n \to S_{n-1})
$$

$$
(s_i : S_n^{\text{op}} \to S_{n+1}^{\text{op}}) = (s_{n-i} : S_n \to S_{n+1}).
$$

**Example 1.3.2.4.** Let $\mathcal{C}$ be a category. For each $n \geq 0$, we can identify $n$-simplices $\sigma$ of $N_\bullet(\mathcal{C})$ with diagrams

$$
C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n
$$

in the category $\mathcal{C}$. Then $\sigma$ determines an $n$-simplex $\sigma'$ of $N_\bullet(\mathcal{C}^{\text{op}})$, given by the diagram

$$
C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0
$$

in the opposite category $\mathcal{C}^{\text{op}}$. The construction $\sigma \mapsto \sigma'$ determines an isomorphism of simplicial sets $N_\bullet(\mathcal{C})^{\text{op}} \simeq N_\bullet(\mathcal{C}^{\text{op}})$. 

Example 1.3.2.5. Let $X$ be a topological space. Then there is a canonical isomorphism of simplicial sets $\text{Sing}_\bullet(X) \simeq \text{Sing}_\bullet(X)^\text{op}$, which carries each singular $n$-simplex $\sigma : |\Delta^n| \to X$ to the composite map

$$|\Delta^n| \xrightarrow{r} |\Delta^n| \xrightarrow{\sigma} X$$

where $r$ is denotes the homeomorphism of $|\Delta^n|$ with itself given by $r(t_0, t_1, \ldots, t_{n-1}, t_n) = (t_n, t_{n-1}, \ldots, t_1, t_0)$.

Proposition 1.3.2.6. Let $\mathcal{C}$ be an $\infty$-category. Then the opposite simplicial set $\mathcal{C}^{\text{op}}$ is also an $\infty$-category.

Proof. Let $\sigma_0 : \Lambda^n_i \to \mathcal{C}^{\text{op}}$ be a map of simplicial sets for $0 < i < n$; we wish to show that $\sigma_0$ can be extended to an $n$-simplex of $\mathcal{C}^{\text{op}}$. Passing to opposite simplicial sets, we are reduced to showing that the map $\sigma_0^{\text{op}} : (\Lambda^n_i)^{\text{op}} \to \mathcal{C}$ can be extended to a map $(\Delta^n)^{\text{op}} \to \mathcal{C}$. This follows from our assumption that $\mathcal{C}$ is an $\infty$-category, since there is a canonical isomorphism $(\Delta^n)^{\text{op}} \simeq \Delta^n$ which carries the simplicial subset $(\Lambda^n_i)^{\text{op}}$ to $\Lambda^n_{n-i}$. \qed

Remark 1.3.2.7. Let $\mathcal{C}$ be an $\infty$-category. We will refer to the $\infty$-category $\mathcal{C}^{\text{op}}$ of Proposition 1.3.2.6 as the opposite of the $\infty$-category $\mathcal{C}$. Note that:

- The objects of $\mathcal{C}^{\text{op}}$ are the objects of $\mathcal{C}$.
- Given a pair of objects $X, Y \in \mathcal{C}$, the datum of a morphism from $X$ to $Y$ in $\mathcal{C}^{\text{op}}$ is equivalent to the datum of a morphism from $Y$ to $X$ in $\mathcal{C}$.

1.3.3 Homotopies of Morphisms

For any topological space $X$, we can view the singular simplicial set $\text{Sing}_\bullet(X)$ as an $\infty$-category, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a continuous path $f : [0, 1] \to X$ satisfying $f(0) = x$ and $f(1) = y$. For many purposes (for example, in the study of the fundamental group $\pi_1(X, x)$), it is useful to work not with paths but with homotopy classes of paths (having fixed endpoints). This notion can be generalized to an arbitrary $\infty$-category:

Definition 1.3.3.1. Let $\mathcal{C}$ be an $\infty$-category and let $f, g : C \to D$ be a pair of morphisms in $\mathcal{C}$ having the same source and target. A homotopy from $f$ to $g$ is a 2-simplex $\sigma$ of $\mathcal{C}$ satisfying $d_0(\sigma) = \text{id}_D$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as depicted in the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\Big\uparrow f & & \Big\downarrow \text{id}_D \\
\end{array}
$$

We will say that $f$ and $g$ are homotopic if there exists a homotopy from $f$ to $g$. 
Example 1.3.3.2. Let $\mathcal{C}$ be an ordinary category. Then a pair of morphisms $f, g : C \to D$ in $\mathcal{C}$ (having the same source and target) are homotopic as morphisms of the $\infty$-category $N\bullet(C)$ if and only if $f = g$.

Example 1.3.3.3. Let $X$ be a topological space. Suppose we are given points $x, y \in X$ and a pair of continuous paths $f, g : [0, 1] \to X$ satisfying $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then $f$ and $g$ are homotopic as morphisms of the $\infty$-category $\text{Sing}\bullet(C)$ (in the sense of Definition 1.3.3.1) if and only if there exists a continuous function $H : [0, 1] \times [0, 1] \to X$ satisfying

$$H(s, 0) = f(s) \quad H(s, 1) = g(s) \quad H(0, t) = x \quad H(1, t) = y$$

(see Exercise 1.3.3.4 for a more precise statement).

Exercise 1.3.3.4. Let $\pi : [0, 1] \times [0, 1] \to |\Delta^2|$ denote the continuous function given by the formula $\pi(s, t) = (1 - s, (1 - t)s, ts)$. For any topological space $X$, the construction $\sigma \mapsto \sigma \circ \pi$ determines a map from the set $\text{Sing}_2(X)$ of singular 2-simplices of $X$ to the set of all continuous functions $H : [0, 1] \times [0, 1] \to X$. Show that, if $f, g : [0, 1] \to X$ are continuous paths satisfying $f(0) = g(0)$ and $f(1) = g(1)$, then the construction $\sigma \mapsto \sigma \circ \pi$ induces a bijection from the set of homotopies from $f$ to $g$ (in the sense of Definition 1.3.3.1) to the set of continuous functions $H$ satisfying the requirements of Example 1.3.3.3.

Proposition 1.3.3.5. Let $\mathcal{C}$ be an $\infty$-category containing objects $X, Y \in \mathcal{C}$, and let $E$ denote the collection of all morphisms from $X$ to $Y$ in $\mathcal{C}$. Then homotopy is an equivalence relation on $E$.

Proof. We first observe that for any morphism $f : X \to Y$ in $\mathcal{C}$, the degenerate 2-simplex $s_1(f)$ is a homotopy from $f$ to itself. It follows that homotopy is a reflexive relation on $E$.

We will complete the proof by establishing the following:

(*) Let $f, g, h : X \to Y$ be three morphisms from $X$ to $Y$. If $f$ is homotopic to $g$ and $f$ is homotopic to $h$, then $g$ is homotopic to $h$.

Let us first observe that assertion (*) implies Proposition 1.3.3.5. Note that in the special case $f = h$, (*) asserts that if $f$ is homotopic to $g$, then $g$ is homotopic to $f$ (since $f$ is always homotopic to itself). That is, the relation of homotopy is symmetric. We can therefore replace the hypothesis that $f$ is homotopic to $g$ in assertion (*) by the hypothesis that $g$ is homotopic to $f$, so that (*) is equivalent to the transitivity of the relation of homotopy.

It remains to prove (*). Let $\sigma_2$ and $\sigma_3$ be 2-simplices of $\mathcal{C}$ which are homotopies from $f$ to $h$ and $f$ to $g$, respectively, and let $\sigma_0$ be the 2-simplex given by the constant map $\Delta^2 \to \Delta^0 \overset{Y}{\to} \mathcal{C}$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$.
(see Exercise 1.1.2.14), depicted informally by the diagram

![Diagram]

here the dotted arrows represent the boundary of the “missing” face of the horn $\Lambda^3_1$. Our hypothesis that $\mathcal{C}$ is an $\infty$-category guarantees that $\tau_0$ can be extended to a 3-simplex $\tau$ of $\mathcal{C}$. We can then regard the face $d_1(\tau)$ as a homotopy from $g$ to $h$. \qedhere

Note that there is a potential asymmetry in Definition 1.3.3.1 if $f, g : X \to Y$ are two morphisms in an $\infty$-category $\mathcal{C}$, then the datum of a homotopy from $f$ to $g$ in the $\infty$-category $\mathcal{C}$ is not equivalent to the datum of a homotopy from $f$ to $g$ in the opposite $\infty$-category $\mathcal{C}^{op}$. Nevertheless, we have the following:

**Proposition 1.3.3.6.** Let $\mathcal{C}$ be an $\infty$-category, and let $f, g : X \to Y$ be morphisms of $\mathcal{C}$ having the same source and target. Then $f$ and $g$ are homotopic if and only if they are homotopic when regarded as morphisms of the opposite $\infty$-category $\mathcal{C}^{op}$. In other words, the following conditions are equivalent:

1. There exists a 2-simplex $\sigma$ of $\mathcal{C}$ satisfying $d_0(\sigma) = \text{id}_Y$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as depicted in the diagram

   ![Diagram]

2. There exists a 2-simplex $\tau$ of $\mathcal{C}$ satisfying $d_0(\tau) = f$, $d_1(\tau) = g$, and $d_2(\tau) = \text{id}_X$, as depicted in the diagram

   ![Diagram]

**Proof.** We will show that (1) implies (2); the proof of the reverse implication is similar. Assume that $f$ is homotopic to $g$. Since the relation of homotopy is symmetric (Proposition 1.3.3.5), it follows that $g$ is also homotopic to $f$. Let $\sigma$ be a homotopy from $g$ to $f$. Then we
can regard the tuple of 2-simplices \((\sigma, s_1(g), \bullet, s_0(g))\) as a map of simplicial sets \(\rho_0 : \Lambda^3_2 \to C\) (see Exercise 1.1.2.14), depicted informally in the diagram

where the dotted arrows indicate the boundary of the “missing” face of the horn \(\Lambda^3_2\). Using our assumption that \(C\) is an \(\infty\)-category, we can extend \(\rho_0\) to a 3-simplex \(\rho\) of \(C\). Then the face \(\tau = d_2(\rho)\) has the properties required by (2).

Using Proposition 1.3.3.6, we can formulate the notion of homotopy in a more symmetric form:

**Corollary 1.3.3.7.** Let \(C\) be an \(\infty\)-category, and let \(f, g : X \to Y\) be morphisms of \(C\) having the same source and target. Then \(f\) and \(g\) are homotopic (in the sense of Definition 1.3.3.1) if and only if there exists a map of simplicial sets \(H : \Delta^1 \times \Delta^1 \to X\) satisfying \(H|_{\{0\} \times \Delta^1} = f\), \(H|_{\Delta^1 \times \{0\}} = \text{id}_X\), and \(H|_{\Delta^1 \times \{1\}} = \text{id}_Y\), as indicated in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & Y \\
\downarrow_{\text{id}_X} & \downarrow_{h} & \downarrow_{\text{id}_Y} \\
X & \xrightarrow{g} & Y
\end{array}
\]

Proof. The “only if” direction is clear: if \(\sigma\) is a homotopy from \(f\) to \(g\) (in the sense of Definition 1.3.3.1), then we can extend \(\sigma\) to a map \(H : \Delta^1 \times \Delta^1 \to C\) by taking \(\tau\) to be the degenerate simplex \(s_0(g)\). Conversely, suppose that there exists a map \(\Delta^1 \times \Delta^1 \to C\), as
indicated in the diagram

\[
\begin{array}{c}
X \\
\downarrow^{\text{id}_X} \quad \downarrow^{\text{id}_Y} \\
X & \xrightarrow{f} & Y \\
\downarrow^{h} & \quad \downarrow^{\text{id}_Y} \\
Y \\
\end{array}
\]

Then the 2-simplex $\sigma$ is a homotopy from $f$ to $h$, and the 2-simplex $\tau$ guarantees that $g$ is homotopic to $h$ (by virtue of Proposition 1.3.3.6). Since homotopy is an equivalence relation (Proposition 1.3.3.5), it follows that $f$ is homotopic to $g$.

\section{Composition of Morphisms}

We now introduce a notion of composition for morphisms in an $\infty$-category.

\begin{definition}
Let $\mathcal{C}$ be an $\infty$-category. Suppose we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. We will say that $h$ is a composition of $f$ and $g$ if there exists a 2-simplex $\sigma$ of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$, and $d_2(\sigma) = f$. In this case, we will also say that the 2-simplex $\sigma$ witnesses $h$ as a composition of $f$ and $g$.
\end{definition}

Beware that, in the situation of Definition 1.3.4.1, the morphism $h$ is not determined by $f$ and $g$. However, it is determined up to homotopy:

\begin{proposition}
Let $\mathcal{C}$ be an $\infty$-category containing morphisms $f : X \to Y$ and $g : Y \to Z$. Then:
\begin{enumerate}
\item There exists a morphism $h : X \to Z$ which is a composition of $f$ and $g$.
\item Let $h : X \to Z$ be a composition of $f$ and $g$, and let $h' : X \to Z$ be another morphism in $\mathcal{C}$ having the same source and target. Then $h'$ is a composition of $f$ and $g$ if and only if $h'$ is homotopic to $h$.
\end{enumerate}
\end{proposition}

\begin{proof}
The tuple $(g, \bullet, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$ (Exercise 1.1.2.14). Since $\mathcal{C}$ is an $\infty$-category, we can extend $\sigma_0$ to a 2-simplex $\sigma$ of $\mathcal{C}$. Then $\sigma$ witnesses the morphism $h = d_1(\sigma)$ as a composition of $f$ and $g$. This proves (1). To prove (2), let us first suppose that $h' : X \to Z$ is some other morphism in $\mathcal{C}$ which is a composition of $f$ and
We will show that $h$ is homotopic to $h'$. Choose a 2-simplex $\sigma'$ which witnesses $h'$ as a composition of $f$ and $g$. Then the tuple $(s_1(g), \bullet, \sigma', \sigma)$ determines a morphism of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram

![Diagram](https://example.com/diagram.png)

where the dotted arrows indicate the boundary of the “missing” face of the horn $\Lambda^3_1$. Using our assumption that $\mathcal{C}$ is an $\infty$-category, we can extend $\tau_0$ to a 3-simplex $\tau$ of $\mathcal{C}$. Then the face $d_1(\tau)$ is a homotopy from $h$ to $h'$.

We now prove the converse. Let $\sigma$ be a 2-simplex of $\mathcal{C}$ which witnesses $h$ as a composition of $f$ and $g$, and let $h' : X \to Z$ be a morphism of $\mathcal{C}$ which is homotopic to $h$. Let $\sigma''$ be a 2-simplex of $\mathcal{C}$ which is a homotopy from $h$ to $h'$. Then the tuple $(s_1(g), \sigma'', \bullet, \sigma)$ determines a map of simplicial sets $\rho_0 : \Lambda^3_2 \to \mathcal{C}$ (Exercise 1.1.2.14), which we depict informally as a diagram

![Diagram](https://example.com/diagram.png)

Our assumption that $\mathcal{C}$ is an $\infty$-category guarantees that we can extend $\rho_0$ to a 3-simplex $\rho$ of $\mathcal{C}$. Then the face $d_2(\rho)$ witnesses $h'$ as a composition of $f$ and $g$. $\Box$

**Notation 1.3.4.3.** Let $\mathcal{C}$ be an $\infty$-category and let $f : X \to Y$ and $g : Y \to Z$ be a pair of morphisms in $\mathcal{C}$. We will write $h = g \circ f$ to indicate that $h$ is a composition of $f$ and $g$ (in the sense of Definition 1.3.4.1). In this case, it should be implicitly understood that we have chosen a 2-simplex that witnesses $h$ as a composition of $f$ and $g$. We will sometimes abuse terminology by referring to $h$ as the composition of $f$ and $g$. However, the reader should beware that only the homotopy class of $h$ is well-defined (Proposition 1.3.4.2).

**Example 1.3.4.4.** Let $\mathcal{C}$ be an ordinary category containing a pair of morphisms $f : X \to Y$ and $g : Y \to Z$. Then there is a unique morphism $h : X \to Z$ in the $\infty$-category $N_\bullet(\mathcal{C})$ which is a composition of $f$ and $g$, given by the usual composition $g \circ f$ in the category $\mathcal{C}$.
Example 1.3.4.5. Let $X$ be a topological space and suppose we are given continuous paths $f, g : [0, 1] \to X$ which are composable in the sense that $f(1) = g(0)$, and let $g \star f : [0, 1] \to X$ denote the path obtained by concatenating $f$ and $g$, given concretely by the formula

$$(g \star f)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2 \\ g(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}.$$

Then $g \star f$ is a composition of $f$ and $g$ in the $\infty$-category $\text{Sing}_\bullet(X)$. More precisely, the continuous map

$$\sigma : |\Delta^2| \to X \quad \sigma(t_0, t_1, t_2) = \begin{cases} f(t_1 + 2t_2) & \text{if } t_0 \geq t_2 \\ g(t_2 - t_0) & \text{if } t_0 \leq t_2 \end{cases}.$$

can be regarded as a 2-simplex of $\text{Sing}_\bullet(X)$ which witnesses $g \star f$ as a composition of $f$ and $g$.

Warning 1.3.4.6. In the situation of Example 1.3.4.5, the concatenation $g \star f$ is not the only path which is a composition of $f$ and $g$ in the $\infty$-category $\text{Sing}_\bullet(X)$. Any path in $X$ which is homotopic to $g \star f$ (with endpoints fixed) has the same property, by virtue of Proposition 1.3.4.2 (and Example 1.3.3.3). For example, we can replace $g \star f$ by a reparametrization, such as the path

$$(s \in [0, 1]) \mapsto \begin{cases} f(3s) & \text{if } 0 \leq s \leq 1/3 \\ g\left(\frac{3}{2}s - \frac{1}{2}\right) & \text{if } 1/3 \leq s \leq 1. \end{cases}$$

When viewing $\text{Sing}_\bullet(X)$ as an $\infty$-category, all of these paths have an equal claim to be regarded as “the” composition of $f$ and $g$.

We now show that composition respects the relation of homotopy:

Proposition 1.3.4.7. Let $C$ be an $\infty$-category. Suppose we are given a pair of homotopic morphisms $f, f' : X \to Y$ in $C$ and a pair of homotopic morphisms $g, g' : Y \to Z$ in $C$. Let $h$ be a composition of $f$ and $g$, and let $h'$ be a composition of $f'$ and $g'$. Then $h$ is homotopic to $h'$.

Proof. Let $h''$ be a composition of $f$ and $g'$. Since homotopy is an equivalence relation (Proposition 1.3.3.5), it will suffice to show that both $h$ and $h'$ are homotopic to $h''$. We will show that $h$ is homotopic to $h''$; the proof that $h'$ is homotopic to $h''$ is similar. Let $\sigma_3$ be a 2-simplex of $C$ which witnesses $h$ as a composition of $f$ and $g$, let $\sigma_2$ be a 2-simplex of $C$ which witnesses $h''$ as a composition of $f$ and $g'$, and let $\sigma_0$ be a 2-simplex of $C$ which is a homotopy from $g$ to $g'$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets...
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$\tau_0 : \Lambda^3_1 \to C$ (Exercise \[1.1.2.14\]), which we depict informally as a diagram

\[
\begin{array}{c}
\Lambda^3_1 \\
\downarrow \downarrow \downarrow \\
\Lambda^2_1 \\
\downarrow \downarrow \downarrow \\
\Lambda^1_1
\end{array}
\]

where the dotted arrows indicate the boundary of the “missing” face of the horn $\Lambda^3_1$. Using our assumption that $C$ is an $\infty$-category, we can extend $\tau_0$ to a 3-simplex $\tau$ of $C$. Then the face $d_1(\tau)$ is a homotopy from $h$ to $h''$. \qed

1.3.5 The Homotopy Category of an $\infty$-Category

To any topological space $X$, one can associate a category $\pi_{\leq 1}(X)$, called the fundamental groupoid of $X$. This category can be described informally as follows:

- The objects of $\pi_{\leq 1}(X)$ are the points of $X$.
- Given a pair of points $x, y \in X$, we can identify $\text{Hom}_{\pi_{\leq 1}(X)}(x, y)$ with the set of homotopy classes of continuous paths $p : [0, 1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.
- Composition in $\pi_{\leq 1}(X)$ is given by concatenation of paths (see Example \[1.3.4.5\]).

All of the concepts needed to define the fundamental groupoid $\pi_{\leq 1}(X)$ (such as points, paths, homotopies, and concatenation) can be formulated in terms of singular $n$-simplices of $X$ (for $n \leq 2$). Consequently, one can view the fundamental groupoid $\pi_{\leq 1}(X)$ as an invariant of the simplicial set $\text{Sing}_{\ast}(X)$, rather than the topological space $X$. In this section, we describe an extension of this invariant, where the simplicial set $\text{Sing}_{\ast}(X)$ is replaced by an arbitrary $\infty$-category $C$. In this case, the fundamental groupoid $\pi_{\leq 1}(X)$ is replaced by a category $hC$ which we call the homotopy category of $C$ (beware that the homotopy category $hC$ is generally not a groupoid: in fact, we will later see that it is a groupoid if and only if $C$ is a Kan complex (Proposition \[4.4.2.1\]).

Construction 1.3.5.1. Let $C$ be an $\infty$-category. For every pair of objects $X, Y \in C$, we let $\text{Hom}_{hC}(X, Y)$ denote the set of homotopy classes of morphisms from $X$ to $Y$ in $C$. For every morphism $f : X \to Y$, we let $[f]$ denote its equivalence class in $\text{Hom}_{hC}(X, Y)$.

It follows from Propositions \[1.3.4.2\] and \[1.3.4.7\] that, for every triple of objects $X, Y, Z \in C$, there is a unique composition law

$$
\circ : \text{Hom}_{hC}(Y, Z) \times \text{Hom}_{hC}(X, Y) \to \text{Hom}_{hC}(X, Z)
$$
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satisfying the identity \([g] \circ [f] = [h]\) whenever \(h : X \to Z\) is a composition of \(f\) and \(g\) in the ∞-category \(C\).

Proposition 1.3.5.2. Let \(C\) be an ∞-category. Then:

1. The composition law of Construction 1.3.5.1 is associative. That is, for every triple of composable morphisms \(f : W \to X\), \(g : X \to Y\), and \(h : Y \to Z\) in \(C\), we have an equality \(([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])\) in \(\text{Hom}_{hC}(W, Z)\).

2. For every object \(X \in C\), the homotopy class \([\text{id}_X]\) ∈ \(\text{Hom}_{hC}(X, X)\) is a two-sided identity with respect to the composition law of Construction 1.3.5.1. That is, for every morphism \(f : W \to X\) in \(C\) and every morphism \(g : X \to Y\) in \(C\), we have \([\text{id}_X] \circ [f] = [f]\) and \([g] \circ [\text{id}_X] = [g]\).

Proof. We first prove (1). Let \(u : W \to Y\) be a composition of \(f\) and \(g\), let \(v : X \to Z\) be a composition of \(g\) and \(h\), and let \(w : W \to Z\) be a composition of \(f\) and \(v\). Then \(([h] \circ [g]) \circ [f] = [w]\) and \([h] \circ ([g] \circ [f]) = [h] \circ [u]\). It will therefore suffice to show that \(w\) is a composition of \(u\) and \(h\). Choose a 2-simplex \(\sigma_0\) of \(C\) which witnesses \(v\) as a composition of \(g\) and \(h\), a 2-simplex \(\sigma_2\) of \(C\) which witnesses \(w\) as a composition of \(f\) and \(v\), and a 2-simplex \(\sigma_3\) of \(C\) which witnesses \(u\) as a composition of \(f\) and \(g\). Then the sequence \((\sigma_0, \bullet, \sigma_2, \sigma_3)\) determines a map of simplicial sets \(\tau_0 : \Delta^3 \to C\) (Exercise 1.1.2.14), which we depict informally as a diagram

```
X  u  Y
  |  v  |
/  f   \
W ---|--- Z
  h   |
```

Using our assumption that \(C\) is an ∞-category, we can extend \(\tau_0\) to a 3-simplex \(\tau\) of \(C\). Then the 2-simplex \(d_1(\tau)\) witnesses \(w\) as a composition of \(u\) and \(h\).

We now prove (2). Fix an object \(X \in C\) and a morphism \(g : X \to Y\) in \(C\); we will show that \([g] \circ [\text{id}_X] = [g]\) (the analogous identity \([\text{id}_X] \circ [f] = [f]\) follows by a similar argument). For this, it suffices to observe that the degenerate 2-simplex \(s_0(g)\) witnesses \(g\) as a composition of \(\text{id}_X\) and \(g\).

Definition 1.3.5.3 (The Homotopy Category). Let \(C\) be an ∞-category. We define a category \(hC\) as follows:

• The objects of \(hC\) are the objects of \(C\).
• For every pair of objects $X, Y \in \mathcal{C}$, we let $\text{Hom}_{\mathcal{C}}(X, Y)$ denote the collection of homotopy classes of morphisms from $X$ to $Y$ in the $\infty$-category $\mathcal{C}$ (as in Construction 1.3.5.1).

• For every object $X \in \mathcal{C}$, the identity morphism from $X$ to itself in $h\mathcal{C}$ is given by the homotopy class $[\text{id}_X]$.

• Composition of morphisms is defined as in Construction 1.3.5.1.

We will refer to $h\mathcal{C}$ as the homotopy category of the $\infty$-category $\mathcal{C}$.

Example 1.3.5.4. Let $\mathcal{C}$ be an ordinary category. Then the homotopy category of the $\infty$-category $N\bullet(\mathcal{C})$ can be identified with $\mathcal{C}$. In particular, for each $n \geq 0$, the homotopy category $h\Delta^n$ can be identified with $[n] = \{0 < 1 < \cdots < n\}$.

Example 1.3.5.5. Let $X$ be a topological space, and regard the singular simplicial set $\text{Sing}\bullet(X)$ as an $\infty$-category. Then the homotopy category $h\text{Sing}\bullet(X)$ can be identified with the fundamental groupoid $\pi_{\leq 1}(X)$. More precisely, we can regard the contents of §1.3, when specialized to $\infty$-categories of the form $\text{Sing}\bullet(X)$, as providing a construction of the fundamental groupoid of $X$. By virtue of Exercise 1.3.3.4 and Example 1.3.4.5 the resulting category $h\text{Sing}\bullet(X)$ matches the informal description of $\pi_{\leq 1}(X)$ given in the introduction to §1.3.5.

Let $\mathcal{C}$ be an $\infty$-category. Beware that we have now introduced two different definitions of the homotopy category $h\mathcal{C}$:

• The homotopy category $h\mathcal{C}$ of Definition 1.3.5.3 defined by an explicit construction using the assumption that $\mathcal{C}$ is an $\infty$-category.

• The homotopy category $h\mathcal{C}$ of Notation 1.2.5.3 defined for any arbitrary simplicial set $S\bullet$ in terms of a universal mapping property.

We conclude this section by showing that these definitions are equivalent (Proposition 1.3.5.7).

Construction 1.3.5.6. Let $\mathcal{C}$ be an $\infty$-category and let $\sigma : \Delta^n \to \mathcal{C}$ be an $n$-simplex of $\mathcal{C}$. For $0 \leq i \leq n$, let $C_i$ denote the object of $\mathcal{C}$ given by the image of the $i$th vertex of $\Delta^n$. For $0 \leq i \leq j \leq n$, let $f_{ij} : C_i \to C_j$ denote the image under $\sigma$ of the edge of $\Delta^n$ joining the $i$th vertex to the $j$th vertex, and let $[f_{ij}] \in \text{Hom}_{h\mathcal{C}}(C_i, C_j)$ denote the homotopy class of $f_{ij}$. Then we can regard $([C_i])_{0 \leq i \leq n}, \{[f_{ij}]\}_{0 \leq i \leq j \leq n}$ as a functor from the linearly ordered set $[n]$ to the homotopy category $h\mathcal{C}$. Let $u(\sigma)$ denote the corresponding $n$-simplex of $N\bullet(h\mathcal{C})$. Then the construction $\sigma \mapsto u(\sigma)$ determines a map of simplicial sets

$$u : \mathcal{C} \to N\bullet(h\mathcal{C}).$$
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The comparison map of Construction 1.3.5.6 has the following universal property:

**Proposition 1.3.5.7.** Let \( \mathcal{C} \) be an \( \infty \)-category and let \( u : \mathcal{C} \to N_\bullet(h\mathcal{C}) \) be as in Construction 1.3.5.6. Then \( u \) exhibits \( h\mathcal{C} \) as a homotopy category of the simplicial set \( \mathcal{C} \), in the sense of Definition 1.2.5.1. In other words, for every category \( \mathcal{D} \), the composite map

\[
\text{Hom}_{\text{Cat}}(h\mathcal{C}, \mathcal{D}) \to \text{Hom}_{\text{Set}_\Delta}(N_\bullet(h\mathcal{C}), N_\bullet(\mathcal{D})) \xrightarrow{\cong} \text{Hom}_{\text{Set}_\Delta}(\mathcal{C}, N_\bullet(\mathcal{D}))
\]

is a bijection.

**Proof.** Let \( F : \mathcal{C} \to N_\bullet(\mathcal{D}) \) be a map of simplicial sets. Then \( F \) induces a functor of homotopy categories \( G : h\mathcal{C} \to hN_\bullet(\mathcal{D}) \simeq \mathcal{D} \) (where the second identification comes from Example 1.3.5.4). By construction, the map of simplicial sets

\[
\mathcal{C} \xrightarrow{u} N_\bullet(h\mathcal{C}) \xrightarrow{N_\bullet(G)} N_\bullet(\mathcal{D})
\]

agrees with \( F \) on the vertices and edges of \( \mathcal{C} \), and therefore coincides with \( F \) (since a simplex of \( N_\bullet(\mathcal{D}) \) is determined by its 1-dimensional facets; see Remark 1.2.1.3). We leave it to the reader to verify that \( G \) is the unique functor with this property. \( \square \)

1.3.6 **Isomorphisms**

Recall that a morphism \( f : X \to Y \) in a category \( \mathcal{C} \) is an *isomorphism* if there exists a morphism \( g : Y \to X \) satisfying \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \). This notion has an \( \infty \)-categorical analogue:

**Definition 1.3.6.1.** Let \( \mathcal{C} \) be an \( \infty \)-category and let \( f : X \to Y \) be a morphism of \( \mathcal{C} \). We will say that \( f \) is an *isomorphism* if the homotopy class \([f]\) is an isomorphism in the homotopy category \( h\mathcal{C} \). We will say that two objects \( X, Y \in \mathcal{C} \) are *isomorphic* if there exists an isomorphism from \( X \) to \( Y \) (that is, if \( X \) and \( Y \) are isomorphic as objects of the homotopy category \( h\mathcal{C} \)).

**Example 1.3.6.2.** Let \( \mathcal{C} \) be an ordinary category. Then a morphism \( f : X \to Y \) of \( \mathcal{C} \) is an isomorphism if and only if it is an isomorphism when regarded as a morphism of the \( \infty \)-category \( N_\bullet(\mathcal{C}) \).

**Remark 1.3.6.3** (Two-out-of-three). Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms in an \( \infty \)-category \( \mathcal{C} \) and let \( h \) be a composition of \( f \) and \( g \). If any two of the morphisms \( f, g, \) and \( h \) is an isomorphism, then so is the third.

**Definition 1.3.6.4.** Let \( \mathcal{C} \) be an \( \infty \)-category and suppose we are given a pair of morphisms \( f : X \to Y \) and \( g : Y \to X \) in \( \mathcal{C} \). We say that \( g \) is a *left homotopy inverse* of \( f \) if the identity morphism \( \text{id}_X \) is a composition of \( f \) and \( g \): that is, if we have an equality \([\text{id}_X] = [g] \circ [f]\) in
the homotopy category $h\mathcal{C}$. We say that $g$ is a right homotopy inverse of $f$ if the identity morphism $\text{id}_Y$ is a composition of $g$ and $f$: that is, if we have an equality $[\text{id}_Y] = [f] \circ [g]$ in the homotopy category $h\mathcal{C}$. We will say that $g$ is a homotopy inverse of $f$ if it is both a left and a right homotopy inverse of $f$.

**Remark 1.3.6.5.** Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an $\infty$-category $\mathcal{C}$. Then the condition that $g$ is a left homotopy inverse (right homotopy inverse, homotopy inverse) to $f$ depends only on the homotopy classes $[f]$ and $[g]$.

**Remark 1.3.6.6.** Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an $\infty$-category $\mathcal{C}$. Then $g$ is left homotopy inverse to $f$ if and only if $f$ is right homotopy inverse to $g$. Both of these conditions are equivalent to the existence of a 2-simplex $\sigma$ of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = \text{id}_X$, and $d_2(\sigma) = f$, as depicted in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & Y \\
\downarrow^{f} & & \downarrow^{g} \\
Y & \xrightarrow{g} & X
\end{array}
$$

**Remark 1.3.6.7.** Let $f : X \to Y$ be a morphism in an $\infty$-category $\mathcal{C}$. Suppose that $f$ admits a left homotopy inverse $g$ and a right homotopy inverse $h$. Then $g$ and $h$ are homotopic: this follows from the calculation

$$
[g] = [g] \circ [\text{id}_Y] = [g] \circ ([f] \circ [h]) = ([g] \circ [f]) \circ [h] = [\text{id}_Y] \circ [h] = [h].
$$

It follows that both $g$ and $h$ are homotopy inverse to $f$.

**Remark 1.3.6.8.** Let $f : X \to Y$ be a morphism in the $\infty$-category $\mathcal{C}$. It follows from Remark 1.3.6.7 that the following conditions are equivalent:

(1) The morphism $f$ is an isomorphism.

(2) The morphism $f$ admits a homotopy inverse $g$.

(3) The morphism $f$ admits both left and right homotopy inverses.

In this case, the morphism $g$ is uniquely determined up to homotopy; moreover, any left or right homotopy inverse of $f$ is homotopic to $g$. We will sometimes abuse notation by writing $f^{-1}$ to denote a homotopy inverse to $f$.

**Warning 1.3.6.9.** Let $f : X \to Y$ be a morphism in an $\infty$-category $\mathcal{C}$, and suppose that $g, h : Y \to X$ are left homotopy inverses to $f$. If $f$ does not admit a right homotopy inverse, then $g$ and $h$ need not be homotopic.
Proposition 1.3.6.10 (Two-out-of-Six). Let $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ be morphisms in an $\infty$-category $C$. If the morphisms $g \circ f$ and $h \circ g$ are isomorphisms, then $f$, $g$, and $h$ are also isomorphisms.

Proof. Let $u$ be a homotopy inverse to $g \circ f$. Then the iterated composition $g \circ (f \circ u)$ is homotopic to the identity, so that $g$ admits a right homotopy inverse. Similarly, $g$ admits a left homotopy inverse. It follows that $g$ is an isomorphism (Remark 1.3.6.7). Since $f \circ u$ is a right homotopy inverse to $g$, it is homotopy inverse to $g$ (Remark 1.3.6.7), and is therefore also an isomorphism. Applying Remark 1.3.6.3, we conclude that $f$ is also an isomorphism. A similar argument shows that $h$ is an isomorphism. □

Proposition 1.3.6.11. Let $C$ be a Kan complex. Then every morphism in $C$ is an isomorphism.

Remark 1.3.6.12. We will see later that the converse to Proposition 1.3.6.11 is also true: if $C$ is an $\infty$-category in which every morphism is an isomorphism, then $C$ is a Kan complex (Proposition 4.4.2.1).

Proof of Proposition 1.3.6.11. Let $f : X \to Y$ be a morphism in $C$. Then the tuple $(\bullet, \text{id}_X, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_0 \to C$ (Exercise 1.1.2.14), which we depict as

![Diagram](image)

If $C$ is a Kan complex, then we can extend $\sigma_0$ to a 2-simplex $\sigma$ of $C$. Then $\sigma$ exhibits the morphism $g = d_0(\sigma)$ as a left homotopy inverse to $f$. A similar argument shows that $f$ admits a right homotopy inverse, so that $f$ is an isomorphism by virtue of Remark 1.3.6.8. □

Definition 1.3.6.13 (The Fundamental Groupoid of a Kan Complex). Let $S_{\bullet}$ be a Kan complex. It follows from Proposition 1.3.6.11 that the homotopy category $\text{h}S_{\bullet}$ of Definition 1.3.5.3 is a groupoid. We will denote this groupoid by $\pi_{\leq 1}(S_{\bullet})$ and refer to it as the fundamental groupoid of $S_{\bullet}$.

Remark 1.3.6.14. Let $S_{\bullet}$ be a Kan complex. By definition, the objects of the fundamental groupoid $\pi_{\leq 1}(S_{\bullet})$ are the vertices of $S_{\bullet}$, and a pair of vertices $x, y \in S_0$ are isomorphic in $\pi_{\leq 1}(S_{\bullet})$ if and only if there exists an edge $e : x \to y$ in $S_{\bullet}$. Applying Proposition 1.1.9.10, we deduce that $x, y \in S_0$ are isomorphic if and only if they belong to the same connected component of $S_{\bullet}$. In other words, we have a canonical bijection

$$\pi_0(S_{\bullet}) \simeq \{\text{Objects of } \pi_{\leq 1}(S_{\bullet})\}/\text{isomorphism}.$$
Example 1.3.6.15. Let $X$ be a topological space. Then the singular simplicial set $\text{Sing} \cdot(X)$ is a Kan complex (Proposition 1.1.9.8), and its fundamental groupoid $\pi_{\leq 1}(\text{Sing} \cdot(X))$ can be identified with the usual fundamental groupoid $\pi_{\leq 1}(X)$ of the topological space $X$ (where objects are the points of $X$ and morphisms are given by homotopy classes of paths in $X$).

1.4 Functors of $\infty$-Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $N \cdot(\mathcal{C})$ and $N \cdot(\mathcal{D})$ denote the corresponding $\infty$-categories. According to Proposition 1.2.2.1, the nerve functor $N \cdot$ induces a bijection

$$\{ \text{Functors } F : \mathcal{C} \to \mathcal{D} \} \simeq \{ \text{Maps of simplicial sets } N \cdot(\mathcal{C}) \to N \cdot(\mathcal{D}) \}.$$ 

Consequently, the notion of functor admits an obvious generalization to the setting of $\infty$-categories:

Definition 1.4.0.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories. A functor from $\mathcal{C}$ to $\mathcal{D}$ is a map of simplicial sets $F : \mathcal{C} \to \mathcal{D}$.

This section is devoted to the study of functors between $\infty$-categories, in the sense of Definition 1.4.0.1. We begin in §1.4.1 with some simple examples, which illustrate the meaning of Definition 1.4.0.1 in the case of $\infty$-categories which arise from ordinary categories (via the construction $\mathcal{E} \mapsto N \cdot(\mathcal{E})$) or topological spaces (via the construction $X \mapsto \text{Sing} \cdot(X)$).

In ordinary category theory, one can think of a functor $F : \mathcal{C} \to \mathcal{D}$ as a kind of commutative diagram in $\mathcal{D}$, having vertices indexed by the objects of $\mathcal{C}$ and arrows indexed by the morphisms of $\mathcal{C}$. This perspective is quite useful: if the category $\mathcal{C}$ is sufficiently small, one can communicate the datum of a functor by drawing a graphical representation of the corresponding diagram. In §1.4.2 we discuss the notion of commutative diagram in an $\infty$-category (Convention 1.4.2.12) and describe some dangers associated with diagrammatic reasoning in the higher-categorical setting (Remark 1.4.2.13).

If $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories, then the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ can itself be organized into a category, which we denote by $\text{Fun}(\mathcal{C}, \mathcal{D})$. In §1.4.3 we describe a counterpart of this construction in the setting of $\infty$-categories. For every pair of simplicial sets $S \cdot$ and $T \cdot$, one can form a new simplicial set $\text{Fun}(S \cdot, T \cdot)$ whose vertices are maps from $S \cdot$ to $T \cdot$ (Construction 1.4.3.1). The main result of this section asserts that if $T \cdot$ is an $\infty$-category, then $\text{Fun}(S \cdot, T \cdot)$ is also an $\infty$-category (Theorem 1.4.3.7). Moreover, our notation is consistent: in the case where $S \cdot$ and $T \cdot$ are isomorphic to the nerves of categories $\mathcal{C}$ and $\mathcal{D}$, the $\infty$-category $\text{Fun}(S \cdot, T \cdot)$ is isomorphic to the nerve of the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$ (Proposition 1.4.3.3).

In order to prove Theorem 1.4.3.7, we will need to introduce some auxiliary ideas. Recall that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms in an $\infty$-category $\mathcal{C}$, then we
can form a composition of \( f \) and \( g \) by choosing a 2-simplex \( \sigma \) of \( C \) which satisfies \( d_0(\sigma) = g \) and \( d_2(\sigma) = f \), as indicated in the diagram

\[
\begin{array}{c}
  Y \\
  \downarrow \quad \downarrow g \\
  X \quad g \circ f \quad Z
\end{array}
\]

We proved in §1.3.4 that the resulting morphism \( g \circ f \) is well-defined up to homotopy (Proposition 1.3.4.2). In §1.4.6, we prove a variant of this assertion which asserts that the 2-simplex \( \sigma \) is “unique up to a contractible space of choices” (see Corollary 1.4.6.2 for a precise statement, and §1.4.7 for an extension to more general path categories). Moreover, we show that a strong version of this uniqueness result is equivalent to the assumption that \( C \) is an \( \infty \)-category (Theorem 1.4.6.1), and deduce the existence of functor \( \infty \)-categories \( \text{Fun}(C, D) \) as a consequence (Theorem 1.4.3.7). The precise formulation and proof of Theorem 1.4.6.1 will require some general ideas about categorical lifting properties and the homotopy theory of simplicial sets, which we develop in §1.4.4 and §1.4.5 respectively.

### 1.4.1 Examples of Functors

Let us begin by illustrating Definition 1.4.0.1 in some special cases.

**Example 1.4.1.1.** Let \( C \) and \( D \) be ordinary categories. It follows from Proposition 1.2.2.1 that the formation of nerves induces a bijection

\[
\{ \text{Functors of ordinary categories from } C \text{ to } D \} \xrightarrow{\sim} \{ \text{Functors of } \infty \text{-categories from } N_\bullet(C) \text{ to } N_\bullet(D) \}.
\]

In other words, Definition 1.4.0.1 can be regarded as a generalization of the usual notion of functor to the setting of \( \infty \)-categories.

**Example 1.4.1.2.** Let \( C \) be an \( \infty \)-category and let \( D \) be an ordinary category. Using Proposition 1.3.5.7 we obtain a bijection

\[
\{ \text{Functors of } \infty \text{-categories from } C \text{ to } N_\bullet(D) \} \xrightarrow{\sim} \{ \text{Functors of ordinary categories from } hC \text{ to } D \}.
\]

**Remark 1.4.1.3.** Let \( F : C \to D \) be a functor of \( \infty \)-categories. Then:
(a) To each object $X \in \mathcal{C}$ the functor $F$ assigns an object of $\mathcal{D}$, which we will denote by $F(X)$ (or sometimes more simply by $FX$).

(b) To each morphism $f : X \to Y$ in the $\infty$-category $\mathcal{C}$, the functor $F$ assigns a morphism $F(f) : F(X) \to F(Y)$ in the $\infty$-category $\mathcal{D}$.

(c) For every object $X \in \mathcal{C}$, the functor $F$ carries the identity morphism $\text{id}_X : X \to X$ in $\mathcal{C}$ to the identity morphism $\text{id}_{F(X)} : F(X) \to F(X)$ in $\mathcal{D}$.

(d) If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$ and $h$ is a composition of $f$ and $g$ (in the sense of Definition 1.3.4.1), then the morphism $F(h) : F(X) \to F(Z)$ is a composition of $F(f)$ and $F(g)$.

**Warning 1.4.1.4.** To define a functor $F$ from an ordinary category $\mathcal{C}$ to an ordinary category $\mathcal{D}$, it suffices to specify the values of $F$ on objects and morphisms (as described in (a) and (b) of Remark 1.4.1.3) and to verify that $F$ is compatible with the formation of composition and identity morphisms (as described in (c) and (d) of Remark 1.4.1.3). In the $\infty$-categorical setting, this is not enough: to give a functor of $\infty$-categories $F : \mathcal{C} \to \mathcal{D}$, one must specify its values on simplices of all dimensions. Roughly speaking, these values encode the requirement that $F$ is compatible with composition “up to coherent homotopy.” For example, suppose that we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. Part (d) of Remark 1.4.1.3 asserts that if $h$ is a composition of $f$ and $g$, then $F(h)$ is a composition of $F(f)$ and $F(g)$. However, we can say more: if $\sigma$ is a 2-simplex of $\mathcal{C}$ which witnesses $h$ as a composition of $f$ and $g$, then $F(\sigma)$ is a 2-simplex of $\mathcal{D}$ which witnesses $F(h)$ as a composition of $F(f)$ and $F(g)$.

**Remark 1.4.1.5.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. If $f, g : X \to Y$ are homotopic morphisms of $\mathcal{C}$, then $F(f), F(g) : F(X) \to F(Y)$ are homotopic morphisms of $\mathcal{D}$. More precisely, the functor $F$ carries homotopies from $f$ to $g$ (viewed as 2-simplices of $\mathcal{C}$) to homotopies from $F(f)$ to $F(g)$ (viewed as 2-simplices of $\mathcal{D}$).

**Remark 1.4.1.6.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $\infty$-categories. If $f : X \to Y$ is a morphism in $\mathcal{C}$ and $g : Y \to X$ is a homotopy inverse to $f$, then $F(g)$ is a homotopy inverse to $F(f)$. In particular, if $f$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is also an isomorphism in $\mathcal{D}$.

**Example 1.4.1.7.** Let $X$ be a topological space and let $\mathcal{C}$ be an ordinary category. To specify a functor of $\infty$-categories $F : \text{Sing}_\bullet(X) \to \text{N}_\bullet(\mathcal{C})$, one must give a rule which assigns to each continuous map $\sigma : |\Delta^n| \to X$ (viewed as an $n$-simplex of $\text{Sing}_\bullet(X)$) a diagram $F(\sigma) = (C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \to \cdots \xrightarrow{f_n} C_n)$. In particular:

(a) To each point $x \in X$, the functor $F$ assigns an object $F(x) \in \mathcal{C}$.
(b) To each continuous path \( f: [0, 1] \to X \) starting at the point \( x = f(0) \) and ending at the point \( y = f(1) \), the functor \( F \) assigns a morphism \( F(f): F(x) \to F(y) \) in the category \( C \). The morphism \( F(f) \) is automatically an isomorphism (by virtue of Proposition 1.3.6.11 and Remark 1.4.1.6).

(c) For each continuous map \( \sigma: |\Delta^2| \to X \) with boundary behavior as depicted in the diagram

\[
\begin{array}{ccc}
  & y & \\
  f \downarrow & & \downarrow g \\
 x & \downarrow h & \rightarrow z,
\end{array}
\]

we have an identity \( F(h) = F(g) \circ F(f) \) in \( \text{Hom}_C(F(x), F(z)) \).

The data of a collection of objects \( \{F(x)\}_{x \in X} \) and isomorphisms \( \{F(f)\}_{f: [0, 1] \to X} \) satisfying (c) is called a \( C \)-valued local system on \( X \). The preceding discussion determines a bijection

\[
\{ \text{Functors of } \infty \text{-categories from } \text{Sing}_\bullet(X) \text{ to } N_\bullet(C) \} \sim \{ \text{C-valued local systems on } X \}.
\]

By virtue of Example 1.4.1.2, we can also identify local systems with functors from the fundamental groupoid \( \pi_{\leq 1}(X) \) into \( C \).

005F \textbf{Remark 1.4.1.8.} Let \( X \) be a topological space and let \( C \) be an arbitrary \( \infty \)-category. Motivated by Example 1.4.1.7, one can define a \( C \)-valued local system on \( X \) to be a functor of \( \infty \)-categories \( \text{Sing}_\bullet(X) \to C \). Beware that this notion generally cannot be reformulated in terms of the fundamental groupoid \( \pi_{\leq 1}(X) \).

005G \textbf{Example 1.4.1.9.} Let \( C \) be an \( \infty \)-category and let \( X \) be a topological space. Then we have a canonical bijection

\[
\{ \text{Functors of } \infty \text{-categories from } C \text{ to } \text{Sing}_\bullet(X) \} \sim \{ \text{Continuous functions from } |C| \text{ to } X \}.
\]

Here \( |C| \) denotes the geometric realization of the simplicial set \( C \) (see Definition 1.1.8.1). Beware that neither side has an obvious interpretation in terms of functors between ordinary categories (even in the special case where \( C \) is the nerve of a category).
1.4.2 Commutative Diagrams

We now consider a variant of the terminology introduced in §1.4.1.

**Definition 1.4.2.1.** Let $C$ be an $\infty$-category. A *diagram in $C$* is a map of simplicial sets $f : K_\bullet \to C$. We will also refer to a map $f : K_\bullet \to C$ as a *diagram in $C$ indexed by $K_\bullet$*, or a $K_\bullet$-indexed *diagram in $C$*.

If $C$ is an ordinary category, then a $(K_\bullet$-indexed) diagram in $C$ is a $(K_\bullet$-indexed) diagram in the $\infty$-category $N_\bullet(C)$.

In the special case where $K_\bullet$ is the nerve $N_\bullet(I)$ of a partially ordered set $I$ (Remark 1.2.1.8), we will refer to a map $f : K_\bullet \to C$ as a *diagram in $C$ indexed by $I$*, or an $I$-indexed *diagram in $C$*.

**Remark 1.4.2.2.** In the case where $K_\bullet$ is an $\infty$-category, Definition 1.4.2.1 is superfluous: a $K_\bullet$-indexed diagram in $C$ (in the sense of Definition 1.4.2.1) is just a functor from $K_\bullet$ to $C$ (in the sense of Definition 1.4.0.1). However, the redundant terminology will be useful to signal a shift in emphasis. We will generally refer to a map $f : C \to D$ as a *functor* when we wish to regard the $\infty$-categories $C$ and $D$ on an equal footing. By contrast, we will refer to a map $f : K_\bullet \to C$ as a *diagram* if we are primarily interested in the $\infty$-category $C$ (in many cases, the source of $f$ will be a very simple simplicial set).

**Remark 1.4.2.3 (Diagrams of Dimension $\leq 1$).** Let $C$ be an $\infty$-category and let $K_\bullet$ be a simplicial set of dimension $\leq 1$, corresponding to a directed graph $G$ (Proposition 1.1.5.9). In this case, a diagram $K_\bullet \to C$ can be identified with a pair $(\{C_v\}_{v \in \text{Vert}(G)}, \{f_e\}_{e \in \text{Edge}(G)})$, where each $C_v$ is an object of the $\infty$-category $C$ and each $f_e : C_{s(e)} \to C_{t(e)}$ is a morphism of $C$ (here $s(e)$ and $t(e)$ denote the source and target of the edge $e$). It is often convenient to specify diagrams $K_\bullet \to C$ by drawing a graphical representation of $G$ (as in Remark 1.1.5.3), where each node is labelled by an object of $C$ and each arrow is labelled by a morphism in $C$ (having the indicated source and target).

**Example 1.4.2.4 (Non-Commuting Squares).** Let $K_\bullet$ denote the boundary of the product $\Delta^1 \times \Delta^1$: that is, the simplicial subset of $\Delta^1 \times \Delta^1$ given by the union of the simplicial subsets $\partial \Delta^1 \times \Delta^1$ and $\Delta^1 \times \partial \Delta^1$. Then $K_\bullet$ is a 1-dimensional simplicial set, corresponding to a directed graph which we can depict as

```
\begin{tikzpicture}
  \node (A) at (0,0) {$\bullet$};
  \node (B) at (1,0) {$\bullet$};
  \node (C) at (0,-1) {$\bullet$};
  \node (D) at (1,-1) {$\bullet$};
  \draw[->] (A) -- (B);
  \draw[->] (A) -- (C);
  \draw[->] (B) -- (D);
  \draw[->] (C) -- (D);
\end{tikzpicture}
```
We can then display a $K_\bullet$-indexed diagram in an $\infty$-category $\mathcal{C}$ pictorially

$$
\begin{array}{ccc}
C_{00} & \xrightarrow{f} & C_{01} \\
\downarrow{g} & & \downarrow{g'} \\
C_{10} & \xrightarrow{f'} & C_{11},
\end{array}
$$

where each $C_{ij}$ is an object of $\mathcal{C}$, $f$ is a morphism in $\mathcal{C}$ from $C_{00}$ to $C_{01}$, $g$ is a morphism in $\mathcal{C}$ from $C_{00}$ to $C_{10}$, $f'$ is a morphism in $\mathcal{C}$ from $C_{10}$ to $C_{11}$, and $g'$ is a morphism in $\mathcal{C}$ from $C_{01}$ to $C_{11}$.

In classical category theory, it is useful to extend the notational conventions of Remark 1.4.2.3 to more general situations by introducing the notion of a commutative diagram.

**Definition 1.4.2.5.** Let $K_\bullet$ be a simplicial set of dimension $\leq 1$, which we will identify with a directed graph $G$ (see Proposition 1.1.5.9). Assume that $G$ satisfies the following additional conditions:

- (a) For every pair of vertices $v, w \in \text{Vert}(G)$, there is at most one edge of $G$ with source $v$ and target $w$. We will denote this edge (if it exists) by $(v, w) \in \text{Edge}(G)$.

- (b) The graph $G$ has no directed cycles. That is, if there exists a sequence of vertices $v_0, v_1, \ldots, v_n \in \text{Vert}(G)$ with the property that the edges $(v_{i-1}, v_i)$ exist for $1 \leq i \leq n$, then either $n = 0$ or $v_0 \neq v_n$.

Let $\mathcal{C}$ be an ordinary category and suppose we are given a diagram $\sigma : K_\bullet \to N_\bullet(\mathcal{C})$, which we identify with a pair $\{C_v\}_{v \in \text{Vert}(G)}, \{f_{w,v} : C_v \to C_w\}_{(v,w) \in \text{Edge}(G)}$. We will say that the diagram $\sigma$ commutes (or that $\sigma$ is a commutative diagram) if the following additional condition is satisfied:

- (c) Let $v$ and $w$ be vertices of $G$ which are joined by directed paths $(v = v_0, v_1, \ldots, v_m = w)$ and $(v = v'_0, v'_1, \ldots, v'_n = w)$ (so that the edges $(v_{i-1}, v_i), (v'_{j-1}, v'_j) \in \text{Edge}(G)$ exist for $1 \leq i \leq m$ and $1 \leq j \leq n$). Then we have an identity

$$
f_{v_m,v_{m-1}} \circ f_{v_{m-1},v_{m-2}} \circ \cdots \circ f_{v_1,v_0} = f_{v'_m,v'_{m-1}} \circ f_{v'_{m-1},v'_{m-2}} \circ \cdots \circ f_{v'_1,v'_0}
$$

in the set $\text{Hom}_{\mathcal{C}}(C_v, C_w)$.

**Proposition 1.4.2.6.** Let $K_\bullet$ be a simplicial set of dimension $\leq 1$, corresponding to a directed graph $G$ which satisfies conditions (a) and (b) of Definition 1.4.2.5. Let $\mathcal{C}$ be an ordinary category, and let $\sigma : K_\bullet \to N_\bullet(\mathcal{C})$ be a diagram. Then:
(1) There is a partial ordering $\leq$ on the vertex set $\text{Vert}(G)$, where we have $v \leq w$ if and only if there exists a sequence of vertices $(v = v_0, v_1, \ldots, v_n = w)$ with the property that the edges $(v_{i-1}, v_i) \in \text{Edge}(G)$ exist for $1 \leq i \leq n$.

(2) There is a unique monomorphism of simplicial sets $K_\bullet \hookrightarrow N_\bullet(\text{Vert}(G))$ which carries each vertex to itself.

(3) The diagram $\sigma$ extends to a map $\bar{\sigma} : N_\bullet(\text{Vert}(G)) \to N_\bullet(C)$ (that is, to a functor $\text{Vert}(G) \to C$) if and only if it is commutative, in the sense of Definition 1.4.2.5. Moreover, if the extension $\bar{\sigma}$ exists, then it is unique.

Proof. It follows immediately from the definitions that the relation $\leq$ defined in (1) is reflexive and transitive. Antisymmetry follows from our assumption that the graph $G$ has no directed loops (condition (b) of Definition 1.4.2.5). By construction, we have $v \leq w$ whenever $v$ and $w$ are connected by an edge $(v, w) \in \text{Edge}(G)$. From the description of the simplicial set $K_\bullet$ given in Remark 1.1.5.10, we immediately see that there is a unique map of simplicial sets $i : K_\bullet \to N_\bullet(\text{Vert}(G))$ which is the identity on vertices. It follows from assumption (a) of Definition 1.4.2.5 that the map $i$ is a monomorphism. Let us henceforth identify $K_\bullet$ with a simplicial subset of $N_\bullet(\text{Vert}(G))$ given by the image of $i$. Let us identify $\sigma$ with a pair $\{\{C_v\}_{v \in \text{Vert}(G)}, \{f_{w,v} : C_v \to C_w\}_{(v, w) \in \text{Edge}(G)}\}$. Suppose that the diagram $\sigma$ extends to a functor $\bar{\sigma} : \text{Vert}(G) \to C$. If $v$ and $w$ are a pair of vertices of $G$ with $v \leq w$, then we can choose a directed path $(v = v_0, v_1, \ldots, v_n = w)$ from $v$ to $w$. The compatibility of $\bar{\sigma}$ with composition then guarantees that $\bar{\sigma}$ must carry the edge $(v, w)$ of $N_\bullet(\text{Vert}(G))$ to the iterated composition $f_{v_n, v_{n-1}} \circ f_{v_{n-1}, v_{n-2}} \circ \cdots \circ f_{v_1, v_0} \in \text{Hom}_C(C_v, C_w)$. Since the morphism $\bar{\sigma}(v, w)$ is independent of the choice of directed path, it follows that the diagram $\sigma$ is commutative. Conversely, if $\sigma$ is commutative, then we can define $\bar{\sigma}$ on morphisms by the formula $\bar{\sigma}(v, w) = f_{v_n, v_{n-1}} \circ f_{v_{n-1}, v_{n-2}} \circ \cdots \circ f_{v_1, v_0}$ to obtain the desired extension of $\sigma$. $\square$

Remark 1.4.2.7. In the situation of Proposition 1.4.2.6, an arbitrary map of simplicial sets $\sigma : K_\bullet \to N_\bullet(C)$ can be identified with a functor $F : \text{Path}[G] \to C$, where $\text{Path}[G]$ denotes the path category of the graph $G$ (Proposition 1.2.6.5). The commutativity of the diagram $\sigma$ is equivalent to the requirement that $F$ factors through the quotient functor $\text{Path}[G] \to \text{Vert}(G)$: that is, the value of the functor $F$ on a path depends only the endpoints of that path.

Example 1.4.2.8 (Commutative Squares in a Category). Let $K_\bullet = \partial(\Delta^1 \times \Delta^1)$ be as in Example 1.4.2.4. For any ordinary category $C$, we can display a diagram $\sigma : K_\bullet \to N_\bullet(C)$ pictorially as

$$
\begin{array}{ccc}
C_{00} & \overset{f}{\longrightarrow} & C_{01} \\
\downarrow{g} & & \downarrow{g'} \\
C_{10} & \overset{f'}{\longrightarrow} & C_{11}.
\end{array}
$$
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The diagram $\sigma$ is commutative if and only if we have $g' \circ f = f' \circ g$ in $\text{Hom}_C(C_{00}, C_{11})$. In this case, Proposition 1.4.2.6 ensures that $\sigma$ extends uniquely to a diagram $\overline{\sigma} : \Delta^1 \times \Delta^1 \to N_\cdot(C)$, or equivalently to a functor of ordinary categories $[1] \times [1] \to C$.

In the setting of $\infty$-categories, assertion (3) of Proposition 1.4.2.6 is false in general.

**Example 1.4.2.9** (Square Diagrams in an $\infty$-Category). Let $I$ denote the partially ordered set $[1] \times [1]$. The simplicial set $N_\cdot(I) \simeq \Delta^1 \times \Delta^1$ has four vertices (given by the elements of $I$), five nondegenerate edges, and two nondegenerate 2-simplices. Unwinding the definitions, we see that an $I$-indexed diagram in an $\infty$-category $C$ is equivalent to the following data:

- A collection of objects $\{C_{ij}\}_{0 \leq i,j \leq 1}$ in $C$.
- A collection of morphisms $f : C_{00} \to C_{01}$, $g : C_{00} \to C_{10}$, $f' : C_{10} \to C_{11}$, $g' : C_{01} \to C_{11}$, and $h : C_{00} \to C_{11}$.
- A 2-simplex $\sigma$ of $C$ which witnesses $h$ as a composition of $f$ with $g'$, and a 2-simplex $\tau$ of $C$ which witnesses $h$ as a composition of $g$ with $f'$.

This data can be depicted graphically as follows:

```
\begin{tikzcd}
C_{00} \arrow{r}{f} \arrow{d}[swap]{g} & C_{01} \arrow{d}{g'} \arrow{dl}{\sigma} \\
C_{10} \arrow{r}[swap]{f'} & C_{11} \arrow{ul}{\tau}
\end{tikzcd}
```

Beware that such a diagram is usually not determined by its restriction to the simplicial subset $K_\bullet \subseteq N_\cdot(I)$ of Example 1.4.2.8.

**Exercise 1.4.2.10.** Let $C$ be an $\infty$-category and let $K_\bullet \subseteq \Delta^1 \times \Delta^1$ be the simplicial subset appearing in Example 1.4.2.8. Suppose we are given a diagram $\sigma : K_\bullet \to C$, which we depict graphically as

```
\begin{tikzcd}
C_{00} \arrow{r}{f} \arrow{d}{g} & C_{01} \arrow{d}{g'} \\
C_{10} \arrow{r}[swap]{f'} & C_{11}
\end{tikzcd}
```
Composing with the unit map $C \to N_\bullet(hC)$, we obtain a diagram $\sigma'$ in the homotopy category $hC$, which we can depict as

$$
\begin{array}{ccc}
C_00 & \xrightarrow{[f]} & C_01 \\
\downarrow{[g]} & & \downarrow{[g']}
\end{array}
$$

$$
\begin{array}{ccc}
C_{10} & \xrightarrow{[f']} & C_{11}
\end{array}
$$

Show that the diagram $\sigma'$ is commutative if and only if $\sigma$ can be extended to a map $\sigma : \Delta^1 \times \Delta^1 \to C$. Beware, however, that this extension is generally not unique.

**Warning 1.4.2.11.** Let $I$ be a partially ordered set and let $C$ be an $\infty$-category. In the case $I = [1] \times [1]$, Exercise 1.4.2.10 implies that every functor of ordinary categories $I \to hC$ can be lifted to a functor of $\infty$-categories $N_\bullet(I) \to C$. Beware that this conclusion is generally false for more complicated partially ordered sets. For example, it fails in the case $I = [1] \times [1] \times [1]$ (see Example [?]).

Example 1.4.2.9 illustrates that the notion of “commutative diagram” becomes considerably more subtle in the setting of $\infty$-categories. To specify an $I$-indexed diagram $F : N_\bullet(I) \to C$ of an $\infty$-category $C$, one generally needs to specify the values of $F$ on all the simplices of the simplicial set $N_\bullet(I)$. In general, it is not feasible to graphically encode all of this data in a comprehensible way. On the other hand, the formalism of commutative diagrams is too useful to completely abandon. We will therefore sacrifice some degree of mathematical precision in favor of clarity of exposition.

**Convention 1.4.2.12.** Let $C$ be an $\infty$-category and let $G$ be a directed graph satisfying conditions $(a)$ and $(b)$ of Definition 1.4.2.5 so that the vertex set $\text{Vert}(G)$ inherits a partial ordering (Proposition 1.4.2.6). We will sometimes refer to the notion of a commutative diagram $\sigma$ in $C$, which we indicate graphically by a collection of objects $\{C_v\}_{v \in \text{Vert}(G)}$ of $C$, connected by arrows which are labelled by morphisms $\{f_e\}_{e \in \text{Edge}(G)}$. In this case, it should be understood that $\sigma$ is a diagram $N_\bullet(\text{Vert}(G)) \to C$, which carries each vertex $v$ of $N_\bullet(\text{Vert}(G))$ to the object $C_v \in C$ and each edge $e = (v, w)$ of $G$ to the morphism $f_e$ in $C$. Beware that in this case, the map $\sigma$ need not be completely determined by the pair $\{(C_v)_{v \in \text{Vert}(G)}, \{f_e\}_{e \in \text{Edge}(G)}\}$ (this pair can instead be identified with the restriction $\sigma|_{K_\bullet}$, where $K_\bullet$ is the 1-dimensional simplicial subset of $N_\bullet(\text{Vert}(G))$ corresponding to $G$).

**Remark 1.4.2.13.** In the situation of Convention 1.4.2.12, suppose that $C = N_\bullet(C_0)$, where $C_0$ is an ordinary category. Then giving a commutative diagram in the $\infty$-category $C$ (in the sense of Convention 1.4.2.12) is equivalent to giving a commutative diagram in the ordinary category $C_0$ (in the sense of Definition 1.4.2.5). In this case, commutativity is a property that the underlying diagram (indexed by a 1-dimensional simplicial set) does or does not possess. For a general $\infty$-category $C$, commutativity of a diagram in $C$ is not a property but
a structure; to promote a diagram to a commutative diagram, one must specify additional data to witness the requisite commutativity.

Example 1.4.2.14. Let $\mathcal{C}$ be an $\infty$-category. If we refer to a commutative diagram $\sigma :$

\[
\begin{matrix}
Y & \xrightarrow{g} & Z \\
\downarrow^{f} & & \downarrow^{g} \\
X & \xrightarrow{h} & Z,
\end{matrix}
\]

then we mean that $\sigma$ is a 2-simplex of $\mathcal{C}$ satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$, and $d_2(\sigma) = f$. In other words, we mean that $\sigma$ is a 2-simplex which witnesses $h$ as a composition of $f$ and $g$, in the sense of Definition 1.3.4.1.

Example 1.4.2.15. Let $\mathcal{C}$ be an $\infty$-category. If we refer to a commutative diagram $\sigma :$

\[
\begin{matrix}
C_0 & \xrightarrow{f} & C_1 \\
\downarrow^{g} & & \downarrow^{g'} \\
C_0' & \xrightarrow{f'} & C_1',
\end{matrix}
\]

we implicitly assume that $\sigma$ is a map from the entire simplicial set $\Delta^1 \times \Delta^1$ to $\mathcal{C}$. In other words, we assume that we have specified another morphism $h : C_{00} \to C_{11}$, which is not indicated in the picture, together with a 2-simplex $\sigma$ witnessing $h$ as the composition of $f$ and $g'$ and a 2-simplex $\tau$ witnessing $h$ as the composition of $g$ and $f'$.

Warning 1.4.2.16. In ordinary category theory, it is sometimes useful to refer to the commutativity of diagrams in situations which do not fit the paradigm of Definition 1.4.2.5. For example, the commutativity of a diagram

\[
\begin{matrix}
X & \xrightarrow{f} & Y \xrightarrow{u} \xrightarrow{v} Z
\end{matrix}
\]

is often understood as the requirement that $u \circ f = v \circ f$. Beware that this usage is potentially ambiguous (from the shape of the diagram alone, it is not clear that commutativity should enforce the identity $u \circ f = v \circ f$, but not the identity $u = v$), so we will take special care when applying similar terminology in the $\infty$-categorical setting.

1.4.3 The $\infty$-Category of Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then we can form a new category Fun($\mathcal{C}, \mathcal{D}$), whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations. In this section, we describe an analogous construction in the setting of $\infty$-categories.
Construction 1.4.3.1. Let $S_\bullet$ and $T_\bullet$ be simplicial sets. Then the construction

$$([n] \in \Delta^{op}) \mapsto \text{Hom}_{\text{Set}}(\Delta^n \times S_\bullet, T_\bullet)$$

determines a functor from the category $\Delta^{op}$ to the category of sets. We regard this functor as a simplicial set which we will denote by $\text{Fun}(S_\bullet, T_\bullet)$.

Note that, given an $n$-simplex $f$ of $\text{Fun}(S_\bullet, T_\bullet)$ and an $n$-simplex $\sigma$ of $S_\bullet$, we can construct an $n$-simplex $\text{ev}(f, \sigma)$ of $T_\bullet$, given by the composition

$$\Delta^n \xrightarrow{\delta} \Delta^n \times \Delta^n \xrightarrow{\text{id} \times \sigma} \Delta^n \times S_\bullet \xrightarrow{f} T_\bullet.$$ 

This construction determines a map of simplicial sets $\text{ev}: \text{Fun}(S_\bullet, T_\bullet) \times S_\bullet \to T_\bullet$, which we will refer to as the evaluation map.

Proposition 1.4.3.2. Let $S_\bullet$, $T_\bullet$, and $U_\bullet$ be simplicial sets. Then the composite map

$$\theta: \text{Hom}_{\text{Set}}(U_\bullet, \text{Fun}(S_\bullet, T_\bullet)) \to \text{Hom}_{\text{Set}}(U_\bullet \times S_\bullet, \text{Fun}(S_\bullet, T_\bullet) \times S_\bullet) \xrightarrow{\text{ev} \circ} \text{Hom}_{\text{Set}}(U_\bullet \times S_\bullet, T_\bullet)$$

is bijective.

Proof. Let $f: U_\bullet \times S_\bullet \to T_\bullet$ be a map of simplicial sets. For each $n$-simplex $\sigma$ of $U_\bullet$, the composite map

$$\Delta^n \times S_\bullet \xrightarrow{\sigma \times \text{id}} U_\bullet \times S_\bullet \xrightarrow{f} T_\bullet$$

can be regarded as an $n$-simplex of $\text{Fun}(S_\bullet, T_\bullet)$, which we will denote by $g(\sigma)$. The construction $\sigma \mapsto g(\sigma)$ determines a map of simplicial sets $g: U_\bullet \to \text{Fun}(S_\bullet, T_\bullet)$. We leave as an exercise for the reader to verify that $g$ is the unique map satisfying $\theta(g) = f$.

Beware that the notation of Construction 1.4.3.1 is potentially confusing, because it conflicts with our use of $\text{Fun}(\mathcal{C}, \mathcal{D})$ to denote the category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. However, these usages are compatible:

Proposition 1.4.3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $e: \text{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D}$ denote the evaluation functor, given on objects by the formula $e(F, C) = F(C)$. Then the composite map

$$N_\bullet(\text{Fun}(\mathcal{C}, \mathcal{D})) \times N_\bullet(\mathcal{C}) \simeq N_\bullet(\text{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C}) \xrightarrow{N_\bullet(e)} N_\bullet(\mathcal{D})$$

corresponds, under the bijection of Proposition 1.4.3.2, to an isomorphism of simplicial sets $\rho: N_\bullet(\text{Fun}(\mathcal{C}, \mathcal{D})) \to \text{Fun}(N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D}))$. 
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Proof. For each \( n \geq 0 \), the map \( \rho \) is given on \( n \)-simplices by the composition

\[
\text{Hom}_{\text{Set}_\Delta}(\Delta^n, N_\bullet(\text{Fun}(\mathcal{C}, \mathcal{D}))) \cong \text{Hom}_{\text{Cat}}([n], \text{Fun}(\mathcal{C}, \mathcal{D})) \\
\cong \text{Hom}_{\text{Cat}}([n] \times \mathcal{C}, \mathcal{D}) \\
\xrightarrow{v} \text{Hom}_{\text{Set}_\Delta}(N_\bullet([n] \times \mathcal{C}), N_\bullet(\mathcal{D})) \\
\cong \text{Hom}_{\text{Set}_\Delta}(\Delta^n \times N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D})) \\
\cong \text{Hom}_{\text{Set}_\Delta}(\Delta^n \times N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D})) \\
\cong \text{Hom}_{\text{Set}_\Delta}(\Delta^n, \text{Fun}(N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D}))).
\]

It will therefore suffice to show that \( v \) is bijective, which is a special case of Proposition 1.2.2.1.

Passing to homotopy categories, we obtain the following weaker result:

Corollary 1.4.3.4. Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. Then there is a canonical isomorphism of categories

\[
\text{Fun}(\mathcal{C}, \mathcal{D}) \cong \text{hFun}(N_\bullet(\mathcal{C}), N_\bullet(\mathcal{D})).
\]

We can also generalize Proposition 1.4.3.3 as follows:

Corollary 1.4.3.5. Let \( S_\bullet \) be a simplicial set having homotopy category \( \text{h}S_\bullet \). Then, for any category \( \mathcal{D} \), the composite map

\[
N_\bullet(\text{Fun}(\text{h}S_\bullet, \mathcal{D})) \times S_\bullet \to N_\bullet(\text{Fun}(\text{h}S_\bullet, \mathcal{D})) \times N_\bullet(\text{h}S_\bullet) \simeq N_\bullet(\text{Fun}(\text{h}S_\bullet, \mathcal{D}) \times \text{h}S_\bullet) \to N_\bullet(\mathcal{D})
\]

induces an isomorphism of simplicial sets \( \rho_{S_\bullet} : N_\bullet(\text{Fun}(\text{h}S_\bullet, \mathcal{D})) \cong \text{Fun}(S_\bullet, N_\bullet(\mathcal{D})) \).

Proof. The construction \( S_\bullet \mapsto \rho_{S_\bullet} \) carries colimits (in the category \( \text{Set}_\Delta \) of simplicial sets) to limits (in the category \( \text{Fun}([1], \text{Set}_\Delta) \) of morphisms between simplicial sets). Since the category \( \text{Set}_\Delta \) is generated under colimits by objects of the form \( \Delta^n \) (Lemma 1.1.8.17), it will suffice to prove Corollary 1.4.3.5 in the special case where \( S_\bullet \simeq \Delta^n \). In this case, the desired result follows from Proposition 1.4.3.3 since \( S_\bullet \) is isomorphic to the nerve of the category \( \mathcal{C} = [n] \).

Corollary 1.4.3.6. The formation of homotopy categories determines a functor \( \text{Set}_\Delta \to \text{Cat} \) which commutes with finite products.

Proof. Since the construction \( S_\bullet \mapsto \text{h}S_\bullet \) preserves final objects, it will suffice to show that for any pair of simplicial sets \( S_\bullet \) and \( T_\bullet \), the canonical map

\[
u : \text{h}(S_\bullet \times T_\bullet) \to \text{h}S_\bullet \times \text{h}T_\bullet
\]
is an isomorphism of categories. In other words, we wish to show that for any category \( \mathcal{C} \), composition with \( u \) induces a bijection

\[
\text{Hom}_{\text{Cat}}(hS \times hT, \mathcal{C}) \rightarrow \text{Hom}_{\text{Cat}}(h(S \times T), \mathcal{C}).
\]

Unwinding the definitions, we see that this map is given by the composition

\[
\text{Hom}_{\text{Cat}}(hS \times hT, \mathcal{C}) \cong \text{Hom}_{\text{Set}_{\Delta}}(S, N_\bullet(\text{Fun}(hT, \mathcal{C})))
\]

\[
\cong \text{Hom}_{\text{Set}_{\Delta}}(S \times T, N_\bullet(\mathcal{C}))
\]

\[
\cong \text{Hom}_{\text{Cat}}(h(S \times T), \mathcal{C}),
\]

where \( \rho_{T_\bullet} \) is the isomorphism appearing in the statement of Corollary \[1.4.3.5\].

We will be primarily interested in the special case of Construction \[1.4.3.1\] where the target simplicial set \( T_\bullet \) is an \( \infty \)-category. In this case, we have the following result:

**Theorem 1.4.3.7.** Let \( S_\bullet \) be a simplicial set and let \( D \) be an \( \infty \)-category. Then the simplicial set \( \text{Fun}(S_\bullet, D) \) is an \( \infty \)-category.

The proof of Theorem 1.4.3.7 will require some combinatorial preliminaries; we defer the proof to \[1.4.6\].

**Definition 1.4.3.8.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories. It follows from Theorem 1.4.3.7 that the simplicial set \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) is also an \( \infty \)-category. We will refer to \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) as the \( \infty \)-category of functors from \( \mathcal{C} \) to \( \mathcal{D} \).

**Remark 1.4.3.9.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories. By definition, the objects of the \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) can be identified with functors from \( \mathcal{C} \) to \( \mathcal{D} \), in the sense of Definition 1.4.0.1 (that is, with maps of simplicial sets from \( \mathcal{C} \) to \( \mathcal{D} \)).

**Remark 1.4.3.10.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories, and suppose we are given a pair of functors \( F, G : \mathcal{C} \rightarrow \mathcal{D} \). We define a natural transformation from \( F \) to \( G \) to be a map of simplicial sets \( u : \Delta^1 \times \mathcal{C} \rightarrow \mathcal{D} \) satisfying \( u|_{\{0\} \times \mathcal{C}} = F \) and \( u|_{\{1\} \times \mathcal{C}} = G \). In other words, a natural transformation from \( F \) to \( G \) is a morphism from \( F \) to \( G \) in the \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \).

**Remark 1.4.3.11.** Let us abuse notation by identifying each ordinary category \( \mathcal{E} \) with the \( \infty \)-category \( N_{\bullet}(\mathcal{E}) \). In this case, Corollary 1.4.3.5 implies that when \( \mathcal{C} \) is an \( \infty \)-category and \( \mathcal{D} \) is an ordinary category, then we have a canonical isomorphism \( \text{Fun}(\mathcal{C}, \mathcal{D}) \cong \text{Fun}(h\mathcal{C}, \mathcal{D}) \). In particular, the functor \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) is an ordinary category.
1.4.4 Digression: Lifting Properties

We now review some categorical terminology which will be useful in the proof of Theorem 1.4.3.7 and in several other parts of this book.

**Definition 1.4.4.1.** Let \( \mathcal{C} \) be a category. A lifting problem in \( \mathcal{C} \) is a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & & \downarrow{p} \\
B & \xrightarrow{v} & Y
\end{array}
\]

in \( \mathcal{C} \). A solution to the lifting problem \( \sigma \) is a morphism \( h : B \to X \) in \( \mathcal{C} \) satisfying \( p \circ h = v \) and \( h \circ f = u \), as indicated in the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & \nearrow{h} & \downarrow{p} \\
B & \xrightarrow{v} & Y
\end{array}
\]

**Remark 1.4.4.2.** In the situation of Definition 1.4.4.1, we will often indicate a lifting problem by a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & \nearrow{h} & \downarrow{p} \\
B & \xrightarrow{v} & Y
\end{array}
\]

which includes a dotted arrow representing a hypothetical solution.

**Definition 1.4.4.3.** Let \( \mathcal{C} \) be a category, and suppose we are given a morphism \( f : A \to B \) and \( p : X \to Y \) in \( \mathcal{C} \). We will say that \( f \) has the left lifting property with respect to \( p \), or that \( p \) has the right lifting property with respect to \( f \), if, for every pair of morphisms \( u : A \to X \) and \( v : B \to Y \) satisfying \( p \circ u = v \circ f \), the associated lifting problem

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & \nearrow{p} & \downarrow{u} \\
B & \xrightarrow{v} & Y
\end{array}
\]

admits a solution (that is, there exists a map \( h : B \to X \) satisfying \( p \circ h = v \) and \( h \circ f = u \)).
If $S$ is a collection of morphisms in $C$, we will say that a morphism $f : A \to B$ has the left lifting property with respect to $S$ if it has the left lifting property with respect to every morphism in $S$. Similarly, we will say that a morphism $p : X \to Y$ has the right lifting property with respect to $S$ if it has the right lifting property with respect to every morphism in $S$.

Let $S$ be a collection of morphisms in a category $C$. We now summarize some closure properties enjoyed by the collection of morphisms which have the left lifting property with respect to $S$.

**Definition 1.4.4.4.** Let $C$ be a category which admits pushouts and let $T$ be a collection of morphisms of $C$. We will say that $T$ is closed under pushouts if, for every pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

in the category $C$ where the morphism $f$ belongs to $T$, the morphism $f'$ also belongs to $T$.

**Proposition 1.4.4.5.** Let $C$ be a category which admits pushouts, let $S$ be a collection of morphisms of $C$, and let $T$ be the collection of all morphisms of $C$ having the left lifting property with respect to $S$. Then $T$ is closed under pushouts.

**Proof.** Suppose we are given a pushout diagram $\sigma :$

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A' \\
\downarrow & & \downarrow \\
B & \xrightarrow{h} & B'
\end{array}
\]

where $f$ belongs to $T$. We wish to show that $f'$ also belongs to $T$. For this, we must show that every lifting problem

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & X \\
\downarrow & & \downarrow \\
B' & \xrightarrow{v} & Y
\end{array}
\]

admits a solution, provided that the morphism $p$ belongs to $S$. Using our assumption that
σ is a pushout square, we are reduced to solving the associated lifting problem

\[
\begin{array}{ccc}
A \xrightarrow{uog} X \\
\downarrow f & \nearrow p \\
B \xrightarrow{voh} Y,
\end{array}
\]

which is possible by virtue of our assumption that \(f\) has the left lifting property with respect to \(p\).

\[\Box\]

**Definition 1.4.4.6.** Let \(\mathcal{C}\) be a category containing a pair of objects \(C\) and \(C'\). We will say that \(C\) is a retract of \(C'\) if there exist maps \(i : C \to C'\) and \(r : C' \to C\) such that \(r \circ i = \text{id}_C\).

**Variant 1.4.4.7.** Let \(\mathcal{C}\) be a category. We will say that a morphism \(f : C \to D\) of \(\mathcal{C}\) is a retract of another morphism \(f' : C' \to D'\) if it is a retract of \(f'\) when viewed as an object of the functor category \(\text{Fun}([1], \mathcal{C})\). In other words, we say that \(f\) is a retract of \(f'\) if there exists a commutative diagram

\[
\begin{array}{ccc}
C \xrightarrow{i} C' \xrightarrow{r} C \\
\downarrow f & \nearrow f' \\
D \xrightarrow{j} D' \xrightarrow{\tau} D
\end{array}
\]

in the category \(\mathcal{C}\), where \(r \circ i = \text{id}_C\) and \(\tau \circ j = \text{id}_D\).

We say that a collection of morphisms \(T\) of \(\mathcal{C}\) is closed under retracts if, for every pair of morphisms \(f, f'\) in \(\mathcal{C}\), if \(f\) is a retract of \(f'\) and \(f'\) belongs to \(T\), then \(f\) also belongs to \(T\).

**Exercise 1.4.4.8.** Let \(\mathcal{C}\) be a category and let \(T\) be the collection of all monomorphisms in \(\mathcal{C}\). Show that \(T\) is closed under retracts.

**Proposition 1.4.4.9.** Let \(\mathcal{C}\) be a category, let \(S\) be a collection of morphisms of \(\mathcal{C}\), and let \(T\) be the collection of all morphisms of \(\mathcal{C}\) having the left lifting property with respect to \(S\). Then \(T\) is closed under retracts.

**Proof.** Let \(f'\) be a morphism of \(\mathcal{C}\) which belongs to \(T\) and let \(f\) be a retract of \(f'\), so that there exists a commutative diagram

\[
\begin{array}{ccc}
C \xrightarrow{i} C' \xrightarrow{r} C \\
\downarrow f & \nearrow f' \\
D \xrightarrow{j} D' \xrightarrow{\tau} D
\end{array}
\]
with \( r \circ i = \text{id}_C \) and \( \tau \circ \bar{i} = \text{id}_D \). We wish to show that \( f \) also belongs to \( T \). Consider a lifting problem \( \sigma : \)

\[
\begin{array}{ccc}
C & \xrightarrow{u} & X \\
\downarrow f & & \downarrow p \\
D & \xrightarrow{v} & Y
\end{array}
\]

where \( p \) belongs to \( S \). Our assumption \( f' \in T \) ensures that the associated lifting problem

\[
\begin{array}{ccc}
C' & \xrightarrow{u \circ r} & X \\
\downarrow f' & & \downarrow p \\
D' & \xrightarrow{v \circ \tau} & Y
\end{array}
\]

admits a solution: that is, we can choose a morphism \( h' : D' \to X \) satisfying \( p \circ h' = v \circ \tau \) and \( h' \circ f' = u \circ r \). Then the morphism \( h = h' \circ \bar{i} \) is a solution to the lifting problem \( \sigma \), by virtue of the calculations

\[
p \circ h = p \circ h' \circ \bar{i} = v \circ \tau \circ \bar{i} = v
\]

\[
h \circ f = h' \circ \bar{i} \circ f = h' \circ f' \circ i = u \circ r \circ i = u.
\]

\[\square\]

**Definition 1.4.4.10.** For every ordinal number \( \alpha \), let \([\alpha] = \{\beta : \beta \leq \alpha\}\) denote the collection of all ordinal numbers which are less than or equal to \( \alpha \), regarded as a linearly ordered set.

Let \( \mathcal{C} \) be a category and let \( T \) be a collection of morphisms of \( \mathcal{C} \). We will say that a morphism \( f \) of \( \mathcal{C} \) is a *transfinite composition of morphisms of \( T \) if* there exists an ordinal number \( \alpha \) and a functor \( F : [\alpha] \to \mathcal{C} \), given by a collection of objects \( \{C_\beta\}_{\beta \leq \alpha} \) and morphisms \( \{f_\gamma,\beta : C_\beta \to C_\gamma\}_{\beta \leq \gamma} \) with the following properties:

(a) For every nonzero limit ordinal \( \lambda \leq \alpha \), the functor \( F \) exhibits \( C_\lambda \) as a colimit of the diagram \( \{C_\beta\}_{\beta \leq \lambda}, \{f_{\gamma,\beta} : C_\beta \to C_\gamma\}_{\beta \leq \gamma < \lambda} \).

(b) For every ordinal \( \beta < \alpha \), the morphism \( f_{\beta+1,\beta} \) belongs to \( T \).

(c) The morphism \( f \) is equal to \( f_{\alpha,0} : C_0 \to C_\alpha \).

We will say that \( T \) is *closed under transfinite composition* if, for every morphism \( f \) which is a transfinite composition of morphisms of \( T \), we have \( f \in T \).
Example 1.4.4.11. Let $C$ be a category and let $T$ be a collection of morphisms of $C$. Then every identity morphism of $C$ is a transfinite composition of morphisms of $T$ (take $\alpha = 0$ in Definition 1.4.4.10). In particular, if $T$ is closed under transfinite composition, then it contains every identity morphism of $C$.

Example 1.4.4.12. Let $C$ be a category and let $T$ be a collection of morphisms of $C$. Then every morphism of $T$ is a transfinite composition of morphisms of $T$ (take $\alpha = 1$ in Definition 1.4.4.10).

Example 1.4.4.13. Let $C$ be a category and let $T$ be a collection of morphisms of $C$ which contains a pair of composable morphisms $f : C_0 \to C_1$ and $g : C_1 \to C_2$. Then the composition $g \circ f$ is a transfinite composition of morphisms of $C$ (take $\alpha = 2$ in Definition 1.4.4.10). In particular, if $T$ is closed under transfinite composition, then it is closed under composition.

Proposition 1.4.4.14. Let $C$ be a category, let $S$ be a collection of morphisms in $C$, and let $T$ be the collection of all morphisms of $C$ which have the left lifting property with respect to $S$. Then $T$ is closed under transfinite composition.

Proof. Let $\alpha$ be an ordinal and suppose we are given a functor $[\alpha] \to C$, given by a pair

$$(\{C_\beta\}_{\beta \leq \alpha}, \{f_{\gamma, \beta}\}_{\beta \leq \gamma \leq \alpha})$$

which satisfies condition (a) of Definition 1.4.4.10. Assume that each of the morphisms $f_{\beta+1, \beta}$ belongs to $T$. We wish to show that the morphism $f_{\alpha, 0}$ also belongs to $T$. For this, we must show that every lifting problem $\sigma :$

$$
\begin{array}{ccc}
C_0 & \xrightarrow{u} & X \\
\downarrow{f_{\alpha, 0}} & & \downarrow{p} \\
C_{\alpha} & \xrightarrow{v} & Y
\end{array}
$$

admits a solution, provided that $p$ belongs to $S$. We construct a collection of morphisms $\{u_{\beta} : C_\beta \to X\}_{\beta \leq \alpha}$, satisfying the requirements $p \circ u_{\beta} = v \circ f_{\alpha, \beta}$ and $u_{\beta} = u_{\gamma} \circ f_{\gamma, \beta}$ for $\beta \leq \gamma$, using transfinite recursion. Fix an ordinal $\gamma \leq \alpha$, and assume that the morphisms $\{u_{\beta}\}_{\beta < \gamma}$ have been constructed. We consider three cases:

- If $\gamma = 0$, we set $u_{\gamma} = u$.
- If $\gamma$ is a nonzero limit ordinal, then our hypothesis that $C_{\gamma}$ is the colimit of the diagram $\{C_\beta\}_{\beta < \gamma}$ guarantees that there is a unique morphism $u_{\gamma} : C_{\gamma} \to X$ satisfying $u_{\beta} = u_{\gamma} \circ f_{\gamma, \beta}$ for $\beta < \gamma$. Moreover, our assumption that the equality $p \circ u_{\beta} = v \circ f_{\alpha, \beta}$ holds for $\beta < \gamma$ guarantees that it also holds for $\beta = \gamma$. 
• Suppose that $\gamma = \beta + 1$ is a successor ordinal. In this case, we take $u_\gamma$ to be any solution to the lifting problem

\[
\begin{array}{ccc}
C_\beta & \xrightarrow{u_\beta} & X \\
\downarrow_{f_{\beta+1,\beta}} & & \downarrow_{p} \\
C_{\beta+1} & \xrightarrow{v_0 f_{\alpha,\beta+1}} & Y,
\end{array}
\]

which exists by virtue of our assumption that $f_{\beta+1,\beta}$ belongs to $T$.

We now complete the proof by observing that $u_\alpha$ is a solution to the lifting problem $\sigma$. □

Motivated by the preceding discussion, we introduce the following:

**Definition 1.4.4.15.** Let $\mathcal{C}$ be a category which admits small colimits and let $T$ be a collection of morphisms of $\mathcal{C}$. We will say that $T$ is *weakly saturated* if it is closed under pushouts (Definition 1.4.4.4), retracts (Variant 1.4.4.7), and transfinite composition (Definition 1.4.4.10).

**Proposition 1.4.4.16.** Let $\mathcal{C}$ be a category which admits small colimits, let $S$ be a collection of morphisms of $\mathcal{C}$, and let $T$ be the collection of all morphisms of $\mathcal{C}$ which have the left lifting property with respect to $S$. Then $T$ is weakly saturated.

*Proof.* Combine Propositions 1.4.4.5, 1.4.4.9, and 1.4.4.14. □

**Remark 1.4.4.17.** Let $\mathcal{C}$ be a category and let $T_0$ be a collection of morphisms of $\mathcal{C}$. Then there exists a smallest collection of morphisms $T$ of $\mathcal{C}$ such that $T_0 \subseteq T$ and $T$ is weakly saturated (for example, we can take $T$ to be the intersection of all the weakly saturated collections of morphisms containing $T_0$). We will refer to $T$ as the *weakly saturated collection of morphisms generated by $T_0$*. It follows from Proposition 1.4.4.16 that if every morphism of $T_0$ has the left lifting property with respect to some collection of morphisms $S$, then every morphism of $T$ also has the left lifting property with respect to $S$.

### 1.4.5 Trivial Kan Fibrations

We now specialize the ideas of §1.4.4 to the category of simplicial sets.
1.4. FUNCTORS OF $\infty$-CATEGORIES

**Definition 1.4.5.1.** Let $p : X \to Y$ be a map of simplicial sets. We say that $p$ is a **trivial Kan fibration** if, for each $n \geq 0$, every lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \to & X \\
\downarrow & & \downarrow p \\
\Delta^n & \to & Y
\end{array}
\]

admits a solution; here $i : \partial \Delta^n \to \Delta^n$ denotes the inclusion map.

**Remark 1.4.5.2.** Suppose we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \to & X \\
p' & & \downarrow p \\
\downarrow & & \downarrow \\
Y' & \to & Y
\end{array}
\]

If $p$ is a trivial Kan fibration, then so is $p'$ (this follows from Proposition 1.4.4.5, applied to the opposite of the category Set$_\Delta$).

**Proposition 1.4.5.3.** Let $p : X \to Y$ be a map of simplicial sets. The following conditions are equivalent:

1. The map $p$ is a trivial Kan fibration (in the sense of Definition 1.4.5.1).
2. The map $p$ has the right lifting property with respect to every monomorphism of simplicial sets $i : A \to B$. In other words, every lifting problem

\[
\begin{array}{ccc}
A & \to & X \\
\downarrow & & \downarrow p \\
B & \to & Y
\end{array}
\]

admits a solution, provided that $i$ is a monomorphism.

We will give the proof of Proposition 1.4.5.3 at the end of this section.

**Corollary 1.4.5.4.** Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets. Then:

1. The map $p$ admits a section: that is, there is a map of simplicial sets $s : Y \to X$ such that the composition $p \circ s$ is the identity map $\text{id}_Y : Y \to Y$. 
(b) Let $s$ be any section of $p$. Then the composition $s \circ p : X_\bullet \to X_\bullet$ is fiberwise homotopic to the identity. That is, there exists a map of simplicial sets $h : \Delta^1 \times X_\bullet \to X_\bullet$, compatible with the projection to $Y_\bullet$, such that $h|_{\{0\} \times X_\bullet} = s \circ p$ and $h|_{\{1\} \times X_\bullet} = \text{id}_{X_\bullet}$.

**Proof.** To prove (a), we observe that a section of $p$ can be described as a solution to the lifting problem

$$
\emptyset \overset{s}{\longrightarrow} X_\bullet \overset{p}{\longrightarrow} Y_\bullet \overset{\text{id}}{\longrightarrow} Y_\bullet,
$$

which exists by virtue of Proposition 1.4.5.3. Given any section $s$, a fiberwise homotopy from $s \circ p$ to the identity can be identified with a solution to the lifting problem

$$
\partial \Delta^1 \times X_\bullet \overset{(s \circ p, \text{id})}{\longrightarrow} X_\bullet \overset{p}{\longrightarrow} \Delta^1 \times X_\bullet \overset{h}{\longrightarrow} Y_\bullet,
$$

which again exists by virtue of Proposition 1.4.5.3. \qed

**Corollary 1.4.5.5.** Let $p : X_\bullet \to Y_\bullet$ be a trivial Kan fibration of simplicial sets and let $i : A_\bullet \to B_\bullet$ be a monomorphism of simplicial sets. Then the canonical map

$$
\theta : \text{Fun}(B_\bullet, X_\bullet) \to \text{Fun}(B_\bullet, Y_\bullet) \times_{\text{Fun}(A_\bullet, Y_\bullet)} \text{Fun}(A_\bullet, X_\bullet)
$$

is also a trivial Kan fibration.

**Proof.** Fix an integer $n \geq 0$; we wish to show that every lifting problem

$$
\partial \Delta^n \longrightarrow \text{Fun}(B_\bullet, X_\bullet) \overset{\theta}{\longrightarrow} \text{Fun}(B_\bullet, Y_\bullet) \times_{\text{Fun}(A_\bullet, Y_\bullet)} \text{Fun}(A_\bullet, X_\bullet)
$$

admits a solution. Unwinding the definitions, we see that this is equivalent to solving an
associated lifting problem

\[
(\partial \Delta^n \times B\bullet) \amalg_{\partial \Delta^n \times A\bullet} (\Delta^n \times A\bullet) \rightarrow X\bullet
\]

\[
\Delta^n \times B\bullet \quad \quad \rightarrow \quad \Delta^n \times A\bullet
\]

\[
\Delta^n \times B\bullet \quad \quad \rightarrow \quad Y\bullet
\]

This is possible by virtue of Proposition [1.4.5.3] since \( p \) is a trivial Kan fibration and \( i \) is a monomorphism.

**Corollary 1.4.5.6.** Let \( p : X\bullet \rightarrow Y\bullet \) be a trivial Kan fibration of simplicial sets. Then, for every simplicial set \( B\bullet \), the induced map \( \text{Fun}(B\bullet, X\bullet) \rightarrow \text{Fun}(B\bullet, Y\bullet) \) is a trivial Kan fibration.

**Proof.** Apply Corollary 1.4.5.5 in the special case \( A\bullet = \emptyset \).

**Definition 1.4.5.7.** Let \( X\bullet \) be a simplicial set. We say that \( X\bullet \) is a contractible Kan complex if the projection map \( X\bullet \rightarrow \Delta^0 \) is a trivial Kan fibration (Definition 1.4.5.1). In other words, \( X\bullet \) is a contractible Kan complex if every map \( \sigma_0 : \partial \Delta^n \rightarrow X\bullet \) can be extended to an \( n \)-simplex of \( X\bullet \).

**Example 1.4.5.8.** Let \( X \) be a topological space. Then the singular simplicial set \( \text{Sing}_\bullet(X) \) is a contractible Kan complex if and only if the space \( X \) is weakly contractible: that is, if and only if every continuous map \( \sigma_0 : S^{n-1} \rightarrow X \) is nullhomotopic (here \( S^{n-1} \simeq |\partial \Delta^n| \) denotes the sphere of dimension \( n - 1 \), so that \( \sigma_0 \) is nullhomotopic if and only if it extends to a continuous map defined on the disk \( D^n \simeq |\Delta^n| \)). In particular, if the topological space \( X \) is contractible, then the simplicial set \( \text{Sing}_\bullet(X) \) is a contractible Kan complex.

**Remark 1.4.5.9.** Let \( p : X\bullet \rightarrow Y\bullet \) be a trivial Kan fibration. Then, for every vertex \( y \) of \( Y\bullet \), the fiber \( X\bullet \times_{Y\bullet} \{y\} \) is a contractible Kan complex (this is a special case of Remark 1.4.5.2). For a partial converse, see Proposition [?].

Applying Proposition 1.4.5.3 in the case \( Y\bullet = \Delta^0 \), we obtain the following:

**Corollary 1.4.5.10.** Let \( X\bullet \) be a simplicial set. The following conditions are equivalent:

1. The simplicial set \( X\bullet \) is a contractible Kan complex.

2. For every monomorphism of simplicial sets \( i : A\bullet \rightarrow B\bullet \) and every map of simplicial sets \( f_0 : A\bullet \rightarrow X\bullet \), there exists a map \( f : B\bullet \rightarrow X\bullet \) such that \( f_0 = f \circ i \).
Corollary 1.4.5.11. Let $X_\bullet$ be a contractible Kan complex. Then $X_\bullet$ is a Kan complex. In particular, $X_\bullet$ is an $\infty$-category.

We will deduce Proposition 1.4.5.3 from the following:

Proposition 1.4.5.12. Let $T$ be the collection of all monomorphisms in the category $\Set_\Delta$ of simplicial sets. Then:

(a) The collection $T$ is weakly saturated, in the sense of Definition 1.4.4.15.

(b) As a weakly saturated collection of morphisms, $T$ is generated by collection of inclusion maps $\{\partial \Delta^n \hookrightarrow \Delta^n\}_{n \geq 0}$ (see Remark 1.4.4.17).

Proof. To prove (a), we must establish the following:

- The collection $T$ is closed under pushouts. That is, if we are given a pushout diagram of simplicial sets
  \[
  \begin{array}{ccc}
  A_\bullet & \longrightarrow & A'_\bullet \\
  \downarrow^f & & \downarrow^{f'} \\
  B_\bullet & \longrightarrow & B'_\bullet
  \end{array}
  \]
  where $f$ is a monomorphism, then $f'$ is also a monomorphism. This is clear, since we have a pushout diagram
  \[
  \begin{array}{ccc}
  A_n & \longrightarrow & A'_n \\
  \downarrow & & \downarrow \\
  B_n & \longrightarrow & B'_n
  \end{array}
  \]
  in the category of sets for each $n \geq 0$ (where the left vertical map is injective, so the right vertical map is injective as well).

- The collection $T$ is closed under retracts. This is a special case of Exercise 1.4.4.8.

- The collection $T$ is closed under transfinite composition. Suppose we are given an ordinal $\alpha$ and a functor $S : [\alpha] \to \Set_\Delta$ given by a collection of simplicial sets $\{S(\beta)_\bullet\}_{\beta \leq \alpha}$ and transition maps $f_{\gamma, \beta} : S(\beta)_\bullet \to S(\gamma)_\bullet$. Assume that the maps $f_{\beta+1, \beta}$ are monomorphisms for $\beta < \alpha$ and that, for every nonzero limit ordinal $\lambda \leq \alpha$, the induced map $\lim_{\beta < \lambda} S(\beta)_\bullet \to S(\lambda)_\bullet$ is an isomorphism. We must show that the map $f_{\alpha, 0} : S(0)_\bullet \to S(\alpha)_\bullet$ is a monomorphism of simplicial sets. In fact, we claim that for each $\gamma \leq \alpha$, the map $f_{\gamma, 0} : S(0)_\bullet \to S(\gamma)_\bullet$ is a monomorphism. The proof proceeds by transfinite induction on $\gamma$. In the case $\gamma = 0$, the map $f_{\gamma, 0} = \text{id}_{S(0)_\bullet}$ is an isomorphism. If $\gamma$ is a nonzero limit ordinal, then the desired result follows from our inductive hypothesis, since the collection of monomorphisms in $\Set_\Delta$ is closed under
filtered colimits. If $\gamma = \beta + 1$ is a successor ordinal, then we can identify $f_{\gamma, 0}$ with the composition

$$S(0) \xrightarrow{f_{\beta, 0}} S(\beta) \xrightarrow{f_{\gamma, \beta}} S(\gamma),$$

where $f_{\gamma, \beta}$ is a monomorphism by assumption and $f_{\beta, 0}$ is a monomorphism by virtue of our inductive hypothesis.

We now prove (b). Let $T'$ be a collection of morphisms in $\text{Set}_\Delta$ which is weakly saturated and contains each of the inclusions $\partial \Delta^n \hookrightarrow \Delta^n$; we wish to show that every monomorphism $i : A \to B$ belongs to $T'$. For each $k \geq -1$, let $B(k) \subseteq B$ denote the simplicial subset given by the union of the skeleton $\text{sk}_k(B)$ (Construction 1.1.3.5) with the image of $i$. Then the inclusion $i$ can be written as a transfinite composition

$$A \simeq B(-1) \hookrightarrow B(0) \hookrightarrow B(1) \hookrightarrow B(2) \hookrightarrow \cdots$$

Since $T'$ is closed under transfinite composition, it will suffice to show that each of the inclusion maps $B(k-1) \hookrightarrow B(k)$ belongs to $T'$. Applying Proposition 1.1.3.13 to both $A$ and $B$, we obtain a pushout diagram

$$\begin{array}{ccc}
\coprod_{\sigma \in Q} \partial \Delta^k & \longrightarrow & \coprod_{\sigma \in Q} \Delta^k \\
\downarrow & & \downarrow \\
B(k-1) & \longrightarrow & B(k).
\end{array}$$

where $Q$ denotes the collection of all nondegenerate $k$-simplices of $B$ which do not belong to the image of $i$. Since $T'$ is closed under pushouts, we are reduced to showing that the inclusion map

$$j : \coprod_{\sigma \in Q} \partial \Delta^k \hookrightarrow \coprod_{\sigma \in Q} \Delta^k$$

belongs to $T'$. Choosing a well-ordering of $Q$, we see that $j$ can be written as a transfinite composition of morphisms

$$j_{\sigma} : \left( \coprod_{\tau \leq \sigma} \partial \Delta^k \right) \coprod \left( \coprod_{\tau > \sigma} \Delta^k \right) \hookrightarrow \left( \coprod_{\tau < \sigma} \partial \Delta^k \right) \coprod \left( \coprod_{\tau \geq \sigma} \Delta^k \right),$$

each of which is a pushout of the inclusion $\partial \Delta^k \hookrightarrow \Delta^k$.

Proof of Proposition 1.4.5.3. Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets and let $T$ be the collection of all morphisms in $\text{Set}_\Delta$ which have the left lifting property with respect to $p$. Then $T$ contains each of the inclusions $\partial \Delta^n \hookrightarrow \Delta^n$ (by virtue of our assumption that $p$ is a trivial Kan fibration) and is weakly saturated (Proposition 1.4.4.16). It follows from Proposition 1.4.5.12 that every monomorphism of simplicial sets $i : A \hookrightarrow B$ belongs to $T$ (and therefore has the left lifting property with respect to $p$).
1.4.6 Uniqueness of Composition

Let \( C \) be an \( \infty \)-category. Given a composable pair of morphisms \( f : X \to Y \) and \( g : Y \to Z \) in \( C \), one can form a composition \( g \circ f \) by choosing a 2-simplex \( \sigma \) with \( d_0(\sigma) = g \) and \( d_2(\sigma) = f \), as indicated in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g \circ f} & Z \\
\downarrow^f & & \downarrow^g \\
Y & & \\
\end{array}
\]

In general, neither the 2-simplex \( \sigma \) nor the resulting morphism \( g \circ f = d_1(\sigma) \) is uniquely determined. However, we saw in §1.3.4 that the composition \( g \circ f \) is unique up to homotopy (Proposition 1.3.4.2). We now prove a stronger result, which asserts that the 2-simplex \( \sigma \) (hence also the composite morphism \( g \circ f = d_1(\sigma) \)) is unique up to a contractible space of choices.

**Theorem 1.4.6.1 (Joyal).** Let \( S_\bullet \) be a simplicial set. The following conditions are equivalent:

1. The simplicial set \( S_\bullet \) is an \( \infty \)-category.
2. The inclusion of simplicial sets \( \Lambda^2_1 \hookrightarrow \Delta^2 \) induces a trivial Kan fibration

\[
\text{Fun}(\Delta^2, S_\bullet) \to \text{Fun}(\Lambda^2_1, S_\bullet).
\]

**Corollary 1.4.6.2.** Let \( f : X \to Y \) and \( g : Y \to Z \) be a composable pair of morphisms in an \( \infty \)-category \( C \), so that the tuple \( (g, \bullet, f) \) determines a map of simplicial sets \( \Lambda^2_1 \to \mathcal{C} \) (see Exercise 1.1.2.14). Then the fiber product

\[
\text{Fun}(\Delta^2, \mathcal{C}) \times_{\text{Fun}(\Lambda^2_1, \mathcal{C})} \{(g, \bullet, f)\}
\]

is a contractible Kan complex.

**Proof.** Combine Theorem 1.4.6.1 with Remark 1.4.5.9. \( \square \)

**Remark 1.4.6.3.** In the situation of Corollary 1.4.6.2, one can think of the simplicial set

\[
Z_\bullet = \text{Fun}(\Delta^2, \mathcal{C}) \times_{\text{Fun}(\Lambda^2_1, \mathcal{C})} \{(g, \bullet, f)\}
\]

as a “parameter space” for all choices of 2-simplex \( \sigma \) satisfying \( d_0(\sigma) = g \) and \( d_2(\sigma) = f \) (note that such 2-simplices can be identified with the vertices of \( Z_\bullet \)). Consequently, we can summarize Corollary 1.4.6.2 informally by saying that this parameter space is contractible.

We will give the proof of Theorem 1.4.6.1 at the end of this section. First, let us note one of its consequences.
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Proof of Theorem 1.4.3.7. Let $S_\bullet$ be a simplicial set and let $D$ be an ∞-category. We wish to show that the simplicial set $\text{Fun}(S_\bullet, D)$ is an ∞-category. By virtue of Theorem 1.4.6.1 it will suffice to show that the restriction map

$$r : \text{Fun}(\Delta^2, \text{Fun}(S_\bullet, D)) \to \text{Fun}(\Lambda^2_1, \text{Fun}(S_\bullet, D))$$

is a trivial Kan fibration. Note that we can identify $r$ with the canonical map

$$\text{Fun}(S_\bullet, \text{Fun}(\Delta^2, D)) \to \text{Fun}(S_\bullet, \text{Fun}(\Lambda^2_1, D)),$$

which is a trivial Kan fibration by virtue of Corollary 1.4.5.6 and Theorem 1.4.6.1.

We now introduce some terminology which will be useful for the proof of Theorem 1.4.6.1.

Definition 1.4.6.4. Let $f : A_\bullet \to B_\bullet$ be a morphism of simplicial sets. We will say that $f$ is inner anodyne if it belongs to the weakly saturated class of morphisms generated by the collection of all inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ (so that $0 < i < n$).

Remark 1.4.6.5. Let $f : A_\bullet \to B_\bullet$ be an inner anodyne map of simplicial sets. Then $f$ is a monomorphism. This follows from the observation that the collection of monomorphisms is weakly saturated (Proposition 1.4.5.12), since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a monomorphism.

Exercise 1.4.6.6. Let $f : A_\bullet \hookrightarrow B_\bullet$ be an inner anodyne morphism of simplicial sets. Show that the underlying map on vertices $A_0 \to B_0$ is a bijection.

Proposition 1.4.6.7. Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:

1. The simplicial set $S_\bullet$ is an ∞-category.

2. For every inner anodyne map of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$ and every map $f_0 : A_\bullet \to S_\bullet$ there exists a map $f : B_\bullet \to S_\bullet$ such that $f_0 = f \circ i$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate (since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is inner anodyne). Conversely, if (1) is satisfied, then every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ has the left lifting property with respect to the projection map $p : S_\bullet \to \Delta^0$. It then follows from Remark 1.4.4.17 that every inner anodyne map has the left lifting property with respect to $p$. □

Variant 1.4.6.8. Let $S_\bullet$ be a simplicial set. The following conditions are equivalent:

1. The simplicial set $S_\bullet$ is isomorphic to the nerve of a category.

2. For every inner anodyne map of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$ and every map $f_0 : A_\bullet \to S_\bullet$ there exists a unique map $f : B_\bullet \to S_\bullet$ such that $f_0 = f \circ i$. □
Proof. Let us regard the simplicial set $S_\bullet$ as fixed, and let $T$ be the collection of all morphisms of simplicial sets $i : A_\bullet \to B_\bullet$ for which the induced map $\text{Hom}_{\text{Set}_\Delta}(B_\bullet, S_\bullet) \to \text{Hom}_{\text{Set}_\Delta}(A_\bullet, S_\bullet)$ is bijective. Then $T$ is weakly saturated (in the sense of Definition 1.4.4.15). It follows that (2) is equivalent to the following \textit{a priori} weaker assertion:

(2') For every pair of integers $0 < i < n$, the map $\text{Hom}_{\text{Set}_\Delta}(\Delta^n, S_\bullet) \to \text{Hom}_{\text{Set}_\Delta}(\Lambda^n_i, S_\bullet)$ is bijective.

The equivalence of (1) and (2') is the content of Proposition 1.2.3.1. 

We will deduce Theorem 1.4.6.1 from the following technical result:

\textbf{Lemma 1.4.6.9 (Joyal).}

(a) For every monomorphism of simplicial sets $i : A_\bullet \hookrightarrow B_\bullet$, the induced map

\[(B_\bullet \times \Lambda^2_1) \coprod_{A_\bullet \times \Lambda^2_1} (A_\bullet \times \Delta^2) \subseteq B_\bullet \times \Delta^2\]

is inner anodyne.

(b) The collection of inner anodyne morphisms is generated (as a weakly saturated class) by the inclusion maps

\[(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2\]

for $m \geq 0$.

\textit{Proof.} Let $T$ be the weakly saturated class of morphisms generated by all inclusions of the form

\[(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2,\]

and let $S$ be the collection of all morphisms of simplicial sets $A_\bullet \to B_\bullet$ for which the map

\[(B_\bullet \times \Lambda^2_1) \coprod_{A_\bullet \times \Lambda^2_1} (A_\bullet \times \Delta^2) \subseteq B_\bullet \times \Delta^2\]

belongs to $T$. By construction, $S$ contains all inclusions of the form $\partial \Delta^m \hookrightarrow \Delta^m$. Moreover, since $T$ is weakly saturated, the class $S$ is also weakly saturated. It follows that every monomorphism of simplicial sets belongs to $S$ (Proposition 1.4.5.12). Consequently, to prove Lemma 1.4.6.9, it will suffice to show that $T$ coincides with the class of inner anodyne morphisms of $\text{Set}_\Delta$. We first show that every inner anodyne morphism belongs to $T$. Since $T$
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is weakly saturated, we are reduced to showing that every inner horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$ belongs to $T$. Since $f$ belongs to $S$, the monomorphism

$$\overline{f} : (\Delta^n \times \Lambda^2_j) \coprod_{\Lambda^n_i \times \Lambda^2_j} (\Lambda^n_i \times \Delta^2) \subseteq \Delta^n \times \Delta^2.$$ 

belongs to $T$. We conclude by observing that the morphism $f$ is a retract of $\overline{f}$. More precisely, we have a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{f} & (\Delta^n \times \Lambda^2_j) \coprod \Lambda^n_i \times \Lambda^2_j (\Lambda^n_i \times \Delta^2) \\
\Delta^n & \xrightarrow{s} & \Delta^n \times \Delta^2
\end{array}
\]

where the maps $s$ and $r$ are given on vertices by the formulae

\[
s(j) = \begin{cases} (j, 0) & \text{if } j < i \\
(j, 1) & \text{if } j = i \\
(j, 2) & \text{if } j > i \end{cases}
\]

\[
r(j, k) = \begin{cases} j & \text{if } j < i, k = 0 \\
(j - 1, 1) & \text{if } i + 1 \leq k \leq j + 1 \\
(j - 1, 2) & \text{if } j + 2 \leq k \leq m + 1. \end{cases}
\]

We now show that every morphism of $T$ is inner anodyne. Since the collection of inner anodyne morphisms is weakly saturated, it will suffice to show that the inclusion map

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

is inner anodyne for each $m \geq 0$. For each $0 \leq i \leq j < m$, we let $\sigma_{ij}$ denote the $(m + 1)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$f_{ij} : [m + 1] \to [m] \times [2]$$

\[
f_{ij}(k) = \begin{cases} (k, 0) & \text{if } 0 \leq k \leq i \\
(k - 1, 1) & \text{if } i + 1 \leq k \leq j + 1 \\
(k - 1, 2) & \text{if } j + 2 \leq k \leq m + 1. \end{cases}
\]

For each $0 \leq i \leq j \leq m$, we let $\tau_{ij}$ denote the $(m + 2)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$g_{ij} : [m + 2] \to [m] \times [2]$$

\[
g_{ij}(k) = \begin{cases} (k, 0) & \text{if } 0 \leq k \leq i \\
(k - 1, 1) & \text{if } i + 1 \leq k \leq j + 1 \\
(k - 1, 2) & \text{if } j + 2 \leq k \leq m + 1. \end{cases}
\]
\[ g_{ij}(k) = \begin{cases} (k, 0) & \text{if } 0 \leq k \leq i \\ (k - 1, 1) & \text{if } i + 1 \leq k \leq j + 1 \\ (k - 2, 2) & \text{if } j + 2 \leq k \leq m + 2. \end{cases} \]

We will regard each \( \sigma_{ij} \) and \( \tau_{ij} \) as a simplicial subset of \( \Delta^m \times \Delta^2 \).

Set \( X(0) = (\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \). For \( 0 \leq j < m \), we let \( X(j + 1) = X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj} \).

We have a chain of inclusions
\[ X(j) \subseteq X(j) \cup \sigma_{0j} \subseteq \cdots \subseteq X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj} = X(j + 1). \]
Each of these inclusions fits into a pushout diagram
\[
\begin{array}{ccc}
\Lambda_{i+1}^{m+1} & \longrightarrow & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{(i-1)j} \\
\downarrow & & \downarrow \\
\sigma_{ij} & \longrightarrow & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{ij},
\end{array}
\]
and is therefore inner anodyne. Set \( Y(0) = X(m) \), so that the inclusion \( X(0) \subseteq Y(0) \) is inner anodyne. We now set \( Y(j + 1) = Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{jj} \) for \( 0 \leq j \leq m \). As before, we have a chain of inclusions
\[ Y(j) \subseteq Y(j) \cup \tau_{0j} \subseteq \cdots \subseteq Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{jj} = Y(j + 1), \]
each of which fits into a pushout diagram
\[
\begin{array}{ccc}
\Lambda_{i+1}^{m+2} & \longrightarrow & Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{(i-1)j} \\
\downarrow & & \downarrow \\
\tau_{ij} & \longrightarrow & Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{ij},
\end{array}
\]
and is therefore inner anodyne. It follows that each inclusion \( Y(j) \subseteq Y(j + 1) \) is inner anodyne. Since the collection of inner anodyne morphisms is closed under composition, we conclude that the inclusion map \( X(0) \hookrightarrow Y(0) \hookrightarrow Y(1) \hookrightarrow \cdots Y(m + 1) = \Delta^m \times \Delta^2 \) is inner anodyne, as desired.

**Proof of Theorem 1.4.6.1.** Let \( S_\bullet \) be a simplicial set and let \( p : \text{Fun}(\Delta^2, S_\bullet) \to \text{Fun}(\Lambda^2_1, S_\bullet) \) denote the restriction map. Then \( p \) is a trivial Kan fibration if and only if every lifting problem
\[
\begin{array}{ccc}
\partial \Delta^m & \longrightarrow & \text{Fun}(\Delta^2, S_\bullet) \\
\downarrow & \nearrow \downarrow p \\
\Delta^m & \longrightarrow & \text{Fun}(\Lambda^2_1, S_\bullet),
\end{array}
\]
admits a solution. Let $T$ be the collection of all morphisms of simplicial sets which have the left lifting property with respect to the projection $S_\bullet \to \Delta^0$. Then $p$ is a trivial Kan fibration if and only if $T$ contains each of the inclusion maps

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2.$$ 

Since $T$ is weakly saturated (Proposition 1.4.4.16), this is equivalent to the requirement that $T$ contains all inner anodyne morphisms (Lemma 1.4.6.9), which is in turn equivalent to the requirement that $S_\bullet$ is an $\infty$-category (Proposition 1.4.6.7). 

1.4.7 Universality of Path Categories

Let $G$ be a directed graph, let $G_\bullet$ denote the associated 1-dimensional simplicial set (see Proposition 1.1.5.9), and let Path$[G]$ denote the path category of $G$ (Construction 1.2.6.1). There is an evident map of simplicial sets $u : G_\bullet \to N_\bullet(\text{Path}[G])$. By virtue of Proposition 1.2.6.5, this map exhibits Path$[G]$ as the homotopy category of the simplicial set $G_\bullet$. In other words, the path category Path$[G]$ is universal among categories $\mathcal{C}$ which are equipped with a $G_\bullet$-indexed diagram (see Definition 1.4.2.1). Our goal in this section is to establish a variant of this statement in the setting of $\infty$-categories:

**Theorem 1.4.7.1.** Let $G$ be a directed graph and let $\mathcal{C}$ be an $\infty$-category. Then composition with the map of simplicial sets $u : G_\bullet \to N_\bullet(\text{Path}[G])$ induces a trivial Kan fibration of simplicial sets $\text{Fun}(N_\bullet(\text{Path}[G]), \mathcal{C}) \to \text{Fun}(G_\bullet, \mathcal{C})$.

More informally, Theorem 1.4.7.1 asserts that any $G$-indexed diagram in an $\infty$-category $\mathcal{C}$ admits an essentially unique extension to a functor of $\infty$-categories $N_\bullet(\text{Path}[G]) \to \mathcal{C}$.

**Example 1.4.7.2.** Let $G$ be the directed graph depicted in the diagram

$$\bullet \to \bullet.$$ 

Then the map $u : G_\bullet \to N_\bullet(\text{Path}[G])$ can be identified with the inclusion of simplicial sets $\Lambda^2_1 \hookrightarrow \Delta^2$. In this case, Theorem 1.4.7.1 reduces to the statement that the map

$$\text{Fun}(\Delta^2, \mathcal{C}) \to \text{Fun}(\Lambda^2_1, \mathcal{C})$$

admits a solution. Unwinding the definitions, we see that this is equivalent to the requirement that every lifting problem of the form

$$
\begin{array}{ccc}
\Delta^m \times \Lambda^2_1 & \coprod & (\partial \Delta^m \times \Delta^2) \\
\downarrow & & \downarrow \\
\Delta^m \times \Delta^2 & \to & \Delta^0
\end{array}
$$

admits a solution. Let $T$ be the collection of all morphisms of simplicial sets which have the left lifting property with respect to the projection $S_\bullet \to \Delta^0$. Then $p$ is a trivial Kan fibration if and only if $T$ contains each of the inclusion maps

$$
(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2.
$$

Since $T$ is weakly saturated (Proposition 1.4.4.16), this is equivalent to the requirement that $T$ contains all inner anodyne morphisms (Lemma 1.4.6.9), which is in turn equivalent to the requirement that $S_\bullet$ is an $\infty$-category (Proposition 1.4.6.7).
is a trivial Kan fibration, which is equivalent to the assumption that $\mathcal{C}$ is an $\infty$-category by virtue of Theorem 1.4.6.1.

We will deduce Theorem 1.4.7.1 from the following more precise assertion.

**Proposition 1.4.7.3.** Let $G$ be a directed graph. Then the map of simplicial sets $u : G_\bullet \hookrightarrow N_\bullet(\text{Path}[G])$ is inner anodyne (Definition 1.4.6.4).

**Remark 1.4.7.4.** Let $G$ be a directed graph and let $\mathcal{C}$ be an ordinary category. Combining Proposition 1.4.7.3 with Variant 1.4.6.8, we deduce that the canonical map

$$\text{Hom}_{\text{Set}}(\Delta(N_\bullet(\text{Path}[G])), N_\bullet(\mathcal{C})) \to \text{Hom}_{\text{Set}}(G_\bullet, N_\bullet(\mathcal{C}))$$

is bijective. Combining this observation with Proposition 1.2.2.1, we obtain a bijection

$$\text{Hom}_{\text{Cat}}(\text{Path}[G], \mathcal{C}) \to \text{Hom}_{\text{Set}}(G_\bullet, N_\bullet(\mathcal{C})).$$

Allowing $\mathcal{C}$ to vary, we recover the assertion that $u : G_\bullet \hookrightarrow N_\bullet(\text{Path}[G])$ exhibits Path$[G]$ as the homotopy category of $G_\bullet$ (Proposition 1.2.6.5).

Let us first show that Proposition 1.4.7.3 implies Theorem 1.4.7.1.

**Lemma 1.4.7.5.** Let $f : X_\bullet \hookrightarrow Y_\bullet$ and $f' : X'_\bullet \hookrightarrow Y'_\bullet$ be monomorphisms of simplicial sets. If $f$ is inner anodyne, then the induced map

$$u_{f,f'} : (Y_\bullet \times X'_\bullet) \coprod_{(X_\bullet \times X'_\bullet)} (X_\bullet \times Y'_\bullet) \hookrightarrow Y_\bullet \times Y'_\bullet$$

is inner anodyne.

**Proof.** Let us regard the morphism $f' : X'_\bullet \hookrightarrow Y'_\bullet$ as fixed. Let $T$ be the collection of all morphisms $f : X_\bullet \to Y_\bullet$ for which the map $u_{f,f'}$ is inner anodyne. Then $T$ is weakly saturated. To prove Lemma 1.4.7.5 we must show that $T$ contains all inner anodyne morphisms of simplicial sets. By virtue of Lemma 1.4.6.9 it will suffice to show that $T$ contains every morphism of the form

$$u_{i,j} : (B_\bullet \times A^2_1) \coprod_{A_\bullet \times A^2_1} (A_\bullet \times \Delta^2) \subseteq B_\bullet \times \Delta^2,$$

where $i : A_\bullet \hookrightarrow B_\bullet$ is a monomorphism of simplicial sets and $j : A^2_1 \hookrightarrow \Delta^2$ is the inclusion. Setting

$$A'_\bullet = (B_\bullet \times X'_\bullet) \coprod_{(A_\bullet \times X'_\bullet)} (A_\bullet \times Y'_\bullet) \quad B'_\bullet = B_\bullet \times Y'_\bullet,$$

we are reduced to the problem of showing that the map

$$u'_{i,j} : (B'_\bullet \times A^2_1) \coprod_{A'_\bullet \times A^2_1} (A'_\bullet \times \Delta^2) \subseteq B'_\bullet \times \Delta^2,$$

is inner anodyne, which follows from Lemma 1.4.6.9. \qed
Proposition 1.4.7.6. Let \( C \) be an \( \infty \)-category and let \( f : X_\bullet \rightarrow Y_\bullet \) be an inner anodyne morphism of simplicial sets. Then the induced map \( p : \text{Fun}(Y_\bullet, C) \rightarrow \text{Fun}(X_\bullet, C) \) is a trivial Kan fibration.

Proof. To show that \( p \) is a trivial Kan fibration, it will suffice to show that it has the right lifting property with respect to every monomorphism of simplicial sets \( f' : X'_\bullet \rightarrow Y'_\bullet \). This is equivalent to the assertion that every map of simplicial sets

\[
g_0 : (Y_\bullet \times X'_\bullet) \coprod_{(X_\bullet \times X'_\bullet)} (X_\bullet \times Y'_\bullet) \rightarrow C
\]

can be extended to a map \( g : Y_\bullet \times Y'_\bullet \rightarrow C \). This follows from Proposition 1.4.7.7, since \( C \) is an \( \infty \)-category and the map

\[
u_{f,f'} : (Y_\bullet \times X'_\bullet) \coprod_{(X_\bullet \times X'_\bullet)} (X_\bullet \times Y'_\bullet) \rightarrow Y_\bullet \times Y'_\bullet
\]
is inner anodyne (Lemma 1.4.7.5).

Proof of Theorem 1.4.7.1. Let \( G \) be a graph and let \( C \) be an \( \infty \)-category; we wish to show that the canonical map

\[
\text{Fun}(N_\bullet(\text{Path}[G]), C) \rightarrow \text{Fun}(G_\bullet, C)
\]
is a trivial Kan fibration. This follows from Proposition 1.4.7.6, since the inclusion \( G_\bullet \hookrightarrow N_\bullet(\text{Path}[G]) \) is inner anodyne (Proposition 1.4.7.3).

Before giving the proof of Proposition 1.4.7.3, let us illustrate its contents with some examples.

Example 1.4.7.7 (The Spine of a Simplex). Let \( n \geq 0 \) and let \( \Delta^n \) be the standard \( n \)-simplex (Construction 1.1.2.1). We let \( \text{Spine}[n] \) denote the simplicial subset of \( \Delta^n \) whose \( k \)-simplices are monotone maps \( \sigma : [k] \rightarrow [n] \) satisfying \( \sigma(k) \leq \sigma(0) + 1 \). We will refer to \( \text{Spine}[n] \) as the spine of the simplex \( \Delta^n \). More informally, it is comprised of all vertices of \( \Delta^n \), together with those edges which join adjacent vertices. The spine \( \text{Spine}[n] \) is a simplicial set of dimension \( \leq 1 \), which we can identify with the directed graph \( G \) depicted in the diagram

\[
0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n.
\]

Under this identification, the map \( u : G_\bullet \rightarrow N_\bullet(\text{Path}[G]) \) corresponds to the inclusion \( \text{Spine}[n] \hookrightarrow \Delta^n \) (see Example 1.2.6.2). Invoking Proposition 1.4.7.3 and Theorem 1.4.7.1 we obtain the following:

(a) The inclusion \( \text{Spine}[n] \hookrightarrow \Delta^n \) is inner anodyne.
For any $\infty$-category $\mathcal{C}$, the restriction map $\text{Fun}(\Delta^n, \mathcal{C}) \to \text{Fun}(\text{Spine}[n], \mathcal{C})$ is a trivial Kan fibration.

**Remark 1.4.7.8** (The Generalized Associative Law). Let $\mathcal{C}$ be an ordinary category and let $n \geq 0$ be an integer. Applying Remark 1.4.7.4 to the inner anodyne inclusion $\text{Spine}[n] \hookrightarrow \Delta^n$ of Example 1.4.7.7, we deduce that every diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n$$

can be extended uniquely to a functor $[n] \to \mathcal{C}$. In particular, it shows that $\mathcal{C}$ satisfies the “generalized associative law”: the iterated composition $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is well-defined (that is, it does not depend on a choice of parenthesization). In essence, Proposition 1.4.7.3 can be regarded as an extension of this generalized associative law to the setting of $\infty$-categories.

**Remark 1.4.7.9.** Let $\mathcal{C}$ be an $\infty$-category and let $\text{h}\mathcal{C}$ denote its homotopy category (Definition 1.3.5.3). Then the canonical map $\mathcal{C} \to \text{N}_\bullet(\text{h}\mathcal{C})$ is an epimorphism of simplicial sets: that is, it induces a surjection on $n$-simplices for each $n \geq 0$. To prove this, we note that there is a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{\text{Set}_\Delta}(\Delta^n, \mathcal{C}) & \longrightarrow & \text{Hom}_{\text{Set}_\Delta}(\Delta^n, \text{N}_\bullet(\text{h}\mathcal{C})) \\
\downarrow & & \downarrow \sim \\
\text{Hom}_{\text{Set}_\Delta}(\text{Spine}[n], \mathcal{C}) & \longrightarrow & \text{Hom}_{\text{Set}_\Delta}(\text{Spine}[n], \text{h}\mathcal{C}),
\end{array}
$$

where the left vertical map is surjective (Example 1.4.7.7) and the right vertical map is bijective (Remark 1.4.7.8). It therefore suffices to show that the bottom horizontal map is surjective: that is, every sequence of composable morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \to \cdots \xrightarrow{f_n} X_n$$

in the homotopy category $\text{h}\mathcal{C}$ can be lifted to a sequence of composable morphisms in $\mathcal{C}$, which is immediate from the definition of $\text{h}\mathcal{C}$.

**Example 1.4.7.10** (The Simplicial Circle). Let $S^1_\bullet$ denote the simplicial set obtained from $\Delta^1$ by collapsing the boundary $\partial \Delta^1$ to a point, so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial \Delta^1 & \longrightarrow & \Delta^1 \\
\downarrow & & \downarrow \\
\Delta^0 & \longrightarrow & S^1_\bullet.
\end{array}
$$

We will refer to $S^1_\bullet$ as the *simplicial circle*; note that the geometric realization $|S^1_\bullet|$ is isomorphic to the standard circle $S^1$ as a topological space. The simplicial set $S^1_\bullet$ has
dimension \( \leq 1 \), and can therefore be identified with the directed graph \( G \) depicted in the diagram

Note that the path category \( \text{Path}[G] \) can be identified with the category \( B\mathbb{Z}_{\geq 0} \) associated to the monoid \( \mathbb{Z}_{\geq 0} \) of nonnegative numbers under addition (Example 1.2.6.4) whose nerve is the simplicial set \( B\mathbb{Z}_{\geq 0} \) of Example 1.2.4.3. Invoking Proposition 1.4.7.3 and Theorem 1.4.7.1 we obtain the following:

(a) The inclusion of simplicial sets \( S^1 \hookrightarrow B\mathbb{Z}_{\geq 0} \) is inner anodyne.

(b) For any \( \infty \)-category \( C \), the restriction map \( \text{Fun}(B\mathbb{Z}_{\geq 0}, C) \to \text{Fun}(S^1, C) \) is a trivial Kan fibration.

Example 1.4.7.11 (Free Monoids). Let \( M \) be the free monoid generated by a set \( E \). Then we can identify \( BM \) with the path category \( \text{Path}[G] \) of a directed graph \( G \) satisfying

\[
\text{Vert}(G) = \{ x \} \quad \text{Edge}(G) = E;
\]

see Example 1.2.6.3 Invoking Proposition 1.4.7.3 and Theorem 1.4.7.1 we obtain the following:

(a) The inclusion of simplicial sets \( G_\bullet \hookrightarrow B\mathbb{M} \) is inner anodyne.

(b) For any \( \infty \)-category \( C \), the restriction map \( \text{Fun}(B\mathbb{M}, C) \to \text{Fun}(G_\bullet, C) \) is a trivial Kan fibration.

Note that if \( C \) is an \( \infty \)-category, then a map of simplicial sets \( \sigma_0 : G_\bullet \to C \) can be identified with a choice of object \( X \in C \) together with a collection of morphisms \( \{ f_e : X \to X \}_{e \in E} \) indexed by \( E \). It follows from (b) that any such map admits an (essentially unique) extension to a functor \( \sigma : B_\bullet M \to C \), which we can interpret as an action of the monoid \( M \) on the object \( X \in C \).

Proof of Proposition 1.4.7.3 Let \( G \) be a directed graph and let \( \text{Path}[G] \) denote its path category. By definition, a morphism from \( x \in \text{Vert}(G) \) to \( y \in \text{Vert}(G) \) in the category \( \text{Path}[G] \) is given by a sequence of edges \( \vec{e} = (e_m, e_{m-1}, \ldots, e_1) \) satisfying

\[
s(e_1) = x \quad t(e_i) = s(e_{i+1}) \quad t(e_m) = y.
\]

In this case, we will refer to \( m \) as the length of the morphism \( \vec{e} \) and write \( m = \ell(\vec{e}) \). If \( \sigma : \Delta^n \to N_\bullet(\text{Path}[G]) \) is an \( n \)-simplex given by a diagram

\[
\begin{array}{ccccccc}
x_0 & \xrightarrow{\vec{e}_1} & x_1 & \xrightarrow{\vec{e}_2} & \cdots & \xrightarrow{\vec{e}_n} & x_n
\end{array}
\]
in $\text{Path}[G]$, we define the \textit{length} $\ell(\sigma)$ to be the sum $\ell(\vec{e}_1) + \cdots + \ell(\vec{e}_n) = \ell(\vec{e}_n \circ \cdots \circ \vec{e}_1)$. For each positive integer $k$, let $N_{\leq k}(\text{Path}[G])$ denote the simplicial subset of $N_{\bullet}(\text{Path}[G])$ consisting of those simplices having length $\leq k$. We then have inclusions

$$N_{\leq 1}(\text{Path}[G]) \subseteq N_{\leq 2}(\text{Path}[G]) \subseteq N_{\leq 3}(\text{Path}[G]) \subseteq \cdots,$$

where $N_{\leq 1}(\text{Path}[G]) = G_{\bullet}$ and $N_{\bullet}(\text{Path}[G]) = \bigcup N_{\leq k}(\text{Path}[G])$. Consequently, to show that the inclusion $G_{\bullet} \hookrightarrow N_{\bullet}(\text{Path}[G])$ is inner anodyne, it will suffice to show that each of the inclusion maps $N_{\leq k}(\text{Path}[G]) \hookrightarrow N_{\leq k+1}(\text{Path}[G])$ is inner anodyne.

We henceforth regard the integer $k \geq 1$ as fixed. Let $\sigma : \Delta^n \to N_{\bullet}(\text{Path}[G])$ be an $n$-simplex of $N_{\bullet}(\text{Path}[G])$ having length $k+1$, corresponding to a diagram

$$x_0 \xrightarrow{\vec{e}_1} x_1 \xrightarrow{\vec{e}_2} \cdots \xrightarrow{\vec{e}_n} x_n$$

as above. Note that $\sigma$ is nondegenerate if and only if each $\vec{e}_i$ has positive length. We will say that $\sigma$ is \textit{normalized} if it is nondegenerate and $\ell(\vec{e}_1) = 1$. Let $S(n)$ be the collection of all normalized $n$-simplices of $N_{\leq k+1}(\text{Path}[G])$ having length $k+1$. We make the following observations:

(i) If $\sigma$ belongs to $S(n)$, then the faces $d_0(\sigma)$ and $d_n(\sigma)$ have length $\leq k$, and are therefore contained in $N_{\leq k}(\text{Path}[G])$.

(ii) If $\sigma$ belongs to $S(n)$ and $1 < i < n$, then the face $d_i(\sigma)$ is a normalized $(n-1)$-simplex of $N_{\leq k+1}(\text{Path}[G])$ of length $k+1$, and therefore belongs to $S(n-1)$.

(iii) If $\sigma$ belongs to $S(n)$, then the face $d_1(\sigma)$ is \textit{not} normalized. Moreover, the construction $\sigma \mapsto d_1(\sigma)$ induces a bijection from $S(n)$ to the collection of $(n-1)$-simplices of $N_{\leq k+1}(\text{Path}[G])$ which are nondegenerate, of length $k+1$, and not normalized.

For each $n \geq 1$, let $X(n)_{\bullet}$ denote the simplicial subset of $N_{\leq k+1}(\text{Path}[G])$ given by the union of the $(n-1)$-skeleton $\text{sk}_{n-1}(N_{\leq k+1}(\text{Path}[G]))$, the simplicial set $N_{\leq k}(\text{Path}[G])$, and the collection of normalized $n$-simplices of $N_{\leq k+1}(\text{Path}[G])$. We have inclusions

$$X(1)_{\bullet} \subseteq X(2)_{\bullet} \subseteq X(3)_{\bullet} \subseteq X(4)_{\bullet} \subseteq \cdots,$$

where $N_{\leq k}(\text{Path}[G]) = X(1)_{\bullet}$ and $N_{\leq k+1}(\text{Path}[G]) = \bigcup_n X(n)_{\bullet}$. It will therefore suffice to show that the inclusion maps $X(n-1)_{\bullet} \hookrightarrow X(n)_{\bullet}$ are inner anodyne for $n \geq 2$. We conclude by observing that (i), (ii), and (iii) guarantee the existence of a pushout diagram of simplicial sets

$$\begin{array}{ccc}
\bigoplus_{\sigma \in S(n)} \Lambda_{\sigma}^n & \longrightarrow & \bigoplus_{\sigma \in S(n)} \Delta^n \\
\downarrow & & \downarrow \\
X(n-1)_{\bullet} & \longrightarrow & X(n)_{\bullet}
\end{array}$$

\qed
Chapter 2

Examples of $\infty$-Categories

In Chapter 1, we introduced the notion of an $\infty$-category: that is, a simplicial set which satisfies the weak Kan extension condition (Definition 1.3.0.1). The theory of $\infty$-categories can be understood as a synthesis of classical category theory and algebraic topology. This perspective is supported by the two main examples of $\infty$-categories that we have encountered so far:

- Every ordinary category $\mathcal{C}$ can be regarded as an $\infty$-category, by identifying $\mathcal{C}$ with the the simplicial set $N_{\bullet}(\mathcal{C})$ of Construction 1.2.1.1.

- Every Kan complex is an $\infty$-category. In particular, for every topological space $X$, the singular simplicial set $\text{Sing}_{\bullet}(X)$ is an $\infty$-category.

Beware that, individually, both of these examples are rather special. An $\infty$-category $\mathcal{C}$ can be regarded as a mathematical structure which encodes information not only about objects and morphisms (given by the vertices and edges of $\mathcal{C}$, respectively), but also about homotopies between morphisms (Definition 1.3.3.1). When $\mathcal{C}$ is (the nerve of) an ordinary category, the notion of homotopy is trivial: two morphisms in $\mathcal{C}$ (having the same source and target) are homotopic if and only if they are identical. On the other hand, if $\mathcal{C}$ is a Kan complex, then every morphism in $\mathcal{C}$ is invertible up to homotopy (Proposition 1.3.6.11); from a category-theoretic perspective, this is a very restrictive condition.

Our goal in this chapter is to supply a larger class of examples of $\infty$-categories, which are more representative of the subject as a whole. To this end, we introduce three variants of the nerve construction $\mathcal{C} \mapsto N_{\bullet}(\mathcal{C})$ which can be used to produce $\infty$-categories out of other (possibly more familiar) mathematical structures. To describe these constructions in a uniform way, it will be convenient to employ the language of enriched category theory, which we review in §2.1. Let $\mathcal{A}$ be a monoidal category: that is, a category equipped with a tensor product operation $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, which is unital and associative up to (specified)
isomorphisms (see Definition 2.1.2.10). A \( \mathcal{A} \)-enriched category is a mathematical structure \( \mathcal{C} \) consisting of the following data (see Definition 2.1.7.1):

- A collection \( \text{Ob}(\mathcal{C}) \) whose elements we refer to as objects of \( \mathcal{C} \).
- For every pair of objects \( X, Y \in \text{Ob}(\mathcal{C}) \), a mapping object \( \text{Hom}_\mathcal{C}(X, Y) \in \mathcal{A} \).
- For every triple of objects \( X, Y, Z \in \text{Ob}(\mathcal{C}) \), a composition law
  \[ \circ : \text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z), \]
  which we require to be unital and associative.

Taking our Examples [?], [?], and [?], we consider three examples of this paradigm:

- Let \( \mathcal{A} = \text{Set}_\Delta \) be the category of simplicial sets, equipped with the monoidal structure given by Cartesian product. In this case, we refer to an \( \mathcal{A} \)-enriched category as a simplicial category (Definition 2.4.1.1). In §2.4, we associate to each simplicial category \( \mathcal{C} \) a simplicial set \( N^{hc}_{hc}(\mathcal{C}) \), which we refer to as the homotopy coherent nerve of \( \mathcal{C} \) (Definition 2.4.3.5). Moreover, we show that if each of the simplicial sets \( \text{Hom}_\mathcal{C}(X, Y) \) is a Kan complex, then the homotopy coherent nerve \( N^{hc}_{hc}(\mathcal{C}) \) is an \( \infty \)-category (Theorem 2.4.5.1).

- Let \( \mathcal{A} = \text{Ch}(\mathbb{Z}) \) be the category of chain complexes of abelian groups, equipped with the monoidal structure given by tensor product of chain complexes. In this case, we refer to an \( \mathcal{A} \)-enriched category as a differential graded category (Definition 2.5.2.1). In §2.5, we associate to each differential graded category \( \mathcal{C} \) a simplicial set \( N^{dg}_{dg}(\mathcal{C}) \), which we refer to as the differential graded nerve of \( \mathcal{C} \) (Definition 2.5.3.7), and show that \( N^{dg}_{dg}(\mathcal{C}) \) is always \( \infty \)-category (Theorem 2.5.3.10).

- Let \( \mathcal{A} = \text{Cat} \) be the category of (small) categories, equipped with the Cartesian product. In this case, we refer to an \( \mathcal{A} \)-enriched category as a strict 2-category (Definition 2.2.0.1). This is a special case of the more general notion of 2-category (or bicategory, in the terminology of Bénabou), which we review in §2.2. In §2.3, we will associate to each 2-category \( \mathcal{C} \) a simplicial set \( N^\bullet_{\text{D}}(\mathcal{C}) \), which we refer to as the Duskin nerve of \( \mathcal{C} \) (Construction 2.3.1.1). Moreover, we show that if each of the categories \( \text{Hom}_\mathcal{C}(X, Y) \) is a groupoid, then \( N^\bullet_{\text{D}}(\mathcal{C}) \) is an \( \infty \)-category (Theorem 2.3.2.1).

Simplicial categories, differential graded categories, and 2-categories are ubiquitous in algebraic topology, homological algebra, and category theory, respectively. Consequently, the constructions of this section furnish a rich supply of examples of \( \infty \)-categories.
2.1 Monoidal Categories

Recall that a monoid is a set $M$ equipped with a map

$$m : M \times M \to M \quad (x, y) \mapsto xy$$

which satisfies the following conditions:

(a) The multiplication $m$ is associative. That is, we have $x(yz) = (xy)z$ for each triple of elements $x, y, z \in M$.

(b) There exists an element $e \in M$ such that $ex = x = xe$ for each $x \in M$ (in this case, the element $e$ is uniquely determined; we refer to it as the unit element of $M$).

Monoids are ubiquitous in mathematics:

Example 2.1.0.1. Let $C$ be a category and let $X$ be an object of $C$. An endomorphism of $X$ is a morphism from $X$ to itself in the category $C$. We let $\text{End}_C(X) = \text{Hom}_C(X, X)$ denote the set of all endomorphisms of $X$. The composition law on $C$ determines a map

$$\text{End}_C(X) \times \text{End}_C(X) \to \text{End}_C(X) \quad (f, g) \mapsto f \circ g,$$

which exhibits $\text{End}_C(X)$ a monoid; the unit element of $\text{End}_C(X)$ is the identity morphism $\text{id}_X : X \to X$. We refer to $\text{End}_C(X)$ as the endomorphism monoid of $X$.

In the setting of category theory, one often encounters multiplication laws which satisfy a more subtle form of associativity.

Example 2.1.0.2. Let $k$ be a field and let $U$, $V$, and $W$ be vector spaces over $k$. Recall that a function $b : U \times V \to W$ is said to be $k$-bilinear if it satisfies the identities

$$b(u + u', v) = b(u, v) + b(u', v) \quad b(u, v + v') = b(u, v) + b(u, v')$$

$$b(\lambda u, v) = \lambda b(u, v) = b(u, \lambda v)$$

for $\lambda \in k$.

We say that a $k$-bilinear map $b : U \times V \to W$ is universal if, for any $k$-vector space $W'$, composition with $b$ induces a bijection

$$\{k \text{-linear maps } W \to W'\} \simeq \{k \text{-bilinear maps } U \times V \to W'\}.$$

If this condition is satisfied, then $W$ is determined (up to unique isomorphism) by $U$ and $V$; we refer to $W$ as the tensor product of $U$ and $V$ and denote it by $U \otimes_k V$. The construction $(U, V) \mapsto U \otimes_k V$ then determines a functor

$$\otimes_k : \text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k,$$

which we will refer to as the tensor product functor. It is associative in the following sense: for every triple of vector spaces $U, V, W \in \text{Vect}_k$, there exists a canonical isomorphism

$$U \otimes_k (V \otimes_k W) \cong (U \otimes_k V) \otimes_k W \quad u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w.$$
Our goal in this section is to review the theory of monoidal categories, which axiomatizes the essential features of Example 2.1.0.2. To simplify the discussion, we begin by developing the nonunital version of this theory.

**Definition 2.1.0.3.** A nonunital monoid is a set $M$ equipped with a map $m : M \times M \to M \quad (x, y) \mapsto xy$

which satisfies the associative law $x(yz) = (xy)z$ for $x, y, z \in M$.

**Warning 2.1.0.4.** The terminology of Definition 2.1.0.3 is not standard. Most authors use the term semigroup for what we call a nonunital monoid.

In § 2.1.1, we generalize Definition 2.1.0.3 by introducing the notion of a nonunital monoidal structure on a category $\mathcal{C}$ (Definition 2.1.1.5). Roughly speaking, a nonunital monoidal structure on $\mathcal{C}$ is a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative up to isomorphism. More precisely, it consists of the functor $\otimes$ together with a choice of isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \sim (X \otimes Y) \otimes Z$ for every triple of objects $X, Y, Z \in \mathcal{C}$ (these isomorphisms are called the associativity constraints of $\mathcal{C}$). The isomorphisms $\alpha_{X,Y,Z}$ are required to depend functorially on $X, Y,$ and $Z$, and to satisfy a further coherence condition called the pentagon identity (this condition was introduced by MacLane in [28], and is sometimes known as MacLane’s pentagon identity).

By definition, a nonunital monoid $M$ is a monoid if and only if there exists an element $e \in M$ satisfying $ex = x = xe$ for each $x \in M$. If this condition is satisfied, then the element $e$ is uniquely determined. The categorical analogue of this statement is a bit more subtle. Let $X$ be an object of a nonunital monoidal category $\mathcal{C}$, and let $\ell_X, r_X : \mathcal{C} \to \mathcal{C}$ denote the functors given by $\ell_X(Y) = X \otimes Y$ and $r_X(Y) = Y \otimes X$. In § 2.1.2, we define a unit in $\mathcal{C}$ to be an object $1$ with the property that the functors $\ell_1$ and $r_1$ are fully faithful, together with a choice of isomorphism $\upsilon : 1 \otimes 1 \sim 1$. In this case, the pair $(1, \upsilon)$ is not unique; however, it is unique up to (unique) isomorphism (Proposition 2.1.2.9). One can use $\upsilon$ to construct natural isomorphisms

$$\lambda_Y : 1 \otimes Y \sim Y \quad \rho_Y : Y \otimes 1 \sim Y,$$

so that $1$ really behaves like a unit for the tensor product $\otimes$ (Construction 2.1.2.17). We define a monoidal category to be a nonunital monoidal category $\mathcal{C}$ together with a choice of unit $(1, \upsilon)$ (Definition 2.1.2.10). A basic prototype is the category $\text{Vect}_k$ of vector spaces over a field $k$ (equipped with the tensor product and associativity constraints given in Example 2.1.0.2 and the unit given by the object $k \in \text{Vect}_k$). We give a more detailed description of this and other examples in § 2.1.3.

The collection of (nonunital) monoids can be organized into a category:
Definition 2.1.0.5. Let $M$ and $M'$ be nonunital monoids. We say that a function $f : M \to M'$ is a nonunital monoid homomorphism if, for every pair of elements $x, y \in M$, we have $f(xy) = f(x)f(y)$. If $M$ and $M'$ are monoids, we say that $f$ is a monoid homomorphism if it is a nonunital monoid homomorphism which carries the unit element $e \in M$ to the unit element $e' \in M'$.

We let $\text{Mon}^{\text{nu}}$ denote the category whose objects are nonunital monoids and whose morphisms are nonunital monoid homomorphisms, and $\text{Mon} \subset \text{Mon}^{\text{nu}}$ the subcategory whose objects are monoids and whose morphisms are monoid homomorphisms.

Most of the rest of this section is devoted to studying category-theoretic analogues of Definition 2.1.0.5. We start in §2.1.4 with the nonunital case. If $\mathcal{C}$ and $\mathcal{C}'$ are nonunital monoidal categories, we define a nonunital monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ to be a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a collection of isomorphisms

$$\mu_{X,Y} : F(X) \otimes F(Y) \cong F(X \otimes Y),$$

which depend functorially on $X, Y \in \mathcal{C}$ and are compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{C}'$ (Definition 2.1.4.4). We also introduce the more general notion of nonunital lax monoidal functor, where we do not require the morphisms $\mu_{X,Y}$ to be isomorphisms (Definition 2.1.4.3). Both of these definitions have unital analogues, which we study in §2.1.6 and §2.1.5 respectively.

We conclude this section in §2.1.7 with a brief review of enriched category theory. If $\mathcal{A}$ is a monoidal category, then an $\mathcal{A}$-enriched category $\mathcal{C}$ consists of a collection $\text{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$, a collection of mapping objects $\text{Hom}_\mathcal{C}(X, Y) \in \mathcal{A}$ for each pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, and a composition law

$$\text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)$$

which is required to be unital and associative (see Definition 2.1.7.1). Enriched category theory will play an important role throughout this chapter: we will be particularly interested in the special case where $\mathcal{A} = \text{Cat}$ is the category of small categories (in which case we recover the notion of strict 2-category, which we study in §2.2), where $\mathcal{A} = \text{Set}_\Delta$ is the category of simplicial sets (in which case we recover the notion of simplicial category, which we study in §2.4), and where $\mathcal{A} = \text{Ch}(\mathbb{Z})(\text{Ab})$ is the category of chain complexes of abelian groups (in which case we recover the notion of differential graded category, which we study in §2.5).

Remark 2.1.0.6. The construction $\mathcal{C} \mapsto \text{End}_\mathcal{C}(X)$ of Example 2.1.0.1 induces an equivalence

$$\{\text{Categories } \mathcal{C} \text{ with } \text{Ob}(\mathcal{C}) = \{X\}\} \cong \{\text{Monoids}\}.$$
More precisely, there is a pullback diagram of categories

\[
\begin{array}{ccc}
\text{Mon} & \xrightarrow{M \mapsto BM} & \text{Cat} \\
\downarrow & & \downarrow \text{Ob} \\
\{\ast\} & \xrightarrow{\ast \mapsto \text{Set}} & \text{Set},
\end{array}
\]

where \(* = \{X\}\) is the set having a single element \(X\). Here the upper horizontal functor assigns to each monoid \(M\) the category \(BM\) of Example [1.2.3] given concretely by

\[
\text{Ob}(BM) = \{X\} \quad \text{Hom}_{BM}(X, X) = M.
\]

### 2.1.1 Nonunital Monoidal Categories

Let \(\text{Cat}\) denote the category whose objects are (small) categories and whose morphisms are functors. Then \(\text{Cat}\) admits finite products. One can therefore consider (nonunital) monoids in \(\text{Cat}\): that is, small categories \(C\) equipped with a strictly associative multiplication \(\otimes : C \times C \to C\). For the convenience of the reader, we spell out this definition in detail (and abandon the smallness assumption on \(C\)):

**Definition 2.1.1.1.** Let \(C\) be a category. A nonunital strict monoidal structure on \(C\) is a functor

\[
\otimes : C \times C \to C \quad (X, Y) \mapsto X \otimes Y
\]

which is strictly associative in the following sense:

- For every triple of objects \(X, Y, Z \in C\), we have an equality \(X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z\) (as objects of \(C\)).

- For every triple of morphisms \(f : X \to X', g : Y \to Y', h : Z \to Z'\), we have an equality

\[
f \otimes (g \otimes h) = (f \otimes g) \otimes h
\]

of morphisms in \(C\) from the object \(X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z\) to the object \(X' \otimes (Y' \otimes Z') = (X' \otimes Y') \otimes Z'\).

A nonunital strict monoidal category is a pair \((C, \otimes)\), where \(C\) is a category and \(\otimes : C \times C \to C\) is a nonunital strict monoidal structure on \(C\).

**Remark 2.1.1.2.** We will often abuse terminology by identifying a nonunital strict monoidal category \((C, \otimes)\) with the underlying category \(C\). If we refer to a category \(C\) as a nonunital strict monoidal category, we implicitly assume that \(C\) is has been endowed with a tensor product functor \(\otimes : C \times C \to C\) which is strictly associative in the sense of Definition 2.1.1.1.
Example 2.1.1.3. Let $M$ be a set, which we regard as a category having only identity morphisms. Then nonunital strict monoidal structures on $M$ (in the sense of Definition 2.1.1.1) can be identified with nonunital monoid structures on $M$ (in the sense of Definition 2.1.0.3). In particular, any nonunital monoid can be regarded as a nonunital strict monoidal category (having only identity morphisms).

Example 2.1.1.4 (Endomorphism Categories). Let $C$ be a category, and let $\text{End}(C) = \text{Fun}(C, C)$ denote the category of functors from $C$ to itself. Then the composition functor $\circ : \text{Fun}(C, C) \times \text{Fun}(C, C) \to \text{Fun}(C, C)$ $(F, G) \mapsto F \circ G$ is a nonunital strict monoidal structure on $\text{End}(C)$.

For many purposes, Definition 2.1.1.1 is too restrictive. Note that if $k$ is a field, then the tensor product functor $\otimes_k : \text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k$ of Example 2.1.0.2 does not quite fit the framework described in Definition 2.1.1.1. Given vector spaces $X$, $Y$, and $Z$ over $k$, there is no reason to expect the iterated tensor products $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ to be identical. In fact, this is impossible to determine based on the definition sketched in Example 2.1.0.2. To construct the functor $\otimes_k$ explicitly, we need to make certain choices: namely, a choice of universal bilinear map $b : U \times V \to U \otimes_k V$ for every pair of vector spaces $U, V \in \text{Vect}_k$. Without an explicit convention for how these choices are to be made, we cannot answer the question of whether the vector spaces $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ are equal. However, this is arguably the wrong question to consider: in the setting of vector spaces, the appropriate notion of “sameness” is not equality, but isomorphism. The iterated tensor products $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ are isomorphic, because they can be characterized by the same universal property: both are universal among vector spaces $W$ equipped with a $k$-trilinear map $t : X \times Y \times Z \to W$. Even better, there is a canonical isomorphism

$$\alpha_{X,Y,Z} : X \otimes_k (Y \otimes_k Z) \to (X \otimes_k Y) \otimes_k Z,$$

which depends functorially on $X$, $Y$, and $Z$. Motivated by this example, we introduce the following generalization of Definition 2.1.1.1:

Definition 2.1.1.5. Let $C$ be a category. A nonunital monoidal structure on $C$ consists of the following data:

- A functor $\otimes : C \times C \to C$, which we will refer to as the tensor product functor.
- A collection of isomorphisms $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$, for $X, Y, Z \in C$, called the associativity constraints of $C$. We demand that the associativity constraints $\alpha_{X,Y,Z}$ depend functorially on $X, Y, Z$ in the following sense: for every triple of morphisms $
$f : X \to X'$, $g : Y \to Y'$, and $h : Z \to Z'$, the diagram

\[
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Y) \otimes Z \\
\downarrow{f \otimes (g \otimes h)} & & \downarrow{(f \otimes g) \otimes h} \\
X' \otimes (Y' \otimes Z') & \xrightarrow{\alpha'_{X', Y', Z'}} & (X' \otimes Y') \otimes Z'
\end{array}
\]

is commutative. In other words, we require that $\alpha = \{\alpha_{X,Y,Z}\}_{X,Y,Z \in C}$ can be regarded as a natural isomorphism from the functor

\[
C \times C \times C \xrightarrow{(X,Y,Z) \mapsto X \otimes (Y \otimes Z)} C
\]

to the functor

\[
C \times C \times C \xrightarrow{(X,Y,Z) \mapsto (X \otimes Y) \otimes Z} C.
\]

The associativity constraints of $C$ are required to satisfy the following additional condition:

(P) For every quadruple of objects $W, X, Y, Z \in C$, the diagram of isomorphisms

\[
\begin{array}{ccc}
W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\alpha_{W \otimes (X \otimes Y), Z}} & (W \otimes (X \otimes Y)) \otimes Z \\
\downarrow{id_W \otimes \alpha_{X,Y,Z}} & & \downarrow{\alpha_{W \otimes X, Y \otimes id_Z}} \\
W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{\sim} & ((W \otimes X) \otimes Y) \otimes Z \\
\downarrow{\sim} & & \downarrow{\sim} \\
(W \otimes X) \otimes (Y \otimes Z)
\end{array}
\]

commutes.

A nonunital monoidal category is a triple $(C, \otimes, \alpha)$, where $C$ is a category and $(\otimes, \alpha)$ is a nonunital monoidal structure on $C$.

**Remark 2.1.1.6.** In the setting of Definition 2.1.1.5, we will refer to (P) as the pentagon identity. It is a prototypical example of a coherence condition: the associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ “witness” the requirement that the tensor product is associative up to isomorphism, and the pentagon identity is a sort of “higher order” associative law required of the witnesses themselves.

**Example 2.1.1.7.** Let $C$ be a category equipped with a nonunital strict monoidal structure $\otimes : C \times C \to C$ (in the sense of Definition 2.1.1.1). Then $\otimes$ determines a nonunital monoidal
structure on $\mathcal{C}$ (in the sense of Definition 2.1.1.5) by taking the associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ to be identity morphisms. Conversely, if $\mathcal{C}$ is equipped with a nonunital monoidal structure $(\otimes, \alpha)$ where each of the associativity constraints $\alpha_{X,Y,Z}$ is an identity morphism, then $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a nonunital strict monoidal structure on $\mathcal{C}$.

**Remark 2.1.1.8.** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \alpha)$. We will often abuse terminology by identifying the nonunital monoidal structure $(\otimes, \alpha)$ with the underlying tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. If we refer to a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as a nonunital monoidal structure on $\mathcal{C}$, we implicitly assume that $\mathcal{C}$ has been equipped with associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ satisfying the pentagon identity of Definition 2.1.1.5. Beware that, in the non-strict case, the associativity constraints are an essential part of the data: it is possible to have inequivalent nonunital monoidal categories $(\mathcal{C}, \otimes, \alpha)$ and $(\mathcal{C}', \otimes', \alpha')$ with $\mathcal{C} = \mathcal{C}'$ and $\otimes = \otimes'$ (see Example 2.1.3.3).

**Remark 2.1.1.9 (Full Subcategories of Nonunital Monoidal Categories).** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \alpha)$, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that, for every pair of objects $X, Y \in \mathcal{C}_0$, the tensor product $X \otimes Y$ also belongs to $\mathcal{C}_0$. Then $\mathcal{C}_0$ inherits a nonunital monoidal structure, with tensor product functor given by the composition

$$\mathcal{C}_0 \times \mathcal{C}_0 \subseteq \mathcal{C} \times \mathcal{C} \overset{\otimes}{\to} \mathcal{C}$$

(which factors through $\mathcal{C}_0$ by hypothesis), and associativity constraints given by those of $\mathcal{C}$.

**Remark 2.1.1.10 (Nonunital Monoidal Structures on Functor Categories).** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then every nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{D}$ determines a nonunital monoidal structure on the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$, whose underlying tensor product is given by the composition

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \times \text{Fun}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}(\mathcal{C}, \mathcal{D} \times \mathcal{D}) \overset{\otimes}{\to} \text{Fun}(\mathcal{C}, \mathcal{D})$$

and whose associativity constraint assigns to each triple of functors $F, G, H : \mathcal{C} \to \mathcal{D}$ the natural isomorphism

$$F \otimes (G \otimes H) \overset{\sim}{\to} (F \otimes G) \otimes H \quad C \mapsto \alpha_{F(C), G(C), H(C)}.$$

### 2.1.2 Monoidal Categories

We now introduce unital versions of Definitions 2.1.1.1 and 2.1.1.5.

**Definition 2.1.2.1.** Let $\mathcal{C}$ be a category. A **strict monoidal structure** on $\mathcal{C}$ is a nonunital strict monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for which there exists an object $\mathbf{1} \in \mathcal{C}$ satisfying the following condition:
(\ast) For every object $X \in \mathcal{C}$, we have $X \otimes 1 = X = 1 \otimes X$ (as objects of $\mathcal{C}$). Moreover, for every morphism $f : X \to X'$ in $\mathcal{C}$, we have $f \otimes \text{id}_1 = f = \text{id}_1 \otimes f$ (as morphisms from $X$ to $X'$).

A strict monoidal category is a pair $(\mathcal{C}, \otimes)$, where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a strict monoidal structure on $\mathcal{C}$.

Remark 2.1.2.2. Let $\mathcal{C}$ be a nonunital strict monoidal category. We will say that an object $1 \in \mathcal{C}$ is a strict unit if it satisfies condition (\ast) of Definition 2.1.2.1. Note that if such an object exists, then it is uniquely determined: it can be characterized as the unit element of the monoid $\text{Ob}(\mathcal{C})$.

It follows from Remark 2.1.2.2 that the notion of strict unit is not invariant under isomorphism. To address this, it will be convenient to consider a more general notion of unit object, which makes sense in the non-strict setting as well. We will use an efficient formulation due to Saavedra ([33]); see also [26]. To motivate the definition, we begin with a simple observation about units in a more elementary setting.

**Proposition 2.1.2.3.** Let $M$ be a nonunital monoid, let $e$ be an element of $M$, and let $\ell_e : M \to M$ denote the function given by the formula $\ell_e(x) = ex$. The following conditions are equivalent:

(a) The element $e$ is a left unit of $M$: that is, $\ell_e$ is the identity function from $M$ to itself.

(b) The element $e$ is idempotent (that is, it satisfies $ee = e$) and the function $\ell_e : M \to M$ is a bijection.

(c) The element $e$ is idempotent and the function $\ell_e : M \to M$ is a monomorphism.

**Proof.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are immediate. To complete the proof, assume that $e$ satisfies condition (c) and let $x$ be an element of $M$. Using the assumption that $e$ is idempotent (and the associativity of the multiplication on $M$), we obtain an identity $\ell_e(x) = ex = (ee)x = e(ex) = \ell_e(ex)$. Since $\ell_e$ is a monomorphism, it follows that $x = ex$. $lacksquare$

**Corollary 2.1.2.4.** Let $M$ be a nonunital monoid. Then an element $e \in M$ is a unit if and only if the following conditions are satisfied:

(i) The element $e$ is idempotent: that is, we have $ee = e$.

(ii) The element $e$ is left cancellative: that is, the function $x \mapsto ex$ is a monomorphism from $M$ to itself.

(iii) The element $e$ is right cancellative: that is, the function $x \mapsto xe$ is a monomorphism from $M$ to itself.
2.1. MONOIDAL CATEGORIES

We now adapt the characterization of Corollary 2.1.2.4 to the setting of nonunital monoidal categories.

Definition 2.1.2.5. Let \( \mathcal{C} \) be a nonunital monoidal category. A unit of \( \mathcal{C} \) is a pair \((1, \upsilon)\), where \( 1 \) is an object of \( \mathcal{C} \) and \( \upsilon : 1 \otimes 1 \xrightarrow{\sim} 1 \) is an isomorphism, which satisfies the following additional condition:

\((*)\) The functors

\[
\begin{align*}
\mathcal{C} \to \mathcal{C} & \quad C \mapsto 1 \otimes C \\
\mathcal{C} \to \mathcal{C} & \quad C \mapsto C \otimes 1
\end{align*}
\]

are fully faithful.

Remark 2.1.2.6. Condition \((*)\) of Definition 2.1.2.5 depends only on the object \( 1 \in \mathcal{C} \), and not on the choice of isomorphism \( \upsilon : 1 \otimes 1 \xrightarrow{\sim} 1 \).

Example 2.1.2.7. Let \( \mathcal{C} \) be a strict monoidal category, and let \( 1 \in \mathcal{C} \) be the strict unit (Remark 2.1.2.2). Then \((1, \text{id}_1)\) is a unit of \( \mathcal{C} \).

Example 2.1.2.8. Let \( M \) be a nonunital monoid, regarded as a (strict) nonunital monoidal category having only identity morphisms (Example 2.1.1.3). Then the converse of Example 2.1.2.7 holds: a pair \((1, \upsilon)\) is a unit structure on \( M \) (in the sense of Definition 2.1.2.5) if and only if \( 1 \) is a unit element of \( M \) and \( \upsilon = \text{id}_1 \). This is a restatement of Corollary 2.1.2.4.

If \( M \) is a nonunital monoid, then a unit element \( e \in M \) is unique if it exists. For nonunital monoidal categories, the analogous statement is more subtle. If a nonunital monoidal category \( \mathcal{C} \) admits a unit \((1, \upsilon)\), then it has many others: we can replace \( 1 \) by any object \( 1' \) which is isomorphic to it, and \( \upsilon \) by any choice of isomorphism \( \upsilon' : 1' \otimes 1' \xrightarrow{\sim} 1' \).

Nevertheless, we have the following strong uniqueness result:

Proposition 2.1.2.9 (Uniqueness of Units). Let \( \mathcal{C} \) be a nonunital monoidal category equipped with units \((1, \upsilon)\) and \((1', \upsilon')\) (in the sense of Definition 2.1.2.5). Then there is a unique isomorphism \( u : 1 \xrightarrow{\sim} 1' \) for which the diagram

\[
\begin{array}{ccc}
1 \otimes 1 & \xrightarrow{\upsilon} & 1 \\
\downarrow u \otimes u & & \downarrow u \\
1' \otimes 1' & \xrightarrow{\upsilon'} & 1'
\end{array}
\]

commutes.

We will give the proof of Proposition 2.1.2.9 at the end of this section.
Definition 2.1.2.10. Let $\mathcal{C}$ be a category. A monoidal structure on $\mathcal{C}$ is a nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{C}$ (Definition 2.1.1.5) together with a choice of unit $(1, v)$ (in the sense of Definition 2.1.2.5). A monoidal category is a category $\mathcal{C}$ together with a monoidal structure $(\otimes, \alpha, 1, v)$ on $\mathcal{C}$. In this case, we refer to $1$ as the unit object of $\mathcal{C}$ and the isomorphism $v : 1 \otimes 1 \cong 1$ as the unit constraint of $\mathcal{C}$.

Remark 2.1.2.11. It is possible to adopt the following variant of Definition 2.1.2.10:

- A monoidal category is a nonunital monoidal category $\mathcal{C}$ which admits a unit, in the sense of Definition 2.1.2.5.

This is essentially equivalent to Definition 2.1.2.10 since a unit $(1, v)$ of $\mathcal{C}$ is uniquely determined up to unique isomorphism (Proposition 2.1.2.9). However, for our purposes it will be more convenient to adopt the convention that a monoidal structure on a category $\mathcal{C}$ includes a choice of unit object $1 \in \mathcal{C}$ and unit constraint $v : 1 \otimes 1 \simeq 1$.

Remark 2.1.2.12. Let $\mathcal{C}$ be a category. We will sometimes abuse terminology by identifying a monoidal structure $(\otimes, \alpha, 1, v)$ with the underlying nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{C}$ (or with the underlying tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$). This is essentially harmless, by virtue of Remark 2.1.2.11. We will also abuse terminology (in a less harmless way) by identifying a monoidal category $(\mathcal{C}, \otimes, \alpha, 1, v)$ with the underlying category $\mathcal{C}$.

Notation 2.1.2.13. Let $\mathcal{C}$ be a monoidal category. We will generally use the symbol $1$ to denote the unit object of $\mathcal{C}$. In situations where this notation is potentially confusing (for example, if we are comparing $\mathcal{C}$ with another monoidal category), we will often disambiguate by instead writing $1_\mathcal{C}$ for the unit object of $\mathcal{C}$.

Example 2.1.2.14. Let $\mathcal{C}$ be a category. Then every strict monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (in the sense of Definition 2.1.1) can be promoted to a monoidal structure $(\otimes, \alpha, 1, v)$ on $\mathcal{C}$, by taking $1$ to be the strict unit of $\mathcal{C}$ and the associativity and unit constraints to be identity morphisms of $\mathcal{C}$. Conversely, if $\mathcal{C}$ is equipped with a monoidal structure $(\otimes, \alpha, 1, v)$ for which the associativity and unit constraints are identity morphisms, then $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a strict monoidal structure on $\mathcal{C}$ and $1$ is the strict unit.

Example 2.1.2.15. Let $\mathcal{C}$ be a monoidal category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Assume that $\mathcal{C}_0$ contains the unit object $1$ and is closed under the formation of tensor products in $\mathcal{C}$. Then $\mathcal{C}_0$ inherits the structure of a monoidal category: the underlying nonunital monoidal structure on $\mathcal{C}_0$ is given by the construction of Remark 2.1.1.9 and the unit $(1, v)$ of $\mathcal{C}_0$ coincides with the unit of $\mathcal{C}$.

Example 2.1.2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then every monoidal structure on $\mathcal{D}$ determines a monoidal structure on the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$, whose underlying
nonunital monoidal structure is given by the construction of Remark 2.1.1.10 and whose unit object is the constant functor \( C \to \{1\} \to D \) (and whose unit constraint \( v : 1 \otimes 1 \simeq 1 \) is the constant natural transformation induced by the unit constraint of \( D \)).

Let \( C \) be a monoidal category. In general, the unit object \( 1 \) of \( C \) need not be strict, in the sense that the functors

\[
\begin{align*}
C & \to C & X & \mapsto 1 \otimes X \\
C & \to C & X & \mapsto X \otimes 1
\end{align*}
\]

need not be equal to the identity functor \( \text{id}_C \). However, they are always (canonically) isomorphic to \( \text{id}_C \).

**Construction 2.1.2.17** (Left and Right Unit Constraints). Let \( C = (C, \otimes, \alpha, 1, v) \) be a monoidal category. For each object \( X \in C \), we have canonical isomorphisms

\[
1 \otimes (1 \otimes X) \xrightarrow{\alpha_{1,1,X}} (1 \otimes 1) \otimes X \xrightarrow{v \otimes \text{id}_X} 1 \otimes X.
\]

Since the functor \( Y \mapsto 1 \otimes Y \) is fully faithful, it follows that there is a unique isomorphism \( \lambda_X : 1 \otimes X \xrightarrow{\sim} X \) for which the diagram

\[
\begin{array}{cccc}
1 \otimes (1 \otimes X) & \xrightarrow{\alpha_{1,1,X}} & (1 \otimes 1) \otimes X \\
\downarrow \sim & & \downarrow \sim \\
1 \otimes X & \xrightarrow{\text{id}_1 \otimes \lambda_X} & (1 \otimes 1) \otimes X \\
\end{array}
\]

commutes. We will refer to \( \lambda_X \) as the **left unit constraint**. Similarly, there is a unique isomorphism \( \rho_X : X \otimes 1 \xrightarrow{\sim} X \) for which the diagram

\[
\begin{array}{cccc}
X \otimes (1 \otimes 1) & \xrightarrow{\alpha_{X,1,1}} & (X \otimes 1) \otimes 1 \\
\downarrow \sim & & \downarrow \sim \\
X \otimes 1 & \xrightarrow{\text{id}_X \otimes \rho_X} & (X \otimes 1) \otimes 1 \\
\end{array}
\]

commutes; we refer to \( \rho_X \) as the **right unit constraint**.

**Remark 2.1.2.18.** Let \( C \) be a monoidal category. Then the left and right unit constraints \( \lambda_X : 1 \otimes X \xrightarrow{\sim} X \) and \( \rho_X : X \otimes 1 \xrightarrow{\sim} X \) depend functorially on \( X \). In other words, for every morphism \( f : X \to Y \), the diagram

\[
\begin{array}{cccc}
1 \otimes X & \xrightarrow{\lambda_X} & X & \xleftarrow{\rho_X} & 1 \otimes 1 \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
1 \otimes Y & \xrightarrow{\lambda_Y} & Y & \xleftarrow{\rho_Y} & Y \otimes 1 \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
1 \otimes Y & \xrightarrow{\lambda_Y} & Y & \xleftarrow{\rho_Y} & Y \otimes 1 \\
\end{array}
\]

is commutative.
**Proposition 2.1.2.19** (The Triangle Identity). Let $\mathcal{C}$ be a monoidal category with unit object $1$. Let $X$ and $Y$ be objects of $\mathcal{C}$, and let $\rho_X : X \otimes 1 \simeq X$ and $\lambda_Y : 1 \otimes Y \rightarrow Y$ be the right and left unit constraints of Construction 2.1.2.17. Then the diagram of isomorphisms

$$
\begin{align*}
X \otimes (1 \otimes Y) & \xrightarrow{\alpha_{X,Y,1}} (X \otimes 1) \otimes Y \\
& \xrightarrow{\sim} \xrightarrow{id_X \otimes \lambda_Y} X \otimes Y \\
& \xleftarrow{\sim} \xleftarrow{\rho_X \otimes id_Y} (X \otimes 1) \otimes Y
\end{align*}
$$

is commutative.

**Proof.** We have a diagram of isomorphisms

$$
\begin{align*}
X \otimes ((1 \otimes 1) \otimes Y) & \xrightarrow{\alpha} (X \otimes (1 \otimes 1)) \otimes Y \\
& \xrightarrow{\sim} (X \otimes 1) \otimes Y \\
& \xrightarrow{\sim} X \otimes Y
\end{align*}
$$

Here the outer cycle commutes by the pentagon identity $(P)$ of Definition 2.1.1.5, the upper rectangle and outer quadrilaterals by the functoriality of the associativity constraint, the side triangles by the definition of the left and right unit constraints, and the lower quadrilateral by the functoriality of the tensor product $\otimes$. It follows that the middle square is also commutative, which is equivalent to the statement of Proposition 2.1.2.19.

**Exercise 2.1.2.20.** Let $\mathcal{C}$ be a monoidal category with unit object $1$. Show that, for every pair of objects $X, Y \in \mathcal{C}$, the diagrams

$$
\begin{align*}
X \otimes (Y \otimes 1) & \xrightarrow{\alpha_{X,Y,1}} (X \otimes Y) \otimes 1 \\
& \xrightarrow{id_X \otimes \rho_Y} X \otimes Y \\
& \xleftarrow{\sim} \xleftarrow{\rho_X \otimes id_Y} (X \otimes Y) \otimes 1
\end{align*}
$$

is commutative.
Corollary 2.1.2.21. Let $C$ be a monoidal category with unit object $1$. Then the left and right unit constraints $\lambda_1, \rho_1 : 1 \otimes 1 \xrightarrow{\sim} 1$ are equal to the unit constraint $\upsilon : 1 \otimes 1 \xrightarrow{\sim} 1$.

Proof. Let $X$ be any object of $C$. Then the left unit constraint $\lambda_X$ is characterized by the commutativity of the diagram

$$
\begin{array}{ccc}
1 \otimes (1 \otimes X) & \xrightarrow{\alpha_{1,1,X}} & (1 \otimes 1) \otimes X \\
\lambda_X & \sim & \upsilon \otimes \text{id}_X
\end{array}
$$

Using Proposition 2.1.2.19, we deduce that $\upsilon \otimes \text{id}_X = \rho_1 \otimes \text{id}_X$ as morphisms from $(1 \otimes 1) \otimes X$ to $1 \otimes X$. In other words, the morphisms $\upsilon, \rho_1 : 1 \otimes 1 \to 1$ have the same image under the functor

$$
C \to C \quad Y \mapsto Y \otimes X.
$$

In the case $X = 1$, this functor is fully faithful; it follows that $\upsilon = \rho_1$. The equality $\upsilon = \lambda_1$ follows by a similar argument. \qed

Proof of Proposition 2.1.2.9. Let $C$ be a nonunital monoidal category equipped with units $(1, \upsilon)$ and $(1', \upsilon')$. We can then regard $C$ as a monoidal category with unit object $1$ and unit constraint $\upsilon$. For each object $X \in C$, let $\lambda_X : 1 \otimes X \xrightarrow{\sim} X$ be the left unit constraint of Construction 2.1.2.17. We wish to show that there is a unique isomorphism $u : 1 \simeq 1'$ for which the outer rectangle in the diagram of isomorphisms

$$
\begin{array}{ccc}
1 \otimes 1 & \xrightarrow{\lambda_1} & 1 \\
\text{id}_1 \otimes u & & u \\
1 \otimes 1' & \xrightarrow{\lambda_{1'}} & 1' \\
w \otimes \text{id}_{1'} & & \text{id}_{1'} \\
1' \otimes 1' & \xrightarrow{\upsilon'} & 1'
\end{array}
$$
is commutative. Since the upper square commutes (Remark \[2.1.2.18]\), this is equivalent to
the commutativity of the lower square. The existence and uniqueness of \(u\) now follows from
the assumption that the functor \(X \mapsto X \otimes 1'\) is fully faithful.

**Remark 2.1.2.22.** Let \(\mathcal{C}\) be a nonunital monoidal category. Suppose we are given objects
\(1, 1' \in \mathcal{C}\) together with isomorphisms
\[ v : 1 \otimes 1 \simeq 1 \quad v' : 1' \otimes 1' \simeq 1'. \]
To carry out the proof of Proposition \[2.1.2.9\], it is sufficient to assume that the functors
\[ \mathcal{C} \to \mathcal{C} \quad X \mapsto 1 \otimes X \]
\[ \mathcal{C} \to \mathcal{C} \quad X \mapsto X \otimes 1' \]
are fully faithful: the first assumption is sufficient to construct the left unit constraints of
Construction \[2.1.2.17\], and the second is used at the end of the proof. This can be regarded
as a categorical analogue of the observation that if a nonunital monoid admits a left unit \(e\)
and a right unit \(e'\), then we must have \(e = e'\).

### 2.1.3 Examples of Monoidal Categories

We now illustrate Definition \[2.1.2.10\] with some examples.

**Example 2.1.3.1.** Let \(k\) be a field and let \(\text{Vect}_k\) denote the category of vector spaces over
\(k\) (where morphisms are \(k\)-linear maps). For every pair of vector spaces \(V, W \in \text{Vect}_k\), let
us *choose* a vector space \(V \otimes_k W\) and a bilinear map
\[ V \times W \to V \otimes_k W \quad (v, w) \mapsto v \otimes w \]
which exhibits \(V \otimes_k W\) as a tensor product of \(V\) and \(W\) (see Example \[2.1.0.2\]). The
construction \((V, W) \mapsto V \otimes_k W\) determines a functor
\[ \otimes_k : \text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k, \]
whose value on a pair of \(k\)-linear maps \(\varphi : V \to V', \psi : W \to W'\) is characterized by the
identity
\[ (\varphi \otimes_k \psi)(v \otimes w) = \varphi(v) \otimes \psi(w). \]

For every triple of vector spaces \(U, V, W \in \text{Vect}_k\), there is a canonical isomorphism
\[ \alpha_{U,V,W} : U \otimes_k (V \otimes_k W) \overset{\sim}{\to} (U \otimes_k V) \otimes_k W, \]
characterized by the identity \(\alpha_{U,V,W}(u \otimes (v \otimes w)) = (u \otimes v) \otimes w\) for \(u \in U, v \in V, \) and \(w \in W\).
The pair \((\otimes_k, \alpha) = (\otimes_k, \{\alpha_{U,V,W}\}_{U,V,W \in \text{Vect}_k})\) is then a nonunital monoidal structure on
the category Vect\(_k\), in the sense of Definition 2.1.1.5. We can upgrade this to a monoidal structure by taking the unit object \(1\) to be the field \(k\) (regarded as a vector space over itself), and the unit constraint \(\nu : 1 \otimes_k 1 \simeq 1\) to be the linear map corresponding to the multiplication on \(k\) (so that \(\nu(a \otimes b) = ab\)).

Example 2.1.3.2 (Cartesian Products). Let \(\mathcal{C}\) be a category. Assume that every pair of objects \(X, Y \in \mathcal{C}\) admits a product in \(\mathcal{C}\). This product is not unique: it is only unique up to (canonical) isomorphism. However, let us choose an object \(X \times Y\) together with a pair of morphisms

\[
X \xleftarrow{\pi_{X,Y}} X \times Y \xrightarrow{\pi'_{X,Y}} Y
\]

which exhibit \(X \times Y\) as a product of \(X\) and \(Y\) in the category \(\mathcal{C}\). Then the construction \((X, Y) \mapsto X \times Y\) determines a functor \(\mathcal{C} \times \mathcal{C} \to \mathcal{C}\), given on morphisms by the construction

\[
((f : X \to X'), (g : Y \to Y')) \mapsto ((f \times g) : (X \times Y) \to (X' \times Y'))
\]

where \(f \times g\) is the unique morphism for which the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\pi_{X,Y}} & X \times Y & \xrightarrow{\pi'_{X,Y}} & Y \\
\downarrow{f} & & \downarrow{f \times g} & & \downarrow{g} \\
X' & \xleftarrow{\pi'_{X',Y'}} & X' \times Y' & \xrightarrow{\pi'_{X',Y'}} & Y'
\end{array}
\]

is commutative.

For every triple of objects \(X, Y, Z \in \mathcal{C}\), there is a canonical isomorphism \(\alpha_{X,Y,Z} : X \times (Y \times Z) \simeq (X \times Y) \times Z\), which is characterized by the commutativity of the diagram

\[
\begin{array}{ccc}
X \times (Y \times Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \times Y) \times Z \\
\pi_{X,Y \times Z} & & \pi'_{X \times Y,Z} \\
\pi_{X,Y} & & \pi'_{X,Y,Z} \\
X & \xleftarrow{\pi_{X,Y}} & X \times Y & \xrightarrow{\pi'_{X,Y}} & Y \\
\downarrow{\pi_{X,Y \times Z}} & & \downarrow{\pi'_{X \times Y,Z}} & & \downarrow{\pi'_{Y,Z}} \\
X & \xleftarrow{\pi_{X,Y}} & X \times Y & \xrightarrow{\pi'_{X,Y}} & Y & \xrightarrow{\pi'_{Y,Z}} & Z.
\end{array}
\]

The category \(\mathcal{C}\) admits a nonunital monoidal structure, with tensor product given by the functor \((X, Y) \mapsto X \times Y\), and associativity constraints given by \((X, Y, Z) \mapsto \alpha_{X,Y,Z}\).

If we assume also that the category \(\mathcal{C}\) has a final object \(1\) (so that \(\mathcal{C}\) admits all finite products), then we can upgrade the nonunital monoidal structure above to a monoidal
structure, where the unit object of $C$ is $1$ and the unit constraint $v$ is the unique morphism from $1 \times 1$ to $1$ in $C$. We refer to this monoidal structure as the **Cartesian monoidal structure** on $C$.

**Example 2.1.3.3** (Group Cocycles). Let $G$ be a group with identity element $1 \in G$, and let $\Gamma$ be an abelian group on which $G$ acts by automorphisms; we denote the action of an element $g \in G$ by $(\gamma \in \Gamma) \mapsto g(\gamma) \in \Gamma$. A $3$-**cocycle on $G$ with values in $\Gamma$** is a map of sets

\[ \alpha : G \times G \times G \to \Gamma \quad (x, y, z) \mapsto \alpha_{x,y,z}, \]

which satisfies the equations

\[ w(\alpha_{x,y,z}) - \alpha_{wx,y,z} + \alpha_{w,xy,z} - \alpha_{w,x,yz} + \alpha_{w,x,y} = 0 \quad (2.1) \]

for every quadruple of elements $w, x, y, z \in G$.

Let $C$ denote the category whose objects are the elements of $G$, and whose morphisms are given by

\[ \text{Hom}_C(g, h) = \begin{cases} \Gamma & \text{if } g = h \\ \emptyset & \text{otherwise}. \end{cases} \]

Using the action of $G$ on $\Gamma$, we can construct a functor

\[ \otimes : C \times C \to C, \]

given on objects by $(g, h) \mapsto gh$ and on morphisms by

\[ ((\gamma : g \to g), (\delta : h \to h)) \mapsto (\gamma + g(\delta) : gh \to gh). \]

Unwinding the definitions, one sees that upgrading the functor $\otimes$ to a nonunital monoidal structure on the category $(\otimes, \alpha)$ on $C$ is equivalent to choosing a $3$-cocycle $\alpha : G \times G \times G \to \Gamma$. More precisely, any map $\alpha : G \times G \times G \to \Gamma$ can be regarded as a natural transformation of functors

\[ \bullet \otimes (\bullet \otimes \bullet) \to (\bullet \otimes \bullet) \otimes \bullet, \]

and pentagon identity $(P)$ of Definition 2.1.1.5 translates to the cocycle condition (2.1) above.

For any choice of cocycle $\alpha : G \times G \times G \to \Gamma$, we can upgrade the associated nonunital monoidal structure $(\otimes, \alpha)$ to a monoidal structure on the category $C$, by taking the unit object of $C$ to be the identity element $1 \in G$ and the unit constraint $v : 1 \otimes 1 \simeq 1$ to be the element $0 \in \Gamma$.

**Example 2.1.3.4** (The Opposite of a Monoidal Category). Let $C$ be a category equipped with a nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in C})$. Then the opposite category $C^{\text{op}}$ inherits a nonunital monoidal structure, which can be described concretely as follows:
• The tensor product on $C^{op}$ is obtained from the tensor product functor $\otimes : C \times C \to C$ by passing to opposite categories.

• Let $X$, $Y$, and $Z$ be objects of $C$, and let us write $X^{op}$, $Y^{op}$, and $Z^{op}$ for the corresponding objects of $C^{op}$. Then the associativity constraint $\alpha_{X^{op},Y^{op},Z^{op}}$ for $C^{op}$ is the inverse of the associativity constraint $\alpha_{X,Y,Z}$ for $C$.

If the nonunital monoidal category $C$ is equipped with a unit structure $(1, \upsilon)$, then we can regard $(1^{op}, \upsilon^{-1})$ as a unit structure for the nonunital monoidal category $C^{op}$. In particular, every monoidal structure on a category $C$ determines a monoidal structure on the opposite category $C^{op}$.

Example 2.1.3.5 (The Reverse of a Monoidal Structure). Let $C$ be a category equipped with a nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in C})$. Then we can equip $C$ with another nonunital monoidal structure $(\otimes^{rev}, \{\alpha^{rev}_{X,Y,Z}\}_{X,Y,Z \in C})$, defined as follows:

• The tensor product functor $\otimes^{rev} : C \times C \to C$ is given on objects by the formula $X \otimes^{rev} Y = Y \otimes X$ (and similarly on morphisms).

• The associativity constraint on $\otimes^{rev}$ is given by the formula $\alpha^{rev}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$.

We will refer to the nonunital monoidal structure $(\otimes^{rev}, \{\alpha^{rev}_{X,Y,Z}\}_{X,Y,Z \in C})$ as the reverse of the nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in C})$. In this case, we will write $C^{rev}$ to denote the nonunital monoidal category whose underlying category is $C$, equipped with the nonunital monoidal structure $(\otimes^{rev}, \{\alpha^{rev}_{X,Y,Z}\}_{X,Y,Z \in C})$.

If the nonunital monoidal category $C$ is equipped with a unit structure $(1, \upsilon)$, then we can also regard $(1, \upsilon)$ as a unit structure for the nonunital monoidal category $C^{rev}$. In other words, if $C$ is a monoidal category, then we can regard $C^{rev}$ as a monoidal category (having the same underlying category and unit object, but “reversed” tensor product).

2.1.4 Nonunital Monoidal Functors

We now study functors between (nonunital) monoidal categories.

Definition 2.1.4.1 (Nonunital Strict Monoidal Functors). Let $C$ and $D$ be nonunital monoidal categories (Definition 2.1.5). A nonunital strict monoidal functor from $C$ to $D$ is a functor $F : C \to D$ with the following properties:

• The diagram of functors

$$
\begin{array}{ccc}
C \times C & \overset{\otimes}{\longrightarrow} & C \\
\downarrow_{F \times F} & & \downarrow_{F} \\
D \times D & \overset{\otimes}{\longrightarrow} & D
\end{array}
$$

We will refer to the nonunital monoidal structure $(\otimes^{rev}, \{\alpha^{rev}_{X,Y,Z}\}_{X,Y,Z \in C})$ as the reverse of the nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in C})$. In this case, we will write $C^{rev}$ to denote the nonunital monoidal category whose underlying category is $C$, equipped with the nonunital monoidal structure $(\otimes^{rev}, \{\alpha^{rev}_{X,Y,Z}\}_{X,Y,Z \in C})$.

If the nonunital monoidal category $C$ is equipped with a unit structure $(1, \upsilon)$, then we can also regard $(1, \upsilon)$ as a unit structure for the nonunital monoidal category $C^{rev}$. In other words, if $C$ is a monoidal category, then we can regard $C^{rev}$ as a monoidal category (having the same underlying category and unit object, but “reversed” tensor product).
is strictly commutative. In particular, for every pair of objects \( X, Y \in \mathcal{C} \), we have an equality \( F(X) \otimes F(Y) = F(X \otimes Y) \) of objects of \( \mathcal{D} \).

- For every triple of objects \( X, Y, Z \in \mathcal{C} \), the functor \( F \) carries the associativity constraint \( \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z \) (for the monoidal structure on \( \mathcal{C} \)) to the associativity constraint \( \alpha_{F(X),F(Y),F(Z)} : F(X) \otimes (F(Y) \otimes F(Z)) \simeq (F(X) \otimes F(Y)) \otimes F(Z) \) (for the monoidal structure on \( \mathcal{D} \)).

**Example 2.1.4.2.** Let \( \mathcal{C} \) be a nonunital monoidal category. Then the identity functor \( \text{id}_\mathcal{C} \) is a nonunital strict monoidal functor from \( \mathcal{C} \) to itself.

For many applications, Definition 2.1.4.1 is too restrictive. In practice, the definition of a (nonunital) monoidal structure \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) on a category \( \mathcal{C} \) often involves constructions which are only well-defined up to isomorphism (see Examples 2.1.3.1 and 2.1.3.2). In such cases, it is unreasonable to require that a functor \( F : \mathcal{C} \to \mathcal{D} \) has the property that \( F(X) \otimes F(Y) \) and \( F(X \otimes Y) \) are the same object of \( \mathcal{D} \). Instead, we should ask for any isomorphism \( \mu_{X,Y} : F(X) \otimes F(Y) \overset{\sim}{\to} F(X \otimes Y) \). To get a well-behaved theory, we should further demand that the isomorphisms \( \mu_{X,Y} \) depend functorially on \( X \) and \( Y \), and are suitably compatible with the associativity constraints on \( \mathcal{C} \) and \( \mathcal{D} \). We begin by considering a slightly more general situation, where the morphisms \( \mu_{X,Y} \) are not required to be invertible.

**Definition 2.1.4.3 (Nonunital Lax Monoidal Functors).** Let \( \mathcal{C} \) and \( \mathcal{D} \) be nonunital monoidal categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be a functor from \( \mathcal{C} \) to \( \mathcal{D} \). A **nonunital lax monoidal structure** on \( F \) is a collection of morphisms \( \mu = \{ \mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \}_{X,Y \in \mathcal{C}} \) which satisfy the following pair of conditions:

(a) The morphisms \( \mu_{X,Y} \) depend functorially on \( X \) and \( Y \): that is, for every pair of morphisms \( f : X \to X' \), \( g : Y \to Y' \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \otimes Y) \\
\downarrow F(f) \otimes F(g) & & \downarrow F(f \otimes g) \\
F(X') \otimes F(Y') & \xrightarrow{\mu_{X',Y'}} & F(X' \otimes Y')
\end{array}
\]

commutes (in the category \( \mathcal{D} \)). In other words, we can regard \( \mu \) as a natural transformation of functors as indicated in the diagram

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\
\downarrow F \times F & & \downarrow F \\
\mathcal{D} \times \mathcal{D} & \xrightarrow{\otimes} & \mathcal{D}
\end{array}
\]
(b) The morphisms $\mu_{X,Y}$ are compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{D}$ in the following sense: for every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & (F(X) \otimes F(Y)) \otimes F(Z) \\
\downarrow \text{id}_{F(X)} \otimes \mu_{Y,Z} & & \downarrow \mu_{X,Y} \otimes \text{id}_{F(Z)} \\
F(X) \otimes F(Y \otimes Z) & & F(X \otimes Y) \otimes F(Z) \\
\downarrow \mu_{X,Y} \otimes \text{id} & & \downarrow \mu_{X \otimes Y, Z} \\
F(X \otimes (Y \otimes Z)) & \xrightarrow{\alpha_{X,Y,Z}} & F((X \otimes Y) \otimes Z)
\end{array}
$$

commutes (in the category $\mathcal{D}$).

A nonunital lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a pair $(F, \mu)$, where $F: \mathcal{C} \to \mathcal{D}$ is a functor and $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ is a nonunital lax monoidal structure on $F$. In this case, we will refer to the morphisms $\{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ as the tensor constraints of $F$.

**Definition 2.1.4.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F: \mathcal{C} \to \mathcal{D}$ be a functor from $\mathcal{C}$ to $\mathcal{D}$. A nonunital monoidal structure on $F$ is a lax nonunital monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ on $F$ with the property that each of the tensor constraints $\mu_{X,Y}: F(X) \otimes F(Y) \to F(X \otimes Y)$ is an isomorphism.

A nonunital monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a pair $(F, \mu)$, where $F: \mathcal{C} \to \mathcal{D}$ is a functor and $\mu$ is a nonunital monoidal structure on $F$.

**Example 2.1.4.5.** Let $k$ be a field and let $\text{Vect}_k$ denote the category of vector spaces over $k$, endowed with the monoidal structure of Example 2.1.3.1. The construction of this monoidal structure involved certain choices: for every pair of vector spaces $U, V \in \text{Vect}_k$, we selected a universal $k$-bilinear map $b_{U,V}: U \times V \to U \otimes_k V$. The collection of functions $b = \{b_{U,V}\}_{U,V \in \text{Vect}_k}$ is then a nonunital lax monoidal structure on the forgetful functor $\text{Vect}_k \to \text{Set}$ (where we equip $\text{Set}$ with the monoidal structure given by Cartesian products; see Example 2.1.3.2). Note that the tensor product functor $\otimes_k: \text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k$ is characterized by the requirement that it is given on objects by $(U, V) \mapsto U \otimes_k V$ and satisfies condition (a) of Definition 2.1.4.3, and the associativity constraint on $\text{Vect}_k$ is characterized by the requirement that it satisfies condition (b) of Definition 2.1.4.3. Note that $b$ is not a nonunital monoidal structure: the bilinear maps $b_{U,V}: U \times V \to U \otimes_k V$ are never bijective, except in the trivial case where $U \simeq 0 \simeq V$.

**Example 2.1.4.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F: \mathcal{C} \to \mathcal{D}$ be a nonunital strict monoidal functor. Then $F$ admits a nonunital monoidal structure
{\mu_{X,Y}}_{X,Y\in\mathcal{C}}$, where we take each $\mu_{X,Y}$ to be the identity morphism from $F(X) \otimes F(Y) = F(X \otimes Y)$ to itself.

Conversely, if $(F,\mu)$ is a nonunital monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ with the property that the tensor constraints $\mu_{X,Y}$ is an identity morphism in $\mathcal{D}$, then $F$ is a nonunital strict monoidal functor.

**Example 2.1.4.7.** Let $M$ and $M'$ be nonunital monoids, regarded as nonunital monoidal categories having only identity morphisms (Example 2.1.1.3). Then nonunital lax monoidal functors from $M$ to $M'$ (in the sense of Definition 2.1.4.3) can be identified with nonunital monoid homomorphisms from $M$ to $M'$ (in the sense of Definition 2.1.0.5). Moreover, every nonunital lax monoidal functor from $M$ to $M'$ is automatically strict.

**Example 2.1.4.8 (The Left Regular Representation).** Let $\mathcal{C}$ be a nonunital monoidal category and let $\text{End}(\mathcal{C}) = \text{Fun}(\mathcal{C}, \mathcal{C})$ be the category of functors from $\mathcal{C}$ to itself, endowed with the strict monoidal structure of Example 2.1.1.4. For each object $X \in \mathcal{C}$, let $\ell_X : \mathcal{C} \to \mathcal{C}$ denote the functor given on objects by the formula $\ell_X(Y) = X \otimes Y$. The construction $X \mapsto \ell_X$ then determines a functor $\ell : \mathcal{C} \to \text{Fun}(\mathcal{C}, \mathcal{C})$. For every pair of objects $X, Y \in \mathcal{C}$, there is a natural isomorphism $\mu_{X,Y} : \ell_X \circ \ell_Y \sim \ell_{X \otimes Y}$, whose value on an object $Z \in \mathcal{C}$ is given by the associativity constraint

$$(\ell_X \circ \ell_Y)(Z) = X \otimes (Y \otimes Z) \xrightarrow{\alpha_{X,Y,Z}} (X \otimes Y) \otimes Z = \ell_{X \otimes Y}(Z).$$

Then $\mu = \{\mu_{X,Y}\}_{X,Y}$ is a nonunital monoidal structure on the functor $X \mapsto \ell_X$: property (a) of Definition 2.1.4.3 follows from the naturality of the associativity constraint on $\mathcal{C}$, and property (b) is a reformulation of the pentagon identity.

**Warning 2.1.4.9.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories. A nonunital strict monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ possessing certain properties. However, a nonunital (lax) monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with additional structure, given by the tensor constraints $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$. We will often abuse terminology by identifying a nonunital (lax) monoidal functor $(F,\mu)$ with the underlying functor $F$; in this case, we implicitly assume that the tensor constraints $\mu_{X,Y}$ have been specified.

**Definition 2.1.4.10.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be functors equipped with nonunital lax monoidal structures $\mu$ and $\mu'$, respectively. We say that a natural transformation of functors $\gamma : F \to F'$ is nonunital monoidal if, for every pair...
of objects $X, Y \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \otimes Y) \\
\downarrow \gamma(X) \otimes \gamma(Y) & & \downarrow \gamma(X \otimes Y) \\
F'(X) \otimes F'(Y) & \xrightarrow{\mu'_{X,Y}} & F'(X \otimes Y)
\end{array}
$$

is commutative.

We let \( \text{Fun}^\text{lax}_{\text{nu}}(\mathcal{C}, \mathcal{D}) \) denote the category whose objects are nonunital lax monoidal functors \((F, \mu)\) from \( \mathcal{C} \) to \( \mathcal{D} \), and whose morphisms are nonunital monoidal natural transformations, and we let \( \text{Fun}^\otimes_{\text{nu}}(\mathcal{C}, \mathcal{D}) \) denote the full subcategory of \( \text{Fun}^\text{lax}_{\text{nu}}(\mathcal{C}, \mathcal{D}) \) spanned by the the nonunital monoidal functors \((F, \mu)\) from \( \mathcal{C} \) to \( \mathcal{D} \).

**Example 2.1.4.11** (Nonunital Algebras). Let \( \mathcal{C} \) be a nonunital monoidal category and let \( A \) be an object of \( \mathcal{C} \). A *nonunital algebra structure on \( A \)* is a map \( m : A \otimes A \to A \) for which the diagram

$$
\begin{array}{ccc}
A \otimes (A \otimes A) & \xrightarrow{\alpha_{A,A,A}} & (A \otimes A) \otimes A \\
\downarrow \text{id} \otimes m & & \downarrow m \otimes \text{id} \\
A \otimes A & \xrightarrow{m} & A \otimes A
\end{array}
$$

is commutative. A *nonunital algebra object of \( \mathcal{C} \)* is a pair \((A, m)\) where \( A \) is an object of \( \mathcal{C} \) and \( m \) is a nonunital algebra structure on \( A \). If \((A, m)\) and \((A', m')\) are nonunital algebra objects of \( \mathcal{C} \), then we say that a morphism \( f : A \to A' \) is a *nonunital algebra homomorphism* if the diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{m} & A \\
\downarrow f \otimes f & & \downarrow f \\
A' \otimes A' & \xrightarrow{m'} & A'
\end{array}
$$

is commutative. We let \( \text{Alg}^\text{nu}_{\text{nu}}(\mathcal{C}) \) denote the category whose objects are nonunital algebra objects of \( \mathcal{C} \) and whose morphisms are nonunital algebra homomorphisms.

Let \( \{e\} \) denote the trivial monoid, regarded as a (strict) monoidal category having only identity morphisms (Example 2.1.1.3). Then we can identify objects \( A \in \mathcal{C} \) with functors \( F : \{e\} \to \mathcal{C} \) (by means of the formula \( A = F(e) \)). Unwinding the definitions, we see that nonunital lax monoidal structures on the functor \( F \) (in the sense of Definition 2.1.4.3) can be identified with nonunital algebra structures on the object \( A = F(e) \). Under this identification,
nonunital monoidal natural transformations correspond to homomorphisms of nonunital algebras. We therefore have an isomorphism of categories $\text{Fun}^{\text{lax}}_{\text{nu}}(\{e\}, C) \simeq \text{Alg}^{\text{nu}}(C)$.

**Example 2.1.4.12.** Let Set denote the category of sets, endowed with the monoidal structure given by Cartesian product of sets (Example 2.1.3.2). For each set $S$, we can identify nonunital algebra structures on $S$ (in the sense of Example 2.1.4.11) with nonunital monoid structures on $S$ (in the sense of Definition 2.1.0.3). This observation supplies an isomorphism of categories $\text{Alg}^{\text{nu}}(\text{Set}) \simeq \text{Mon}^{\text{nu}}$, where $\text{Mon}^{\text{nu}}$ is the category of Definition 2.1.0.5.

**Example 2.1.4.13.** Let $C$ and $D$ be nonunital monoidal categories, and let $C^\text{rev}$ and $D^\text{rev}$ denote the same categories with the reversed nonunital monoidal structure (Example 2.1.3.5). Then every functor $F : C \to D$ can be also regarded as a functor from $C^\text{rev}$ to $D^\text{rev}$, which we will denote by $F^\text{rev}$. There is a canonical bijection

$$\{\text{Nonunital lax monoidal structures on } F\} \sim \{\text{Nonunital lax monoidal structures on } F^\text{rev}\},$$

which carries a nonunital lax monoidal structure $\mu$ to the the nonunital lax monoidal structure $\mu^\text{rev}$ given by the formula $\mu^\text{rev}_{X,Y} = \mu_{Y,X}^{-1}$. Using these bijections, we obtain a canonical isomorphism of categories $\text{Fun}^{\text{lax}}_{\text{nu}}(C, D) \simeq \text{Fun}^{\text{lax}}_{\text{nu}}(C^\text{rev}, D^\text{rev})$, which restricts to an isomorphism $\text{Fun}^{\ominus}_{\text{nu}}(C, D) \simeq \text{Fun}^{\ominus}_{\text{nu}}(C^\text{rev}, D^\text{rev})$.

**Example 2.1.4.14.** Let $C$ and $D$ be nonunital monoidal categories, and regard the opposite categories $C^\text{op}$ and $D^\text{op}$ as equipped with the nonunital monoidal structures of Example 2.1.3.4. Then every functor $F : C \to D$ determines a functor $F^\text{op} : C^\text{op} \to D^\text{op}$. There is a canonical bijection

$$\{\text{Nonunital monoidal structures on } F\} \simeq \{\text{Nonunital monoidal structures on } F^\text{op}\},$$

which carries a nonunital monoidal structure $\mu$ on $F$ to a nonunital monoidal structure $\mu'$ on $F^\text{op}$, given concretely by $\mu'_{X,Y} = \mu_{Y,X}^{-1}$. Using these bijections, we obtain a canonical isomorphism of categories $\text{Fun}^{\ominus}_{\text{nu}}(C, D)^{\text{op}} \simeq \text{Fun}^{\ominus}_{\text{nu}}(C^\text{op}, D^\text{op})$.

**Warning 2.1.4.15.** The analogue of Example 2.1.4.14 for nonunital lax monoidal functors is false. The notion of nonunital lax monoidal functor is not self-opposite: in general, there is no simple relationship between the categories $\text{Fun}^{\text{lax}}_{\text{nu}}(C, D)$ and $\text{Fun}^{\text{lax}}_{\text{nu}}(C^\text{op}, D^\text{op})$.

Motivated by Warning 2.1.4.15, we introduce the following:
2.1. MONOIDAL CATEGORIES

**Variant 2.1.4.16.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *nonunital colax monoidal structure* on $F$ is a nonunital lax monoidal structure on the opposite functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ (Definition 2.1.4.3). In other words, a colax monoidal structure on $F$ is a collection of morphisms $\mu = \{\mu_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathcal{C}}$ which satisfy the following pair of conditions:

(a) The morphisms $\mu_{X,Y}$ depend functorially on $X$ and $Y$: that is, for every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{\mu_{X,Y}} & F(X) \otimes F(Y) \\
\downarrow F(f \otimes g) & & \downarrow (f \otimes g) \\
F(X' \otimes Y') & \xrightarrow{\mu_{X',Y'}} & F(X') \otimes F(Y')
\end{array}
$$

commutes (in the category $\mathcal{D}$).

(b) For every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
F(X \otimes (Y \otimes Z)) & \xrightarrow{F(\alpha_{X,Y,Z})} & F((X \otimes Y) \otimes Z) \\
\downarrow \mu_{X,Y \otimes Z} & & \downarrow \mu_{X \otimes Y,Z} \\
F(X) \otimes F(Y \otimes Z) & & F(X \otimes Y) \otimes F(Z) \\
\downarrow \text{id} \otimes \mu_{Y,Z} & & \downarrow \mu_{X,Y} \otimes \text{id} \\
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & (F(X) \otimes F(Y)) \otimes F(Z)
\end{array}
$$

commutes.

**Construction 2.1.4.17** (Composition of Nonunital Monoidal Functors). Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be nonunital monoidal categories, and suppose we are given a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$. If $\mu = \{\mu_{X,Y} : X,Y \in \mathcal{C}\}$ is a nonunital lax monoidal structure on the functor $F$ and $\nu = \{\nu_{U,V} : U,V \in \mathcal{D}\}$ is a nonunital lax monoidal structure on $G$, then the composite functor $G \circ F$ inherits a nonunital lax monoidal structure, which associates to each pair of objects $X, Y \in \mathcal{C}$ the composite map

$$
(G \circ F)(X) \otimes (G \circ F)(Y) \xrightarrow{\nu_{F(X),F(Y)}} G(F(X) \otimes F(Y)) \xrightarrow{G(\mu_{X,Y})} (G \circ F)(X \otimes Y).
$$

This construction determines a composition law

$$
\circ : \text{Fun}_{\text{lax}}(\mathcal{E}, \mathcal{D}) \times \text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D}) \to \text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{E}).
$$
Remark 2.1.4.18. In the situation of Construction 2.1.4.17, suppose that $\mu$ and $\nu$ are nonunital monoidal structures on $F$ and $G$, respectively: that is, assume that all of the tensor constraints
\[ \mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \quad \nu_{U,V} : G(U) \otimes G(V) \to G(U \otimes V) \]
are isomorphisms. Then Construction 2.1.4.17 supplies a nonunital monoidal structure on the composite functor $G \circ F$. We therefore obtain a composition law
\[ \circ : \Fun_{\text{nu}}(\mathcal{D}, \mathcal{E}) \times \Fun_{\text{nu}}(\mathcal{C}, \mathcal{D}) \to \Fun_{\text{nu}}(\mathcal{C}, \mathcal{E}). \]

We close this section by describing an alternative perspective on nonunital lax monoidal functors. First, we need to review a bit of terminology.

Notation 2.1.4.19 (The Comma Category). Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and suppose we are given a pair of functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. We let $(F \downarrow G)$ denote the comma category of the functors $F$ and $G$, which is defined as the fiber product
\[ (F \downarrow G) = \mathcal{C} \times_{\Fun([0], \mathcal{E})} \Fun([1], \mathcal{E}) \times_{\Fun([1], \mathcal{D})} \mathcal{D}. \]

More concretely:

- An object of the comma category $(F \downarrow G)$ is a triple $(C, D, \eta)$ where $C$ is an object of the category $\mathcal{C}$, $D$ is an object of the category $\mathcal{D}$, and $\eta : F(C) \to G(D)$ is a morphism in the category $\mathcal{E}$.

- If $(C, D, \eta)$ and $(C', D', \eta')$ are objects of the comma category $(F \downarrow G)$, then a morphism from $(C, D, \eta)$ to $(C', D', \eta')$ is a pair $(u, v)$, where $u : C \to C'$ is a morphism in the category $\mathcal{C}$, $v : D \to D'$ is a morphism in the category $\mathcal{D}$, and the diagram
\[ \begin{array}{ccc} F(C) & \xrightarrow{\eta} & G(D) \\ \downarrow F(u) & & \downarrow G(v) \\ F(C') & \xrightarrow{\eta'} & G(D') \end{array} \]
commutes in the category $\mathcal{E}$.

We will be particularly interested in the special case of this construction where $\mathcal{C} = \mathcal{E}$ and $F = \text{id}_\mathcal{C}$ is the identity functor; in this case, we denote the comma category $(F \downarrow G)$ by $(\mathcal{C} \downarrow G)$. Similarly, if $\mathcal{D} = \mathcal{E}$ and $G = \text{id}_\mathcal{D}$ is the identity functor, then we denote the comma category $(F \downarrow G)$ by $(F \downarrow \mathcal{D})$.

Proposition 2.1.4.20. Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, let $G : \mathcal{D} \to \mathcal{C}$ be a functor, and let $(\mathcal{C} \downarrow G)$ denote the comma category of Notation 2.1.4.19. Then:
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- Let $\mu = \{\mu_{D,D'}\}_{D,D' \in \mathcal{D}}$ be a nonunital lax monoidal structure on the functor $G$. Then there is a unique nonunital monoidal structure $\otimes_\mu$ on the comma category $(C \downarrow G)$ with the following properties:

  1. The forgetful functor $U : (C \downarrow G) \to C \times \mathcal{D}$ is a strict nonunital monoidal functor.

  2. On objects, the tensor product $\otimes_\mu$ is given by the formula
     
     $$(C,D,\eta) \otimes_{\mu} (C',D',\eta') = (C \otimes C', D \otimes D', t(\eta,\eta')),$$
     
     where $t(\eta,\eta')$ is the composition $C \otimes C' \xrightarrow{\eta \otimes \eta'} G(D) \otimes G(D') \xrightarrow{\mu_{D,D'}} G(D \otimes D')$.

- The construction $\mu \mapsto \otimes_\mu$ induces a bijection

  \[
  \{ \text{Nonunital lax monoidal structures on } G \} \to \{ \text{Nonunital monoidal structures on } (C \downarrow G) \text{ satisfying (1)} \}.
  \]

**Remark 2.1.4.21.** Let $C$ and $D$ be nonunital monoidal categories. We can summarize Proposition 2.1.4.20 more informally as follows: for any functor $G : \mathcal{D} \to C$, choosing a nonunital lax monoidal structure on $G$ is equivalent to choosing a nonunital monoidal structure on the comma category $(C \downarrow G)$ which is compatible with the existing nonunital monoidal structures on $C$ and $D$, respectively.

**Proof of Proposition 2.1.4.20.** Unwinding the definitions, we see that to describe nonunital monoidal structure on the category $(C \downarrow G)$ satisfying condition (1), one must give the following data:

- For every pair of objects $(C,D,\eta)$ and $(C',D',\eta')$ of the comma category $(C \downarrow G)$, we must supply a tensor product $(C,D,\eta) \otimes (C',D',\eta')$. By virtue of the assumption that $U$ is a nonunital strict monoidal, this tensor product must be given as a triple $(C \otimes C', D \otimes D', t(\eta,\eta'))$, for some morphism $t(\eta,\eta') : C \otimes C' \to G(D \otimes D')$ in the category $\mathcal{D}$.

- For every pair of morphisms $(u,v) : (C,D,\eta) \to (C',D,\pi)$ and $(u',v') : (C',D',\eta') \to (C',D',\pi')$ in the comma category $(C \downarrow G)$, we must supply a tensor product morphism $(C \otimes C', D \otimes D', t(\eta,\eta')) \to (C \otimes C', D \otimes D', t(\eta,\eta'))$. Note that this morphism is uniquely determined: for $U$ to be a nonunital strict monoidal functor, it must be the pair $(u \otimes u', v \otimes v')$. However, the existence of this morphism imposes the following condition:
(i) If the diagrams
\[
\begin{array}{ccc}
C & \xrightarrow{\eta} & G(D) \\
\downarrow{u} & & \downarrow{G(v)} \\
\overline{C} & \xrightarrow{\eta'} & G(D')
\end{array}
\quad \begin{array}{ccc}
C' & \xrightarrow{\eta'} & G(D') \\
\downarrow{u'} & & \downarrow{G(v')}
\end{array}
\quad \begin{array}{ccc}
C'' & \xrightarrow{\eta''} & G(D'') \\
\downarrow{u''} & & \downarrow{G(v'')}
\end{array}
\]
commute (in the category \(C\)), then the diagram
\[
\begin{array}{ccc}
C \otimes C' & \xrightarrow{t(\eta,\eta')} & G(D \otimes D') \\
\downarrow{u \otimes u'} & & \downarrow{G(v \otimes v')}
\end{array}
\quad \begin{array}{ccc}
\overline{C} \otimes \overline{C'} & \xrightarrow{t(\overline{\eta},\overline{\eta}'')} & G(D \otimes D')
\end{array}
\]
also commutes.

• For every triple of objects \((C, D, \eta), (C', D', \eta'), (C'', D'', \eta'')\) of the comma category \((\mathcal{C} \downarrow G)\), we must supply an associativity constraint
\[
(C \otimes (C' \otimes C'')) \simeq ((C \otimes (C' \otimes (C'', D'', \eta''))) \simeq ((C, D, \eta) \otimes (C', D', \eta')) \otimes (C'', D'', \eta'')
\]
in \((\mathcal{C} \downarrow G)\). By virtue of our assumption that \(U\) is nonunital strict monoidal, this associativity constraint is uniquely determined: it must be the pair \((\alpha_{C,C',C''}, \alpha_{D,D',D''})\) given by the associativity constraints for the nonunital monoidal structures on \(\mathcal{C}\) and \(\mathcal{D}\), respectively. However, the existence of this morphism imposes the following condition:

(ii) For every triple of morphisms \(\eta : C \to G(D), \eta' : C' \to G(D'), \text{ and } \eta'' : C'' \to G(D'')\), the diagram
\[
\begin{array}{ccc}
C \otimes (C' \otimes C'') & \xrightarrow{\alpha_{C,C',C''}} & (C \otimes C') \otimes C'' \\
\downarrow{t(\eta, t(\eta', \eta''))} & & \downarrow{t(t(\eta, \eta'), \eta'')}
\end{array}
\quad \begin{array}{ccc}
G(D \otimes (D' \otimes D'')) & \xrightarrow{G(\alpha_{D,D',D''})} & G((D \otimes D') \otimes D'')
\end{array}
\]
commutes (in the category \(\mathcal{C}\)).

If this condition is satisfied, then the associativity constraints are automatically functorial and satisfy the pentagon identity (since the analogous conditions hold in the categories \(\mathcal{C}\) and \(\mathcal{D}\), respectively).
Given a collection of morphisms $t(\eta, \eta')$ satisfying these conditions, we define $\mu = \{\mu_{D,D'}\}_{D,D' \in D}$ by the formula $\mu_{D,D'} = t(\id_{G(D)}, \id_{G(D')})$. Note that, if $(C, D, \eta)$ and $(C', D', \eta')$ are arbitrary objects of the comma category $(C \downarrow G)$, then we have canonical maps

$$(\eta, \id_D) : (C, D, \eta) \rightarrow (G(D), D, \id_{G(D)}) \quad (\eta', \id_{D'}) : (C', D', \eta') \rightarrow (G(D'), D', \id_{G(D')}).$$

Applying condition $(i)$, we see that the morphism $t(\eta, \eta')$ can then be recovered as the composition

$$C \otimes C' \xrightarrow{\eta \otimes \eta'} G(D) \otimes G(D') \xrightarrow{\mu_{D,D'}} G(D \otimes D').$$

To complete the proof, it will suffice to show that if we are given any system of morphisms $\mu = \{\mu_{D,D'} : G(D) \otimes G(D') \rightarrow G(D \otimes D')\}_{D,D' \in D}$ and we define $t(\eta, \eta')$ as above, then $\mu$ is a nonunital lax monoidal structure on $G$ if and only if conditions $(i)$ and $(ii)$ are satisfied.

Using the formula for $t(\eta, \eta')$ in terms of $\mu$, we can rewrite condition $(i)$ as follows:

$(i')$ If the diagrams

$$\begin{array}{ccc}
C & \xrightarrow{\eta} & G(D) \\
\downarrow u & & \downarrow G(v) \\
\overline{C} & \xrightarrow{\overline{\eta}} & G(D)
\end{array} \quad \begin{array}{ccc}
C' & \xrightarrow{\eta'} & G(D') \\
\downarrow u' & & \downarrow G(v') \\
\overline{C'} & \xrightarrow{\overline{\eta'}} & G(D')
\end{array}$$

commute (in the category $C$), then the outer rectangle in the diagram

$$\begin{array}{ccc}
C \otimes C' & \xrightarrow{\eta \otimes \eta'} & G(D) \otimes G(D') \\
\downarrow u \otimes u' & & \downarrow G(v) \otimes G(v') \\
\overline{C} \otimes \overline{C'} & \xrightarrow{\overline{\eta} \otimes \overline{\eta'}} & G(D) \otimes G(D')
\end{array} \quad \begin{array}{ccc}
\mu_{D,D'} & & \\
\downarrow G(v) \otimes G(v') & & \downarrow G(v \otimes v') \\
\mu_{\overline{D},\overline{D'}} & & \\
\downarrow G(v \otimes v') & & \downarrow G(v \otimes v')
\end{array} \quad \begin{array}{ccc}
C \otimes C' & \xrightarrow{\eta \otimes \eta'} & G(D) \otimes G(D') \\
\downarrow u \otimes u' & & \downarrow G(v) \otimes G(v') \\
\overline{C} \otimes \overline{C'} & \xrightarrow{\overline{\eta} \otimes \overline{\eta'}} & G(D) \otimes G(D')
\end{array}$$

commutes.

Note that the left square appearing in this diagram is automatically commutative. Assertion $(i')$ is therefore a consequence of the following:

$(a)$ For every pair of morphisms $v : D \rightarrow \overline{D}$ and $v' : D' \rightarrow \overline{D}'$ in the category $D$, the diagram

$$\begin{array}{ccc}
G(D) \otimes G(D') & \xrightarrow{\mu_{D,D'}} & G(D \otimes D') \\
\downarrow G(v) \otimes G(v') & & \downarrow G(v \otimes v') \\
G(\overline{D}) \otimes G(\overline{D'}) & \xrightarrow{\mu_{\overline{D},\overline{D'}}} & G(\overline{D} \otimes \overline{D'})
\end{array}$$

commutes (in the category $C$).
Conversely, if \((i')\) is satisfied, then \((a)\) can be deduced by specializing to the case \(\eta = \text{id}_{G(D)}\), \(\eta' = \text{id}_{G(D')}\), and \(\eta'' = \text{id}_{G(D'')}\). It follows that \((i)\) is satisfied if and only if \((a)\) is satisfied: that is, if and only if \(\mu = \{\mu_{D,D'}\}_{D,D' \in D}\) is a natural transformation.

Using \((a)\), we can reformulate condition \((ii)\) as follows:

\[(ii')\] For every triple of morphisms \(\eta : C \to G(D), \eta' : C' \to G(D'),\) and \(\eta'' : C'' \to G(D'')\), the outer rectangle in the diagram

\[
\begin{array}{ccc}
C \otimes (C' \otimes C'') & \xrightarrow{\alpha_{C,C',C''}} & (C \otimes C') \otimes C'' \\
\downarrow \eta \otimes (\eta' \otimes \eta'') & & \downarrow (\eta \otimes \eta') \otimes \eta'' \\
G(D) \otimes (G(D') \otimes G(D'')) & \xrightarrow{\alpha_{G(D),G(D'),G(D'')}} & (G(D) \otimes G(D')) \otimes G(D'') \\
\downarrow \text{id}_{G(D)} \otimes \mu_{D,D'} & & \downarrow \mu_{D,D'} \otimes \text{id}_{G(D'')} \\
G(D) \otimes G(D' \otimes D'') & & (G(D) \otimes G(D')) \otimes G(D'') \\
\downarrow \mu_{D,D' \otimes D''} & & \downarrow \mu_{D \otimes D', D''} \\
G(D \otimes (D' \otimes D'')) & \xrightarrow{G(\alpha_{D,D',D''})} & G((D \otimes D') \otimes D'')
\end{array}
\]

commutes (in the category \(\mathcal{C}\)).

Since the upper square in this diagram automatically commutes (by the naturality of the associativity constraints on \(\mathcal{C}\)), assertion \((ii')\) is a consequence of the following simpler assertion:

\[(b)\] For every triple of objects \(D, D', D'' \in D\), the diagram

\[
\begin{array}{ccc}
G(D) \otimes (G(D') \otimes G(D'')) & \xrightarrow{\alpha_{G(D),G(D'),G(D'')}} & (G(D) \otimes G(D')) \otimes G(D'') \\
\downarrow \text{id}_{G(D)} \otimes \mu_{D,D'} & & \downarrow \mu_{D,D'} \otimes \text{id}_{G(D'')} \\
G(D) \otimes G(D' \otimes D'') & & (G(D) \otimes G(D')) \otimes G(D'') \\
\downarrow \mu_{D,D' \otimes D''} & & \downarrow \mu_{D \otimes D', D''} \\
G(D \otimes (D' \otimes D'')) & \xrightarrow{G(\alpha_{D,D',D''})} & G((D \otimes D') \otimes D'')
\end{array}
\]

commutes (in the category \(\mathcal{C}\)).
Conversely, if \((ii')\) is satisfied, then \((b)\) can be deduced by specializing to the case \(\eta = \text{id}_{G(D)}\), \(\eta' = \text{id}_{G(D')}\), and \(\eta'' = \text{id}_{G(D'')}\). We conclude by observing that conditions \((a)\) and \((b)\) assert precisely that \(\mu\) is a nonunital lax monoidal structure (Definition 2.1.4.3).

\[\text{Remark 2.1.4.22 (Adjoint Functors).}\] Let \(\mathcal{C}\) and \(\mathcal{D}\) be nonunital monoidal categories and suppose we are given a pair of adjoint functors \(\mathcal{C} \xrightarrow{F} \xleftarrow{G} \mathcal{D}\), given by an isomorphism of comma categories \((\mathcal{C} \downarrow G) \simeq (F \downarrow \mathcal{D})\) (see Notation 2.1.4.19). Applying Proposition 2.1.4.20 (and the dual characterization of nonunital colax monoidal functors), we see that the following are equivalent:

- The datum of a nonunital lax monoidal structure on the functor \(G : \mathcal{D} \to \mathcal{C}\).
- The datum of a nonunital colax monoidal structure on the functor \(F : \mathcal{C} \to \mathcal{D}\).
- The datum of a nonunital monoidal structure on the comma category \((\mathcal{C} \downarrow G) \simeq (F \downarrow \mathcal{D})\) which is compatible with the nonunital monoidal structures on \(\mathcal{C}\) and \(\mathcal{D}\) (meaning that the projection map \((\mathcal{C} \downarrow G) \to \mathcal{C} \times \mathcal{D}\) is a nonunital strict monoidal functor).

\[\text{2.1.5 Lax Monoidal Functors}\]

We now introduce a unital version of Definition 2.1.4.3. To motivate the discussion, we begin with a special case.

\[\text{Definition 2.1.5.1.}\] Let \(\mathcal{C}\) be a monoidal category with unit object \(1\), and let \(A\) be a nonunital algebra object of \(\mathcal{C}\) (Example 2.1.4.11) with multiplication \(m : A \otimes A \to A\). We say that a morphism \(\epsilon : 1 \to A\) is a left unit for \(A\) if the composite map

\[A \xrightarrow{\lambda_A^{-1}} 1 \otimes A \xrightarrow{\epsilon \otimes \text{id}_A} A \otimes A \xrightarrow{m} A\]

is the identity map from \(A\) to itself; here \(\lambda_A : 1 \otimes A \xrightarrow{\sim} A\) denotes the left unit constraint of Construction 2.1.2.17. We say that \(\epsilon\) is a right unit of \(A\) if the composite map

\[A \xrightarrow{\rho_A^{-1}} A \otimes 1 \xrightarrow{\text{id}_A \otimes \epsilon} A \otimes A \xrightarrow{m} A\]

is equal to the identity. We say that \(\epsilon\) is a unit of \(A\) if it is both a left and a right unit of \(A\).

By virtue of Example 2.1.4.11, we can view the theory of nonunital algebras as a special case of the theory of nonunital lax monoidal functors \(F : \mathcal{C} \to \mathcal{D}\), where we take \(\mathcal{C}\) to be the trivial monoid \(\{e\}\) (regarded as a category having only identity morphisms). Definition 2.1.5.1 has an analogue for nonunital lax monoidal functors in general.
Definition 2.1.5.2. Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories with unit objects \( 1_{\mathcal{C}} \) and \( 1_{\mathcal{D}} \), respectively. Let \( F : \mathcal{C} \to \mathcal{D} \) be a nonunital lax monoidal functor with tensor constraints \( \mu = \{ \mu_{X,Y} \}_{X,Y \in \mathcal{C}} \). Let \( \epsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}}) \) be a morphism in \( \mathcal{D} \). We say that \( \epsilon \) is a left unit for \( F \) if, for every object \( X \in \mathcal{C} \), the left unit constraint \( \lambda_{F(X)} : 1_{\mathcal{D}} \otimes F(X) \xrightarrow{\sim} F(X) \) in the category \( \mathcal{D} \) is equal to the composition

\[
1_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon \otimes \text{id}_{F(X)}} F(1_{\mathcal{C}}) \otimes F(X) \xrightarrow{\mu_{1_{\mathcal{C}},X}} F(1_{\mathcal{C}} \otimes X) \xrightarrow{F(\lambda_X)} F(X),
\]

where \( \lambda_X : 1_{\mathcal{C}} \otimes X \xrightarrow{\sim} X \) is the left unit constraint in the monoidal category \( \mathcal{C} \). We say that \( \epsilon \) is a right unit for \( F \) if, for every object \( X \in \mathcal{C} \), the right unit constraint \( \rho_{F(X)} : F(X) \otimes 1_{\mathcal{D}} \xrightarrow{\sim} F(X) \) is equal to the composition

\[
F(X) \otimes 1_{\mathcal{D}} \xrightarrow{\text{id}_{F(X)} \otimes \epsilon} F(X) \otimes F(1_{\mathcal{C}}) \xrightarrow{\mu_{X,1_{\mathcal{C}}}} F(X \otimes 1_{\mathcal{C}}) \xrightarrow{F(\rho_X)} F(X).
\]

We say that \( \epsilon \) is a unit for \( F \) if it is both a left and a right unit for \( F \).

Example 2.1.5.3. Let \( \mathcal{C} \) be a monoidal category and let \( A \) be a nonunital algebra object in \( \mathcal{C} \), which we identify with a nonunital lax monoidal functor \( F : \{ \epsilon \} \to \mathcal{C} \) as in Example 2.1.4.11. Then a map \( \epsilon : 1 \to A = F(\epsilon) \) is a unit (left unit, right unit) for \( A \) (in the sense of Definition 2.1.5.1) if and only if it is a unit (left unit, right unit) for \( F \) (in the sense of Definition 2.1.5.2).

We now show that if a nonunital lax monoidal functor \( F \) admits a unit \( \epsilon \), then \( \epsilon \) is uniquely determined. This is a consequence of the following:

Proposition 2.1.5.4. Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories with unit objects \( 1_{\mathcal{C}} \) and \( 1_{\mathcal{D}} \), respectively, and let \( F : \mathcal{C} \to \mathcal{D} \) be a nonunital lax monoidal functor. Suppose that \( F \) admits a left unit \( \epsilon_L : 1_{\mathcal{D}} \to F(1_{\mathcal{C}}) \) and a right unit \( \epsilon_R : 1_{\mathcal{D}} \to F(1_{\mathcal{C}}) \). Then \( \epsilon_L = \epsilon_R \).

Proof. We first observe that there is a commutative diagram

\[
\begin{array}{ccc}
1_{\mathcal{D}} \otimes 1_{\mathcal{D}} & \xrightarrow{id \otimes \epsilon_R} & 1_{\mathcal{D}} \otimes F(1_{\mathcal{C}}) \\
\downarrow{\lambda_{1_{\mathcal{D}}}} & & \downarrow{\lambda_{F(1_{\mathcal{C}})}} \\
1_{\mathcal{D}} & \xrightarrow{\epsilon_L \otimes \text{id}} & F(1_{\mathcal{C}}) \otimes F(1_{\mathcal{C}}) \\
\end{array}
\]

the left square commutes by the naturality of the left unit constraints for \( \mathcal{C} \) (Remark 2.1.2.18), and the right square commutes by virtue of our assumption that \( \epsilon_L \) is a left unit for \( \mathcal{C} \). Using Corollary 2.1.2.21 we see that the unit constraints

\[
v_C : 1_{\mathcal{C}} \otimes 1_{\mathcal{C}} \xrightarrow{\sim} 1_{\mathcal{C}} \quad v_D : 1_{\mathcal{D}} \otimes 1_{\mathcal{D}} \xrightarrow{\sim} 1_{\mathcal{D}}
\]

of \( \mathcal{D} \) and the interchange law (Remark 2.1.2.19).
are equal to the left unit constraints $\lambda_1_C$ and $\lambda_1_D$, respectively. It follows that the composition $\epsilon_R \circ v_D$ coincides with the composition

$$1_D \otimes 1_D \xrightarrow{\epsilon_L \otimes \epsilon_R} F(1_C) \otimes F(1_C) \xrightarrow{\mu_{1_C,1_C}} F(1_C \otimes 1_C) \xrightarrow{F(v_C)} F(1_C).$$

A similar argument shows that this composition coincides with $\epsilon_L \circ v_D$. Since $v_D$ is an isomorphism, it follows that $\epsilon_R = \epsilon_L$.

**Corollary 2.1.5.5.** Let $C$ and $D$ be monoidal categories and let $F : C \to D$ be a nonunital lax monoidal functor. Then $F$ admits a unit $\epsilon : 1_D \to F(1_C)$ if and only if it has both a left unit and a right unit. In this case, the unit $\epsilon$ is unique.

**Proposition 2.1.5.6.** Let $C$ and $D$ be monoidal categories with unit objects $1_C$ and $1_D$, respectively. Let $G : D \to C$ be a functor equipped with a nonunital lax monoidal structure, which we will identify with the corresponding nonunital monoidal structure on the comma category $(C \downarrow G)$ (see Proposition 2.1.4.20). Let $\epsilon : 1_C \to F(1_D)$ be a morphism in $C$, and regard the triple $1 = (1_C, 1_D, \epsilon)$ as an object of $(C \downarrow G)$. Then:

1. The morphism $\epsilon$ is a left unit for $G$ if and only if, for every object $(C, D, \eta)$ of the comma category $(C \downarrow G)$, the left unit constraints $\lambda_C : 1_C \otimes C \simeq C$ and $\lambda_D : 1_D \otimes D \simeq D$ determine an isomorphism $(\lambda_C, \lambda_D) : 1 \otimes (C, D, \eta) \simeq (C, D, \eta)$ in the category $(C \downarrow G)$.

2. The morphism $\epsilon$ is a right unit for $G$ if and only if, for every object $(C, D, \eta)$ of the comma category $(C \downarrow G)$, the right unit constraints $\rho_C : C \otimes 1_C \simeq C$ and $\rho_D : D \otimes 1_D \simeq D$ determine an isomorphism $(\rho_C, \rho_D) : (C, D, \eta) \otimes 1 \simeq (C, D, \eta)$ in the category $(C \downarrow G)$.

**Proof.** We will prove (1); the proof of (2) is similar. Fix an object $(C, D, \eta)$ of the comma category $(C \downarrow G)$. Unwinding the definitions, we see that the pair $(\lambda_C, \lambda_D)$ determines a morphism from $1 \otimes (C, D, \eta)$ to $(C, D, \eta)$ in $(C \downarrow G)$ if and only if the outer rectangle of the diagram

$$
\begin{array}{ccc}
1_C \otimes C & \xrightarrow{\lambda_C} & C \\
| \quad \quad | \quad \quad | \\
1_C \otimes G(D) & \xrightarrow{\lambda_G(D)} & G(D) \\
| \quad \quad \quad | \quad \quad \quad | \\
G(1_D) \otimes G(D) & \xrightarrow{\mu} & G(1_D \otimes D) \\
| \quad \quad \quad \quad | \quad \quad \quad \quad | \\
G(1_D \otimes G(D)) & \xrightarrow{G(\lambda_D)} & G(D).
\end{array}
$$

Here the upper square commutes by the functoriality of the left unit constraints in $C$ (Remark 2.1.2.18), and the commutativity of the lower rectangle follows from the assumption that $\epsilon$.
is a left unit. This proves the “only if” direction of (1). The converse follows by specializing to the case where \( C = G(D) \) and \( \eta \) is the identity map.

**Corollary 2.1.5.7.** Let \( C \) and \( D \) be monoidal categories with units \((1_C, \upsilon_C)\) and \((1_D, \upsilon_D)\), respectively. Let \( G : D \to C \) be a nonunital lax monoidal functor. Let \( \epsilon : 1_C \to G(1_D) \) be a morphism in \( C \) and regard the triple \( 1 = (1_C, 1_D, \epsilon) \) as an object of the comma category \( (C \downarrow G) \). The following conditions are equivalent:

1. The morphism \( \epsilon \) is a unit for \( G \) (in the sense of Definition 2.1.5.2).
2. The pair \( \upsilon = (\upsilon_C, \upsilon_D) \) is a morphism from \( 1 \otimes 1 \) to \( 1 \) in the comma category \( (C \downarrow G) \), and the pair \( (1, \upsilon) \) is a unit with respect to the tensor product \( \otimes_\mu \) of Proposition 2.1.4.20.

**Proof.** Assume first that (1) is satisfied. Then Proposition 2.1.5.6 implies that the functors
\[
(C \downarrow G) \to (C \downarrow G) \quad X \mapsto 1 \otimes X, X \mapsto X \otimes 1
\]
are naturally isomorphic to the identity, and are therefore fully faithful. To complete the proof of (2), it will suffice to show that the pair \((\upsilon_C, \upsilon_D)\) is a morphism from \( 1 \otimes 1 \) to \( 1 \) in \((C \downarrow G)\). This also follows from Proposition 2.1.5.6 by virtue of the identities \( \upsilon_C = \lambda_{1_C} \) and \( \upsilon_D = \lambda_{1_D} \) (Corollary 2.1.2.21).

Now suppose that (2) is satisfied, so that we can regard \((C \downarrow G)\) as a monoidal category with unit \((1, \upsilon)\). It follows that the forgetful functor \((C \downarrow G) \to C \times D\) carries the left and right unit constraints of \((C \downarrow G)\) to the left and right unit constraints of \( C \) and \( D \). Applying Proposition 2.1.5.6, we conclude that \( \epsilon \) is both a left and right unit for the nonunital lax monoidal functor \( G \).

**Definition 2.1.5.8.** Let \( C \) and \( D \) be monoidal categories and let \( F : C \to D \) be a functor. A lax monoidal structure on \( F \) is a nonunital lax monoidal structure \( \mu = \{ \mu_{X,Y} \}_{X,Y \in C} \) (Definition 2.1.4.3) for which there exists a unit \( \epsilon : 1_D \to F(1_C) \).

A lax monoidal functor from \( C \) to \( D \) is a pair \((F, \mu)\), where \( F : C \to D \) is functor and \( \mu \) is a lax monoidal structure on \( F \). In this case, we will refer to the morphism \( \epsilon : 1_D \to F(1_C) \) as the unit of \( F \).

**Remark 2.1.5.9.** Let \( C \) and \( D \) be monoidal categories and let \( F : C \to D \) be a nonunital lax monoidal functor. The condition that \( F \) is a lax monoidal functor depends only on the underlying nonunital monoidal structures on \( C \) and \( D \), and not on the particular choice of units \((1_C, \upsilon_C)\) and \((1_D, \upsilon_D)\) for \( C \) and \( D \), respectively (see Remark 2.1.2.11).

Combining Proposition 2.1.4.20 with Corollary 2.1.5.7, we obtain the following:
Let \( C \) and \( D \) be monoidal categories, let \( G : D \to C \) be a functor, let \((C ↓ G)\) be the comma category of Notation 2.1.4.19, and let \( U : (C ↓ G) \to C \times D \) denote the forgetful functor \((C, D, η) \mapsto (C, D)\). Then the construction \( \mu \mapsto \otimes_\mu \) of Proposition 2.1.4.20 restricts to a bijection
\[
\{ \text{Lax monoidal structures on } G \} \quad \downarrow
\{ \text{Monoidal structures on } (C ↓ G) \text{ with } U \text{ strict monoidal} \}
\]
(see Example 2.1.6.5).

A colax monoidal structure on \( F \) is a lax monoidal structure on the opposite functor \( F^{op} : C^{op} \to D^{op} \): that is, a collection of maps \( μ = \{ μ_{X,Y} : F(X ⊗ Y) \to F(X) ⊗ F(Y) \}_{X,Y \in C} \) satisfying the requirements of Variant 2.1.4.16, together with the additional condition that there exists a counit \( ϵ : F(1_C) \to 1_D \) having the property that, for every object \( X \in C \), the left and right unit constraints of \( F(X) \) the inverses of the composite maps
\[
F(X) \xrightarrow{F(λ_X)} F(1_C ⊗ X) \xrightarrow{η_{1_C,X}} F(1_C) ⊗ F(X) \xrightarrow{id ⊗ ϵ} 1_D ⊗ F(X)
\]
\[
F(X) \xrightarrow{F(ρ_X)} F(X ⊗ 1_C) \xrightarrow{μ_X,1_C} F(X) ⊗ F(1_C) \xrightarrow{id ⊗ ϵ} F(X) ⊗ 1_C.
\]

The datum of a lax monoidal structure on the functor \( G : D \to C \).

The datum of a colax monoidal structure on the functor \( F : C \to D \).

The datum of a monoidal structure on the comma category \((C ↓ G) \simeq (F ↓ D)\) which is compatible with the monoidal structures on \( C \) and \( D \).

The compatibility conditions appearing in Definition 2.1.5.2 can be formulated more directly in terms of the unit constraints of \( C \) and \( D \) (without referring the left and right unit constraints of Construction 2.1.2.17).

Let \( C \) and \( D \) be monoidal categories with unit objects \( 1_C \) and \( 1_D \), respectively, let \( F : C \to D \) be a nonunital lax monoidal functor, and let \( ε : 1_D \to F(1_C) \) be a morphism in \( C \). Then \( ε \) is a left unit for \( F \) if and only if it satisfies the following pair of conditions:
(1) The diagram

\[
\begin{array}{ccc}
1_D \otimes 1_D & \xrightarrow{\epsilon \otimes \epsilon} & F(1_C) \otimes F(1_C) \\
\downarrow v_D & & \downarrow \mu_{1_C,1_C} \\
1_D & \xrightarrow{\epsilon} & F(1_C)
\end{array}
\]

commutes (in the category \(D\)). Here \(v_C\) and \(v_D\) denote the unit constraints of \(C\) and \(D\), respectively.

(2) For every object \(X \in C'\), the composite map

\[
1_D \otimes F(X) \xrightarrow{\epsilon \otimes \text{id}_{F(X)}} F(1_C) \otimes F(X) \xrightarrow{\mu_{1_C,X}} F(1_C \otimes X)
\]

is a monomorphism in the category \(C\).

Moreover, if these conditions are satisfied, then the map

\[
1_D \otimes F(X) \xrightarrow{\epsilon \otimes \text{id}_{F(X)}} F(1_C) \otimes F(X) \xrightarrow{\mu_{1_C,X}} F(1_C \otimes X)
\]

is an isomorphism for each \(X \in C'\).

Example 2.1.5.14. In the special case where \(C = \{e\}\), we can identify a nonunital lax monoidal functor \(F : C \to D\) with a nonunital algebra object \(A\) of \(D\). In this case, Proposition 2.1.5.13 asserts that a morphism \(\epsilon : 1_D \to A\) is a left unit (in the sense of Definition 2.1.5.1) if and only if the diagram

\[
\begin{array}{ccc}
1_D \otimes 1_D & \xrightarrow{\epsilon \otimes \epsilon} & A \otimes A \\
\downarrow v & & \downarrow m \\
1_D & \xrightarrow{\epsilon} & A
\end{array}
\]

is commutative (that is, \(\epsilon\) is idempotent) and the map

\[
1_D \otimes A \xrightarrow{\epsilon \otimes \text{id}_A} A \otimes A \xrightarrow{m} A
\]

is a monomorphism in \(D\) (that is, \(\epsilon\) is left cancellative). When \(D\) is the category of sets (equipped with the Cartesian monoidal structure of Example 2.1.3.2), this reduces to the statement of Proposition 2.1.2.3.

Proof of Proposition 2.1.5.13. To simplify the notation, let us use the symbol \(1\) to denote the unit objects of both \(C\) and \(D\), \(v : 1 \otimes 1 \xrightarrow{\sim} 1\) for the unit constraints of both \(C\) and \(D\),
and $\lambda$ for the unit constraints of both $\mathcal{C}$ and $\mathcal{D}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor equipped with a nonunital lax monoidal structure $\mu = \{ \mu_{X,Y} \}_{X,Y \in \mathcal{C}}$. Suppose first that $\epsilon : 1 \to F(1)$ is a left unit for $F$. Then the diagram

\[
\begin{array}{ccccc}
1 \otimes 1 & \xrightarrow{id_1 \otimes \epsilon} & 1 \otimes F(1) & \xrightarrow{\epsilon \otimes id_{F(1)}} & F(1) \otimes F(1) \\
\downarrow \lambda_1 & & \downarrow \lambda_{F(1)} & & \downarrow \mu_{1,1} \\
1 & \xrightarrow{\epsilon} & F(1) & \xrightarrow{F(\epsilon)} & F(\lambda_1) \\
\end{array}
\]

commutes: the region on the left commutes by the naturality of the left unit constraints for $\mathcal{D}$ (Remark 2.1.2.18), and the region on the right commutes by virtue of our assumption that $\epsilon$ is a left unit. The commutativity of the outer square shows that $\epsilon$ satisfies condition (1) of Proposition 2.1.5.13 (by virtue of the fact that the unit constraints of $\mathcal{C}$ and $\mathcal{D}$ are given by $\upsilon = \lambda_1$; see Corollary 2.1.2.21). For every object $X \in \mathcal{C}$, the composition

\[
1 \otimes F(X) \xrightarrow{\epsilon \otimes id_{F(X)}} F(1) \otimes F(X) \xrightarrow{\mu_{1,X}} F(1 \otimes X) \xrightarrow{F(\lambda_X)} F(X)
\]

is the left unit constraint $\lambda_{F(X)}$, which is an isomorphism. Since $F(\lambda_X)$ is also an isomorphism, it follows that the composition $\mu_{1,X} \circ (\epsilon \otimes id_{F(X)})$ is an isomorphism.

Now suppose that $\epsilon$ satisfies conditions (1) and (2); we wish to show that it is a left unit for $F$. Fix an object $X \in \mathcal{C}$, and let $f : 1 \otimes F(X) \to F(X)$ denote the composition

\[
1 \otimes F(X) \xrightarrow{\epsilon \otimes id_{F(X)}} F(1) \otimes F(X) \xrightarrow{\mu_{1,X}} F(1 \otimes X) \xrightarrow{F(\lambda_X)} F(X).
\]

We wish to show that $f$ is equal to the left unit constraint $\lambda_{F(X)}$ for the monoidal category $\mathcal{D}$. Unwinding the definitions, this is equivalent to the assertion that $id_1 \otimes f$ is equal to the composition

\[
1 \otimes (1 \otimes F(X)) \xrightarrow{\alpha_{1,1,X}} (1 \otimes 1) \otimes F(X) \xrightarrow{\upsilon \otimes id_{F(X)}} 1 \otimes F(X).
\]

By virtue of assumption (2), it will suffice to prove that these morphisms agree after postcomposition with the monomorphism

\[
1 \otimes F(X) \xrightarrow{\epsilon \otimes id_{F(X)}} F(1) \otimes F(X) \xrightarrow{\mu_{1,X}} F(1 \otimes X).
\]
This is equivalent to the commutativity of the outer rectangle in the diagram

\[
\begin{array}{ccc}
1 \otimes (1 \otimes F(X)) & \xrightarrow{\epsilon} & 1 \otimes (F(1) \otimes F(X)) \\
\downarrow \alpha & & \downarrow \alpha \\
(1 \otimes 1) \otimes F(X) & \xrightarrow{\epsilon \otimes \epsilon} & (F(1) \otimes F(1)) \otimes F(X) \\
\downarrow \nu & & \downarrow \mu \\
1 \otimes F(X) & \xrightarrow{\epsilon} & F(1) \otimes F(X) \\
\end{array}
\]

In fact, the whole diagram commutes: the rectangle on the lower left commutes by virtue of our assumption that \( \epsilon \) satisfies (1), the rectangle in the middle commutes by virtue of the compatibility of the \( \mu \) with the associativity constraints of \( C \) and \( D \), the square on the lower right commutes by the construction of the left unit constraint \( \lambda_X \), and the remaining regions commute by naturality.

\[\square\]

**Example 2.1.5.15.** Let \( k \) be a field, let \( \text{Vect}_k \) denote the category of vector spaces over \( k \), and let \( F : \text{Vect}_k \to \text{Set} \) be the forgetful functor, endowed with the nonunital lax monoidal structure described in Example 2.1.4.5. Then \( F \) is a lax monoidal functor: the function

\[ \epsilon : \{\ast\} \to F(1) \quad \epsilon(\ast) = 1 \in k \]

is a left and right unit for \( F \).

**Example 2.1.5.15** illustrates a special case of a general phenomenon:

**Example 2.1.5.16.** Let \( C \) be a monoidal category, and let \( F : C \to \text{Set} \) denote the functor corepresented by the unit object \( 1 \in C \), given concretely by the formula \( F(X) = \text{Hom}_C(1, X) \).

For every pair of objects \( X, Y \in C \), we have a canonical map

\[ \mu_{X,Y} : F(X) \times F(Y) \to F(X \otimes Y), \]

which carries a pair of elements \( x \in F(X), y \in F(Y) \) to the composite map

\[ 1 \xrightarrow{\nu^{-1}} 1 \otimes 1 \xrightarrow{x \otimes y} X \otimes Y. \]

The collection of maps \( \{\mu_{X,Y}\}_{X,Y \in C} \) determines a lax monoidal structure on the functor \( F \), with unit given by the map

\[ \epsilon : \{\ast\} \to F(1) = \text{Hom}_C(1, 1) \quad \epsilon(\ast) = \text{id}_1. \]
Example 2.1.5.17. Let $C$ and $D$ be categories which admit finite products, and regard $C$ and $D$ as endowed with the Cartesian monoidal structures described in Example 2.1.3.2. Let $F : C \to D$ be any functor, and let $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$ be the induced functor of opposite categories. Then the functor $F^{\text{op}}$ admits a lax monoidal structure, which associates to each pair of objects $X, Y \in C$ the canonical map $\mu_{X,Y} : F(X \times Y) \to F(X) \times F(Y)$ in the category $D$ (which we can view as a morphism from $F^{\text{op}}(X) \otimes F^{\text{op}}(Y) \rightarrow F^{\text{op}}(X \otimes Y)$ in the category $D^{\text{op}}$). The unit for $F$ is given by the unique morphism $\epsilon : F(1_C) \to 1_D$ in the category $D$ (where $1_C$ and $1_D$ are final objects of $C$ and $D$, respectively).

Definition 2.1.5.18. Let $C$ and $D$ be monoidal categories and let $F, F' : C \to D$ be lax monoidal functors from $C$ to $D$. We will say that a natural transformation $\gamma : F \to F'$ is monoidal if it satisfies the following pair of conditions:

- The natural transformation $\gamma$ is nonunital monoidal, in the sense of Definition 2.1.4.10. That is, for every pair of objects $X, Y \in C$, the diagram
  \[
  \begin{array}{ccc}
  F(X) \otimes F(Y) & \xrightarrow{\mu_{X,Y}} & F(X \otimes Y) \\
  \downarrow{\gamma(X) \otimes \gamma(Y)} & & \downarrow{\gamma(X \otimes Y)} \\
  F'(X) \otimes F'(Y) & \xrightarrow{\mu'_{X,Y}} & F'(X \otimes Y)
  \end{array}
  \]
  commutes, where $\mu$ and $\mu'$ are the tensor constraints of $F$ and $F'$, respectively.

- The unit of $F'$ is equal to the composition $1_D \xrightarrow{\epsilon} F(1_C) \xrightarrow{\gamma(1_C)} F'(1_C)$, where $\epsilon$ is the unit of $F$.

We let $\text{Fun}^{\text{lax}}(C, D)$ denote the category whose objects are lax monoidal functors from $C$ to $D$ and whose morphisms are monoidal natural transformations, which we regard as a (non-full) subcategory of the category $\text{Fun}^{\text{nu}}(C, D)$ introduced in Definition 2.1.4.10.

Remark 2.1.5.19 (Compatibility with Reversal). Let $C$ and $D$ be monoidal categories, let $F : C \to D$ be a nonunital lax monoidal functor, and let $F^{\text{rev}} : C^{\text{rev}} \to D^{\text{rev}}$ be as in Example 2.1.4.13. Then $F$ is a lax monoidal functor if and only if $F^{\text{rev}}$ is a lax monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories $\text{Fun}^{\text{lax}}(C, D) \simeq \text{Fun}^{\text{lax}}(C^{\text{rev}}, D^{\text{rev}})$.

Remark 2.1.5.20 (Closure under Composition). Let $C$, $D$, and $E$ be monoidal categories and let $F : C \to D$ and $G : D \to E$ be functors equipped with nonunital lax monoidal structures $\mu$ and $\nu$, respectively, so that the composite functor $G \circ F$ inherits a nonunital lax monoidal structure (Construction 2.1.4.17). If $F$ and $G$ admit units
\[
\delta : 1_D \to F(1_C) \quad \epsilon : 1_E \to G(1_D),
\]
then the composite map
\[ 1_{\mathcal{E}} \xrightarrow{\epsilon} G(1_{\mathcal{D}}) \xrightarrow{G(\delta)} (G \circ F)(1_{\mathcal{C}}) \]
is a unit for the composite functor \( G \circ F \). This observation (and its counterpart for monoidal natural transformations) imply that the composition law of Construction 2.1.4.17 restricts to a functor
\[ \circ : \text{Fun}^{\text{lax}}(\mathcal{D}, \mathcal{E}) \times \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \to \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{E}). \]

**Example 2.1.5.21** (Algebra Objects). Let \( \mathcal{C} \) be a monoidal category. An **algebra object of** \( \mathcal{C} \) is a pair \((A, m)\), where \( A \) is an object of \( \mathcal{C} \) and \( m : A \otimes A \to A \) is a nonunital algebra structure on \( A \) (Example 2.1.4.11) for which there exists a unit \( \epsilon : 1 \to A \) (in the sense of Definition 2.1.5.1). If \((A, m)\) and \((A', m')\) are algebra objects of \( \mathcal{C} \) with units \( \epsilon \) and \( \epsilon' \), then we say that a morphism \( f : A \to A' \) is an **algebra homomorphism** if it is a nonunital algebra homomorphism (Example 2.1.4.11) which satisfies \( \epsilon' = f \circ \epsilon \). We let \( \text{Alg}(\mathcal{C}) \) denote the category whose objects are algebra objects of \( \mathcal{C} \) and whose morphisms are algebra homomorphisms. We regard \( \text{Alg}(\mathcal{C}) \) as a (non-full) subcategory of the category \( \text{Alg}^{\text{nu}}(\mathcal{C}) \) of nonunital algebra objects of \( \mathcal{C} \) defined in Example 2.1.4.11.

Let \( \{e\} \) denote the trivial monoid, regarded as a (strict) monoidal category having only identity morphisms (Example 2.1.1.3). Then algebra objects of \( \mathcal{C} \) can be identified with lax monoidal functors \( \{e\} \to \mathcal{C} \). More precisely, the isomorphism \( \text{Fun}^{\text{lax}}_{\text{nu}}(\{e\}, \mathcal{C}) \simeq \text{Alg}^{\text{nu}}(\mathcal{C}) \) of Example 2.1.4.11 specializes to an isomorphism of (non-full) subcategories \( \text{Fun}^{\text{lax}}(\{e\}, \mathcal{C}) \simeq \text{Alg}(\mathcal{C}) \).

**Example 2.1.5.22.** Let Set denote the category of sets, equipped with the Cartesian monoidal structure of Example 2.1.3.2. Then we can identify algebra objects of Set with monoids. More precisely, there is a canonical isomorphism of categories \( \text{Alg}(\text{Set}) \simeq \text{Mon} \), where Mon denotes the category of monoids (Definition 2.1.0.5).

### 2.1.6 Monoidal Functors

We now introduce the unital analogue of Definition 2.1.4.4.

**Definition 2.1.6.1.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. A **monoidal structure** on \( F \) is a nonunital lax monoidal structure \( \mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}} \) (Definition 2.1.4.3) which satisfies the following additional conditions:

- For every pair of objects \( X, Y \in \mathcal{C} \), the tensor constraint \( \mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \) is an isomorphism in \( \mathcal{D} \) (that is, \( \mu \) is a nonunital monoidal structure on \( F \)).

- There exists an isomorphism \( \epsilon : 1_{\mathcal{D}} \xrightarrow{\sim} F(1_{\mathcal{C}}) \) which is a unit for \( F \) (in the sense of Definition 2.1.5.2).
A monoidal functor from $C$ to $D$ is a pair $(F, \mu)$, where $F$ is a functor from $C$ to $D$ and $\mu$ is a monoidal structure on $F$.

**Remark 2.1.6.2.** Let $C$ and $D$ be monoidal categories. We will generally abuse terminology by identifying a monoidal functor $(F, \mu)$ from $C$ to $D$ with the underlying functor $F : C \to D$. If we refer to $F$ as a monoidal functor, we implicitly assume that it has been equipped with a monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in C}$.

**Warning 2.1.6.3.** Let $C$ and $D$ be monoidal categories, and let $F : C \to D$ be a nonunital lax monoidal functor. If $F$ is a monoidal functor from $C$ to $D$, then it is both a nonunital monoidal functor (that is, the tensor constraints $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ are isomorphisms) and a lax monoidal functor (that is, it admits a unit $\epsilon : 1_D \to F(1_C)$). However, the converse is false: to qualify as a monoidal functor, $F$ must satisfy the additional condition that $\epsilon$ is an isomorphism.

**Remark 2.1.6.4.** Let $C$ and $D$ be monoidal categories and let $F : C \to D$ be a nonunital monoidal functor. Let $\epsilon : 1_D \to F(1_C)$ be an isomorphism in the category $C$. Then $\epsilon$ automatically satisfies condition (2) of Proposition 2.1.5.13. for each $X \in C$, both of the maps

$$1_D \otimes F(X) \xrightarrow{\epsilon \otimes \text{id}_F(X)} F(1_C) \otimes F(X) \xrightarrow{\mu_{1_C, X}} F(1_C \otimes X)$$

are isomorphisms. It follows that $\epsilon$ is a unit for $F$ if and only if it satisfies condition (1) of Proposition 2.1.5.13, that is, if and only if the diagram

$$
\begin{array}{ccc}
1_D \otimes 1_D & \xrightarrow{\epsilon \otimes \epsilon} & F(1_C) \otimes F(1_C) \\
\downarrow \nu_D & & \downarrow \mu_{1_C, 1_C} \\
1_D & \xrightarrow{F(\nu_C)} & F(1_C) \\
\end{array}
$$

is commutative. By virtue of Proposition 2.1.2.9, there exists an isomorphism $\epsilon$ satisfying this condition if and only if the pair $(F(1_C), F(\nu_C) \circ \mu_{1_C, 1_C})$ is a unit of $C$ (in the sense of Definition 2.1.2.5).

In other words, a nonunital monoidal functor $F : C \to D$ is monoidal if and only if the functors

$$
\begin{align*}
& D \to D & X & \mapsto F(1_C) \otimes X \\
& D \to D & X & \mapsto X \otimes F(1_C)
\end{align*}
$$

are fully faithful (in which case they are both canonically isomorphic to the identity functor $\text{id}_D : D \simeq D$).
CHAPTER 2. EXAMPLES OF ∞-CATEGORIES

Example 2.1.6.5 (Strict Monoidal Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be strict monoidal categories (Definition 2.1.2.1). We say that a functor $F: \mathcal{C} \to \mathcal{D}$ is strict monoidal if it is a nonunital strict monoidal functor (Definition 2.1.4.1) which carries the strict unit object $1_\mathcal{C}$ to the strict unit object $1_\mathcal{D}$.

Every strict monoidal functor $F: \mathcal{C} \to \mathcal{D}$ can be regarded as a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$, by taking each tensor constraint $\mu_{X,Y}$ to be the identity morphisms from $F(X) \otimes F(Y) = F(X \otimes Y)$ to itself. Conversely, if $(F,\mu)$ is a monoidal functor for which the tensor constraints $\mu_{X,Y}$ and the unit morphism $\epsilon: 1_\mathcal{D} \to F(1_\mathcal{C})$ are identity morphisms in $\mathcal{D}$, then $F$ is a strict monoidal functor from $\mathcal{C}$ to $\mathcal{D}$.

Example 2.1.6.6. Let $M$ and $M'$ be monoids, regarded as monoidal categories having only identity morphisms (Example 2.1.2.8). Then lax monoidal functors from $M$ to $M'$ (in the sense of Definition 2.1.5.8) can be identified with monoid homomorphisms from $M$ to $M'$ (in the sense of Definition 2.1.0.5). Moreover, every lax monoidal functor from $M$ to $M'$ is automatically strict monoidal (and therefore monoidal).

Example 2.1.6.7. Let $\mathcal{C}$ be a monoidal category, and let $\ell: \mathcal{C} \to \text{Fun}(\mathcal{C},\mathcal{C})$ be the nonunital monoidal functor of Example 2.1.4.8 (carrying each object $X \in \mathcal{C}$ to the functor $\ell_X: \mathcal{C} \to \mathcal{C}$ given by $\ell_X(Y) = X \otimes Y$). Then $\ell$ is a monoidal functor: it admits a unit $\epsilon: \text{id}_\mathcal{C} \to \ell_1$ given by the inverse of the left unit constraint of Construction 2.1.2.17. To prove this, it suffices to verify that $\epsilon$ satisfies property (1) of Proposition 2.1.5.13 (Remark 2.1.6.4). Unwinding the definitions, this is equivalent to the the assertion that for every object $X \in \mathcal{C}$, the outer cycle of the diagram

\[
\begin{array}{c}
X & \xleftarrow{\lambda_X} & 1 \otimes X & \xleftarrow{\text{id}_1 \otimes \lambda_X} & 1 \otimes (1 \otimes X) \\
\downarrow & & \downarrow & \uparrow & \downarrow \\
& & & \alpha_{1,1,X} & \\
X & \xleftarrow{\lambda_X} & 1 \otimes X & \xleftarrow{\text{id}_1 \otimes \lambda_X} & (1 \otimes 1) \otimes X \\
\downarrow & & \downarrow & \uparrow & \downarrow \\
& & & \upsilon \otimes \text{id}_X & \\
& & & & 1 \otimes X
\end{array}
\]

is commutative. In fact, the whole diagram commutes: for the inner cycle on the left this is immediate, and for the inner cycle on the right it follows from the definition of the left unit constraining $\lambda_X$ (Construction 2.2.1.11).

Example 2.1.6.8 (2-Cochains as Monoidal Structures). Let $G$ be a group and let $\Gamma$ be an abelian group equipped with an action of $G$. Let $\mathcal{C}$ be the category introduced in Example 2.1.3.3 whose objects are the elements of $G$ and morphisms are given by

$$\text{Hom}_\mathcal{C}(g,h) = \begin{cases} 
\Gamma & \text{if } g = h \\
\emptyset & \text{otherwise.}
\end{cases}$$
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Then every 3-cocycle \( \alpha : G \times G \times G \to \Gamma \) can be regarded as the associativity constraint for a monoidal structure \((\otimes, \alpha)\) on \( \mathcal{C} \). Let us write \( \mathcal{C}(\alpha) \) to indicate the category \( \mathcal{C} \), endowed with the monoidal structure \((\otimes, \alpha)\).

Suppose that we are given a pair of cocycles \( \alpha, \alpha' : G \times G \times G \to \Gamma \). Unwinding the definitions, we see that monoidal structures on the identity functor \( \text{id}_\mathcal{C} : \mathcal{C}(\alpha) \to \mathcal{C}(\alpha') \) are given by functions \( \mu : G \times G \to \Gamma \) which satisfy the identity

\[
\alpha_{x,y,z} + \mu_{x,yz} + x(\mu_{y,z}) = \mu_{xy,z} + \mu_{x,y} + \alpha'_{x,y,z}
\]

for \( x, y, z \in G \). We can rewrite this identity more compactly as an equation \( \alpha + d\mu = \alpha' \), where \( d : \{2\text{-Cochains } G \times G \to \Gamma\} \to \{3\text{-Cochains } G \times G \times G \to \Gamma\} \) is defined by the formula

\[
(d\mu)_{x,y,z} = x(\mu_{y,z}) - \mu_{xy,z} + \mu_{x,yz} - \mu_{x,y}.
\]

In particular, the identity functor \( \text{id}_\mathcal{C} \) can be promoted to a monoidal functor from \( \mathcal{C}(\alpha) \) to \( \mathcal{C}(\alpha') \) if and only if the cocycles \( \alpha \) and \( \alpha' \) are cohomologous: that is, they represent the same element of the cohomology group \( H^3(G; \Gamma) \).

**Notation 2.1.6.9.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories, and let \( F, F' : \mathcal{C} \to \mathcal{D} \) be monoidal functors. We say that a natural transformation \( \gamma : F \to F' \) is monoidal if it is monoidal when viewed as a natural transformation of lax monoidal functors (Definition 2.1.5.18). We let \( \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \) denote the category whose objects are monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \) and whose morphisms are monoidal natural transformations. We regard \( \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \) as a full subcategory of the category \( \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \) of Definition 2.1.5.18 (or as a non-full subcategory of the category \( \text{Fun}^{\text{nu}}(\mathcal{C}, \mathcal{D}) \) of nonunital monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \)).

**Warning 2.1.6.10.** We will not be consistent in our usage of Notation 2.1.6.9. For example, if \( \mathcal{C} \) and \( \mathcal{D} \) are symmetric monoidal categories ([?]), then we will sometimes write \( \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \) to denote the category of symmetric monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \) which is a full subcategory of the category of monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \) defined in Notation 2.1.6.9.

**Remark 2.1.6.11** (Compatibility with Reversal). Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories, let \( F : \mathcal{C} \to \mathcal{D} \) be a nonunital lax monoidal functor, and let \( F^{\text{rev}} : \mathcal{C}^{\text{rev}} \to \mathcal{D}^{\text{rev}} \) be as in Example 2.1.4.13. Then \( F \) is a monoidal functor if and only if \( F^{\text{rev}} \) is a monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories \( \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\otimes}(\mathcal{C}^{\text{rev}}, \mathcal{D}^{\text{rev}}) \).

**Remark 2.1.6.12** (Opposite Functors). Let \( \mathcal{C} \) and \( \mathcal{D} \) be monoidal categories, let \( F : \mathcal{C} \to \mathcal{D} \) be a nonunital monoidal functor, and let \( F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}} \) be the induced nonunital monoidal
functor on opposite categories (Example 2.1.4.14). Then $F$ is a monoidal functor if and only if $F^{\text{op}}$ is a monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})^{\text{op}} \simeq \text{Fun}^\otimes(\mathcal{C}^{\text{op}}, \mathcal{D}^{\text{op}})$.

Remark 2.1.6.13 (Composition of Monoidal Functors). Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors equipped with nonunital lax monoidal structures $\mu$ and $\nu$, respectively, so that the composite functor $G \circ F$ inherits a nonunital lax monoidal structure (Construction 2.1.4.17). If $\mu$ and $\nu$ are monoidal structures on $F$ and $G$, then $G \circ F$ inherits a monoidal structure. This observation (and its counterpart for monoidal natural transformations) imply that the composition law of Construction 2.1.4.17 restricts to a functor

$$\circ : \text{Fun}^\otimes(\mathcal{D}, \mathcal{E}) \times \text{Fun}^\otimes(\mathcal{C}, \mathcal{D}) \to \text{Fun}^\otimes(\mathcal{C}, \mathcal{E}).$$

Example 2.1.6.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite products, endowed with the Cartesian monoidal structure described in Example 2.1.3.2. For any functor $F : \mathcal{C} \to \mathcal{D}$, we can regard the opposite functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ as endowed with the lax monoidal structure described in Example 2.1.5.17. This lax monoidal structure is a monoidal structure if and only if the functor $F$ preserves finite products. If this condition is satisfied, then the original functor $F$ inherits a monoidal structure (Remark 2.1.6.12).

Example 2.1.6.15 (1-Cochains as Natural Transformations). Let $G$ be a group, let $\Gamma$ be an abelian group equipped with an action of $G$, and choose a pair of 3-cocycles $\alpha, \alpha' : G \times G \times G \to \Gamma$, which we can regard as associativity constraints for monoidal categories $\mathcal{C}(\alpha)$ and $\mathcal{C}(\alpha')$ having the same underlying category $\mathcal{C}$ (Example 2.1.6.8). Suppose we are given a pair of monoidal structures $\mu$ and $\mu'$ on the identity functor $\text{id}_\mathcal{C}$, which we can identify with 2-cochains $\mu, \mu' : G \times G \to \Gamma$ satisfying

$$\alpha + d\mu = \alpha' \quad \alpha + d\mu' = \alpha'.$$

Then the difference $\nu = \mu - \mu'$ is a 2-cocycle: that is, it satisfies the identity

$$x\nu_{y,z} - \nu_{xy,z} + \nu_{x,yz} - \nu_{x,y} = 0$$

for every triple of elements $x, y, z \in G$.

Note that a natural transformation from the identity functor $\text{id}_\mathcal{C}$ to itself can be identified with a function

$$\gamma : G \to \Gamma \quad x \mapsto \gamma_x;$$
that is, with a 1-cochain on $G$ taking values in the group $\Gamma$. Unwinding the definitions, we see that the natural transformation $\gamma$ is monoidal (with respect to the monoidal structures supplied by $\mu$ and $\mu'$, respectively) if and only if it satisfies the identity

$$\mu'_{x,y} + x\gamma_y + \gamma_x = \mu_{x,y} + \gamma_{xy}$$

for every pair of elements $x, y \in G$. We can rewrite this identity more conceptually as $\mu' + d\gamma = \mu$, where

$$d : \{1\text{-Chains } G \to \Gamma\} \to \{2\text{-Chains } G \times G \to \Gamma\}$$

is defined by the formula $(d\gamma)_{x,y} = x(\gamma_y) - \gamma_{xy} + \gamma_x$. In particular, the identity transformation $\text{id}_C$ can be promoted to a monoidal isomorphism from $(\text{id}_C, \mu)$ to $(\text{id}_C, \mu')$ if and only if the 2-cocycle $\nu = \mu - \mu'$ is a coboundary: that is, it has vanishing image in the cohomology group $H^2(G; \Gamma)$.

### 2.1.7 Enriched Category Theory

Let $\mathcal{C}$ be a category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\text{Hom}_\mathcal{C}(X, Y)$ denote the set of morphisms from $X$ to $Y$ in $\mathcal{C}$. In many cases of interest, the sets $\text{Hom}_\mathcal{C}(X, Y)$ can be endowed with additional structure, which are respected by the composition law on $\mathcal{C}$. To give a systematic discussion of this phenomenon, it is convenient to use the formalism of enriched category theory.

**Definition 2.1.7.1.** Let $\mathcal{A}$ be a monoidal category with unit object $1$. An $\mathcal{A}$-enriched category $\mathcal{C}$ consists of the following data:

1. A collection $\text{Ob}(\mathcal{C})$, whose elements we refer to as objects of $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that $X$ is an element of $\text{Ob}(\mathcal{C})$.
2. For every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, an object $\text{Hom}_\mathcal{C}(X, Y)$ of the monoidal category $\mathcal{A}$.
3. For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C})$, a morphism

   $$c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)$$

   in the category $\mathcal{A}$, which we will refer to as the composition law.
4. For every object $X \in \text{Ob}(\mathcal{C})$, a morphism $e_X : 1 \to \text{Hom}_\mathcal{C}(X, X)$ in the category $\mathcal{A}$, which we refer to as the identity of $X$.

These data are required to satisfy the following conditions:
(A) For every quadruple of objects \( W, X, Y, Z \in \text{Ob}(\mathcal{C}) \), the diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(W, Y) & \xrightarrow{\text{id} \otimes \text{c}_{Y,X,W}} & \text{Hom}_\mathcal{C}(Y, Z) \otimes (\text{Hom}_\mathcal{C}(X, Y) \otimes \text{Hom}_\mathcal{C}(W, X)) \\
\downarrow \alpha & & \downarrow \text{c}_{Z,Y,W} \\
(\text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(X, Y)) \otimes \text{Hom}_\mathcal{C}(W, X) & \xrightarrow{\text{c}_{Z,Y,X} \otimes \text{id}} & \text{Hom}_\mathcal{C}(X, Z) \otimes \text{Hom}_\mathcal{C}(W, X)
\end{array}
\]

commutes. Here \( \alpha \) denotes the associativity constraint on the monoidal category \( \mathcal{A} \).

(U) For every pair of objects \( X, Y \in \text{Ob}(\mathcal{C}) \), the diagrams

\[
\begin{array}{ccc}
1 \otimes \text{Hom}_\mathcal{C}(X, Y) & \xrightarrow{\text{e}_Y \otimes \text{id}} & \text{Hom}_\mathcal{C}(Y, Y) \otimes \text{Hom}_\mathcal{C}(X, Y) \\
\downarrow \lambda & & \downarrow \text{c}_{Y,Y,X} \\
\text{Hom}_\mathcal{C}(X, Y) & & \text{Hom}_\mathcal{C}(X, Y)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(X, Y) \otimes 1 & \xrightarrow{\text{id} \otimes \text{e}_X} & \text{Hom}_\mathcal{C}(X, Y) \otimes \text{Hom}_\mathcal{C}(X, X) \\
\downarrow \rho & & \downarrow \text{c}_{Y,X,X} \\
\text{Hom}_\mathcal{C}(X, Y) & & \text{Hom}_\mathcal{C}(X, Y)
\end{array}
\]

commute, where \( \lambda \) and \( \rho \) denote the left and right unit constraints on \( \mathcal{A} \) (see Construction [2.1.2.17]).

**Example 2.1.7.2** (Categories Enriched Over Sets). Let \( \mathcal{A} = \text{Set} \) be the category of sets, endowed with the monoidal structure given by the Cartesian product (see Example [2.1.3.2]). Then an \( \mathcal{A} \)-enriched category (in the sense of Definition [2.1.7.1]) can be identified with a category in the usual sense.

**Example 2.1.7.3.** Let \( \mathcal{A} \) be a monoidal category. If \( \mathcal{C} \) is a category enriched over \( \mathcal{A} \) and \( X \) is an object of \( \mathcal{C} \), then the composition law

\[
c_{X,X,X} : \text{Hom}_\mathcal{C}(X, X) \otimes \text{Hom}_\mathcal{C}(X, X) \to \text{Hom}_\mathcal{C}(X, X)
\]
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exhibits $\text{Hom}_C(X, X)$ as an algebra object of $A$, in the sense of Example 2.1.5.21. Moreover, this construction induces a bijection

$$\{A\text{-Enriched Categories } C \text{ with } \text{Ob}(C) = \{X\}\} \simeq \{\text{Algebra objects of } A\}.$$  

Consequently, the theory of enriched categories can be regarded as a generalization of the theory of associative algebras (See Example 2.1.7.14 for a more precise statement).

**Remark 2.1.7.4** (Functoriality). Let $A$ and $A'$ be monoidal categories, and let $F: A \to A'$ be a lax monoidal functor (with tensor constraints $\mu_{A,B}: F(A) \otimes F(B) \to F(A \otimes B)$ and unit $\epsilon: 1_{A'} \to F(1_A)$). Then every $A$-enriched category $C$ determines an $A'$-enriched category $C'$, which can be described concretely as follows:

- The objects of $C'$ are the objects of $C$: that is, we have $\text{Ob}(C') = \text{Ob}(C)$.
- For every pair of objects $X, Y \in \text{Ob}(C')$, we set $\text{Hom}_{C'}(X, Y) = F(\text{Hom}_C(X, Y))$.
- For every triple of objects $X, Y, Z \in \text{Ob}(C')$, the composition law $\epsilon'_{Z,Y,X}$ for $C'$ is given by the composition

$$\text{Hom}_{C'}(Y, Z) \otimes \text{Hom}_{C'}(X, Y) \xrightarrow{\mu} F(\text{Hom}_C(Y, Z) \otimes \text{Hom}_C(X, Y)) \xrightarrow{F(\epsilon_{Z,Y,X})} F(\text{Hom}_C(X, Z)) = \text{Hom}_{C'}(X, Z).$$

- For every object $X \in \text{Ob}(C')$, the identity morphism $\epsilon'_{X}$ for $X$ in $C'$ is given by the composition

$$1_{A'} \xrightarrow{\epsilon} F(1_A) \xrightarrow{F(\text{ex})} F(\text{Hom}_{C'}(X, X)) \simeq \text{Hom}_{C'}(X, X).$$

**Example 2.1.7.5** (The Underlying Category of an Enriched Category). Let $A$ be a monoidal category and let $F: A \to \text{Set}$ be the functor given by $F(A) = \text{Hom}_A(1, A)$, endowed with the lax monoidal structure of Example 2.1.5.16. If $C$ is a category enriched over $A$, then we can apply the construction of Remark 2.1.7.4 to obtain a Set-enriched category, which we can identify with an ordinary category (Example 2.1.7.2). We will refer to this category as the underlying category of the $A$-enriched category $C$, and we will generally abuse notation by denoting it also by $C$. Concretely, this underlying category has the same objects as the enriched category $C$, with morphism sets given by the formula $\text{Hom}_C(X, Y) = \text{Hom}_A(1, \text{Hom}_C(X, Y))$.

**Remark 2.1.7.6.** Let $A$ be a monoidal category and let $C$ be an ordinary category. We define an $A$-enrichment of $C$ to be an $A$-enriched category $\tilde{C}$ together with an identification of $C$ with the underlying category of $\tilde{C}$, in the sense of Example 2.1.7.5.
Example 2.1.7.7 (Enrichment in Vector Spaces). Let $k$ be a field and let $\text{Vect}_k$ denote the category of vector spaces over $k$, endowed with the monoidal structure given by tensor product over $k$ (Example 2.1.3.1). Then choosing an $\text{Vect}_k$-enrichment of $\mathcal{C}$ is equivalent to endowing each of the sets $\text{Hom}_\mathcal{C}(X,Y)$ with the structure of a $k$-vector space, for which the composition maps

$$\text{Hom}_\mathcal{C}(Y,Z) \times \text{Hom}_\mathcal{C}(X,Y) \rightarrow \text{Hom}_\mathcal{C}(X,Z)$$

are $k$-bilinear.

Example 2.1.7.8 (Topologically Enriched Categories). Let $\text{Top}$ denote the category of topological spaces, endowed with the monoidal structure given by the Cartesian product (Example 2.1.3.2). We will refer to a $\text{Top}$-enriched category as a topologically enriched category. Note that the functor $F$ of Example 2.1.7.5 is (canonically isomorphic to) the forgetful functor $\text{Top} \rightarrow \text{Set}$. Consequently, if $\mathcal{C}$ is a topologically enriched category, then the underlying ordinary category $\mathcal{C}_0$ can be described concretely as follows:

- The objects of the ordinary category $\mathcal{C}_0$ as the same as the objects of the Top-enriched category $\mathcal{C}$.
- Given a pair of objects $X, Y \in \mathcal{C}_0$, a morphism $f$ from $X$ to $Y$ (in the ordinary category $\mathcal{C}_0$) is a point of the topological space $\text{Hom}_\mathcal{C}(X,Y)$.
- Given a pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in $\mathcal{C}_0$, the composition $g \circ f$ is given by the image of $(g,f)$ under the continuous map

$$c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y,Z) \otimes \text{Hom}_\mathcal{C}(X,Y) \rightarrow \text{Hom}_\mathcal{C}(X,Z).$$

It follows that, for any ordinary category $\mathcal{C}_0$, promoting $\mathcal{C}_0$ to a topologically enriched category $\mathcal{C}$ is equivalent to endowing each of the morphism sets $\text{Hom}_{\mathcal{C}_0}(X,Y)$ with a topology for which the composition maps $\circ : \text{Hom}_{\mathcal{C}_0}(Y,Z) \times \text{Hom}_{\mathcal{C}_0}(X,Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(X,Z)$ are continuous.

Exercise 2.1.7.9 (Uniqueness of Identities). Let $\mathcal{A}$ be a monoidal category. A nonunital $\mathcal{A}$-enriched category $\mathcal{C}$ consists of a collection $\text{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$, together with objects $\{\text{Hom}_\mathcal{C}(X,Y)\}_{X,Y \in \text{Ob}(\mathcal{C})}$ of the category $\mathcal{A}$ and composition laws

$$c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y,Z) \otimes \text{Hom}_\mathcal{C}(X,Y) \rightarrow \text{Hom}_\mathcal{C}(X,Z)$$

which satisfy the associative law $(A)$ appearing in Definition 2.1.7.1. Show that, if a nonunital $\mathcal{A}$-enriched category $\mathcal{C}$ can be promoted to an $\mathcal{A}$-enriched category $\mathcal{C}$, then $\mathcal{C}$ is unique: that is, the identity maps $e_X : 1 \rightarrow \text{Hom}_\mathcal{C}(X,X)$ are determined by axiom $(U)$ of Definition 2.1.7.1.
Definition 2.1.7.10. Let $\mathcal{A}$ be a monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{A}$-enriched categories. An $\mathcal{A}$-enriched functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

1. For every object $X \in \text{Ob}(\mathcal{C})$, and object $F(X) \in \text{Ob}(\mathcal{D})$.
2. For every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a morphism
   
   \[ F_{X,Y} : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{D}(F(X), F(Y)) \]

   in the category $\mathcal{A}$.

These data are required to satisfy the following conditions:

- For every object $X \in \text{Ob}(\mathcal{C})$, the morphism $e_{F(X)} : 1 \to \text{Hom}_\mathcal{D}(F(X), F(X))$ factors as a composition

\[ 1 \xrightarrow{e_X} \text{Hom}_\mathcal{C}(X, X) \xrightarrow{F_{X,X}} \text{Hom}_\mathcal{D}(F(X), F(X)). \]

- For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C})$, the diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(Y, Z) \otimes \text{Hom}_\mathcal{C}(X, Y) & \longrightarrow & \text{Hom}_\mathcal{C}(X, Z) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{Hom}_\mathcal{D}(F(Y), F(Z)) \otimes \text{Hom}_\mathcal{D}(F(X), F(Y)) & \longrightarrow & \text{Hom}_\mathcal{D}(F(X), F(Z))
\end{array}
\]

commutes (in the category $\mathcal{A}$); here the horizontal maps are given by the composition laws on $\mathcal{C}$ and $\mathcal{D}$.

Notation 2.1.7.11 (The Category of Enriched Categories). Let $\mathcal{A}$ be a monoidal category. We say that an $\mathcal{A}$-enriched category $\mathcal{C}$ is small if the collection of objects $\text{Ob}(\mathcal{C})$ is small. The collection of small $\mathcal{A}$-enriched categories can itself be organized into a category $\text{Cat}(\mathcal{A})$, whose morphisms are given by $\mathcal{A}$-enriched functors (in the sense of Definition 2.1.7.10).

Example 2.1.7.12. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, which we regard as Set-enriched categories by means of Example 2.1.7.2. Then Set-enriched functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.1.7.10) can be identified with functors from $\mathcal{C}$ to $\mathcal{D}$ in the usual sense. This identification determines an isomorphism of categories $\text{Cat}(\mathcal{A}) \simeq \text{Cat}(\text{Set})$.

Remark 2.1.7.13. Let $F : \mathcal{A} \to \mathcal{A}'$ be a lax monoidal functor between monoidal categories. Then the construction of Remark 2.1.7.4 determines a functor $\text{Cat}(\mathcal{A}) \to \text{Cat}(\mathcal{A}')$. In the special case where $\mathcal{A}' = \text{Set}$ and $F$ is the functor $A \mapsto \text{Hom}_\mathcal{A}(1, A)$ corespresented by the unit object $1 \in \mathcal{A}$, we obtain a forgetful functor

\[ \text{Cat}(\mathcal{A}) \to \text{Cat}(\text{Set}) \simeq \text{Cat}, \]

which assigns to each (small) $\mathcal{A}$-enriched category $\mathcal{C}$ its underlying ordinary category (Example 2.1.7.5).
Given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ with the same source and target, we are usually not interested in the question of whether or not $F$ and $G$ are equal. Instead, we should regard $F$ and $G$ as interchangeable if there exists a natural isomorphism $\alpha : F \simeq G$. This sort of information is not encoded in the structure of the category $\text{Cat}$.

(ii) Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, we are usually not interested in the question of whether or not $\mathcal{C}$ and $\mathcal{D}$ are isomorphic. Instead, we should regard $\mathcal{C}$ and $\mathcal{D}$ as interchangeable if there exists an equivalence of categories from $F : \mathcal{C} \to \mathcal{D}$. In this case, the functor $F$ need not be invertible when regarded as a morphism in $\text{Cat}$.

To remedy the situation, it is useful to contemplate a more elaborate mathematical structure.

**Definition 2.2.0.1.** A strict 2-category $\mathcal{C}$ consists of the following data:

- A collection $\text{Ob}(\mathcal{C})$, whose elements we refer to as objects of $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that $X$ is an element of $\text{Ob}(\mathcal{C})$. 

**Example 2.1.7.14.** Let $\mathcal{A}$ be a monoidal category, let $A$ be an algebra object of $\mathcal{A}$, which we can identify with an $\mathcal{A}$-enriched category $\mathcal{C}_A$ having a single object $X$ (Example 2.1.7.3). For any $\mathcal{A}$-enriched category $\mathcal{D}$ containing an object $Y$, we have a canonical bijection

$$\{\text{$\mathcal{A}$-Enriched Functors } F : \mathcal{C}_A \to \mathcal{D} \text{ with } F(X) = Y\} \sim \{\text{Algebra homomorphisms } A \to \text{Hom}_\mathcal{D}(Y,Y)\}.$$ 

In particular, if $\mathcal{D} = \mathcal{C}_B$ for some other algebra object $B \in \text{Alg}(\mathcal{D})$, we obtain a bijection

$$\text{Hom}_{\text{Cat}(\mathcal{A})}(\mathcal{C}_A, \mathcal{C}_B) \simeq \text{Hom}_{\text{Alg}(\mathcal{A})}(A, B).$$

In other words, the construction $A \mapsto \mathcal{C}_A$ induces a fully faithful embedding $\text{Alg}(\mathcal{A}) \to \text{Cat}(\mathcal{A})$, whose essential image is spanned by those $\mathcal{A}$-enriched categories having a single object.

**2.2 The Theory of 2-Categories**

The collection of (small) categories can itself be organized into a (large) category $\text{Cat}$, whose objects are small categories and whose morphisms are functors. However, the structure of $\text{Cat}$ as an abstract category fails to capture many of the essential features of category theory:

(i) Given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ with the same source and target, we are usually not interested in the question of whether or not $F$ and $G$ are equal. Instead, we should regard $F$ and $G$ as interchangeable if there exists a natural isomorphism $\alpha : F \simeq G$. This sort of information is not encoded in the structure of the category $\text{Cat}$.

(ii) Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, we are usually not interested in the question of whether or not $\mathcal{C}$ and $\mathcal{D}$ are isomorphic. Instead, we should regard $\mathcal{C}$ and $\mathcal{D}$ as interchangeable if there exists an equivalence of categories from $F : \mathcal{C} \to \mathcal{D}$. In this case, the functor $F$ need not be invertible when regarded as a morphism in $\text{Cat}$.
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- For every pair of objects \(X, Y \in \mathcal{C}\), a category \(\text{Hom}_C(X, Y)\). We refer to objects \(f\) of the category \(\text{Hom}_C(X, Y)\) as 1-morphisms from \(X\) to \(Y\) and write \(f : X \to Y\) to indicate that \(f\) is a 1-morphism from \(X\) to \(Y\). Given a pair of 1-morphisms \(f, g \in \text{Hom}_C(X, Y)\), we refer to morphisms from \(f\) to \(g\) in the category \(\text{Hom}_C(X, Y)\) as 2-morphisms from \(f\) to \(g\).

- For every triple of objects \(X, Y, Z \in \mathcal{C}\), a composition functor \(\circ : \text{Hom}_C(Y, Z) \times \text{Hom}_C(X, Y) \to \text{Hom}_C(X, Z)\).

- For every object \(X \in \mathcal{C}\), an identity 1-morphism \(\text{id}_X \in \text{Hom}_C(X, X)\).

These data are required to satisfy the following conditions:

1. For each object \(X \in \mathcal{C}\), the identity 1-morphism \(\text{id}_X\) is a unit for both right and left composition. That is, for every object \(Y \in \mathcal{C}\), the functors
   \[
   \text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y) \quad f \mapsto f \circ \text{id}_X
   \]
   \[
   \text{Hom}_C(Y, X) \to \text{Hom}_C(Y, X) \quad g \mapsto \text{id}_X \circ g
   \]
   are both equal to the identity.

2. The composition law of \(\mathcal{C}\) is strictly associative. That is, for every quadruple of objects \(W, X, Y, Z \in \mathcal{C}\), the diagram of categories
   \[
   \text{Hom}_C(Y, Z) \times \text{Hom}_C(X, Y) \times \text{Hom}_C(W, X) \xrightarrow{\text{id} \times \circ} \text{Hom}_C(Y, Z) \times \text{Hom}_C(W, Y)
   \]
   \[
   \text{Hom}_C(X, Z) \times \text{Hom}_C(W, X) \xrightarrow{\circ} \text{Hom}_C(W, Z)
   \]
   commutes (in the ordinary category \(\text{Cat}\)).

**Remark 2.2.0.2** (Strict 2-Categories as Enriched Categories). Let \(\text{Cat}\) denote the category whose objects are (small) categories and whose morphisms are functors. Then \(\text{Cat}\) admits finite products, and therefore admits a monoidal structure given by the formation of Cartesian products (Example 2.1.3.2). Neglecting set-theoretic technicalities, a strict 2-category (in the sense of Definition 2.2.0.1) can be identified with a \(\text{Cat}\)-enriched category (in the sense of Definition 2.1.7.1).

**Remark 2.2.0.3.** To every strict 2-category \(\mathcal{C}\), we can associate an ordinary category \(\mathcal{C}_0\), whose objects and morphisms are given by
\[
\text{Ob}(\mathcal{C}_0) = \text{Ob}(\mathcal{C}) \quad \text{Hom}_{\mathcal{C}_0}(X, Y) = \text{Ob}(\text{Hom}_C(X, Y)).
\]
We will refer to $C_0$ as the *underlying ordinary category* of $C$ (note that $C_0$ can be obtained from $C$ by the general procedure of Example [2.1.7.5](#2.1.7.5)). More informally, the underlying category $C_0$ is obtained from $C$ by “forgetting” its 2-morphisms.

**Example 2.2.0.4.** We define a strict 2-category $\textbf{Cat}$ as follows:

- The objects of $\textbf{Cat}$ are (small) categories.
- For every pair of small categories $\mathcal{C}, \mathcal{D} \in \text{Cat}$, we take $\text{Fun}(\mathcal{C}, \mathcal{D})$ to be the category $\text{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$.
- The composition law on $\textbf{Cat}$ is given by the usual composition of functors.

We will refer to $\textbf{Cat}$ as the *strict 2-category of (small) categories*. Note that the underlying ordinary category of $\textbf{Cat}$ is the category $\text{Cat}$ (whose objects are small categories and morphisms are functors).

We can obtain many more examples by studying categories equipped with additional structure.

**Example 2.2.0.5.** We define a strict 2-category $\textbf{MonCat}$ as follows:

- The objects of $\textbf{MonCat}$ are (small) monoidal categories.
- For every pair of small monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we take $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ to be the category $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ (Notation [2.1.6.9](#2.1.6.9)).
- The composition law on $\textbf{MonCat}$ is given by the composition of monoidal functors described in Remark [2.1.6.13](#2.1.6.13).

There are several obvious variants on this construction: for example, we can work with nonunital monoidal categories in place of monoidal categories, or lax monoidal functors in place of monoidal functors.

**Example 2.2.0.6 (Ordinary Categories).** Every ordinary category can be regarded as a strict 2-category. More precisely, to each category $\mathcal{C}$ we can associate a strict 2-category $\mathcal{C}'$ as follows:

- The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, objects of the category $\text{Hom}_\mathcal{C}(X, Y)$ are elements of the set $\text{Hom}_\mathcal{C}(X, Y)$, and every morphism in $\text{Hom}_\mathcal{C}(X, Y)$ is an identity morphism.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition functor
  $$\circ : \text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)$$
  is given on objects by the composition map $\text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)$.  


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- For every object \( X \in \mathcal{C} \), the identity object \( \text{id}_X \in \text{Hom}_\mathcal{C}(X, X) \) coincides with the identity morphism \( \text{id}_X \in \text{Hom}_\mathcal{C}(X, X) \).

In this situation, we will generally abuse terminology by identifying the strict 2-category \( \mathcal{C}' \) with the ordinary category \( \mathcal{C} \) (see Example 2.2.5.7).

**Remark 2.2.0.7** (Endomorphism Categories). Let \( \mathcal{C} \) be a strict 2-category and let \( X \) be an object of \( \mathcal{C} \). We will write \( \text{End}_\mathcal{C}(X) \) for the category \( \text{Hom}_\mathcal{C}(X, X) \). Then the composition law

\[
\circ : \text{Hom}_\mathcal{C}(X, X) \times \text{Hom}_\mathcal{C}(X, X) \to \text{Hom}_\mathcal{C}(X, X)
\]
determines a strict monoidal structure on the category \( \text{End}_\mathcal{C}(X) \).

Note that, if \( \mathcal{C} \) is an ordinary category (regarded as a strict 2-category by means of Example 2.2.0.6), then the endomorphism category \( \text{End}_\mathcal{C}(X) \) can be identified with the endomorphism monoid \( \text{End}_\mathcal{C}(X) \) of Example 2.1.0.1, regarded as a (strict) monoidal category via Example 2.1.2.8.

**Example 2.2.0.8** (Delooping). Let \( \mathcal{M} \) be a category equipped with a strict monoidal structure \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) (Definition 2.1.2.1). We define a strict 2-category \( B\mathcal{M} \) as follows:

- The set of objects \( \text{Ob}(B\mathcal{M}) \) is the singleton set \( \{X\} \).
- The category \( \text{Hom}_{B\mathcal{M}}(X, X) \) is equal to \( \mathcal{M} \).
- The composition functor \( \circ : \text{Hom}_{B\mathcal{M}}(X, X) \times \text{Hom}_{B\mathcal{M}}(X, X) \to \text{Hom}_{B\mathcal{M}}(X, X) \) is equal to the tensor product \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \).
- The identity morphism \( \text{id}_X \) is the strict unit object of \( \mathcal{M} \).

We will refer to \( B\mathcal{M} \) as the delooping of \( \mathcal{M} \).

Note that the constructions

\[
\mathcal{M} \mapsto B\mathcal{M} \quad \mathcal{C} \mapsto \text{End}_\mathcal{C}(X)
\]

induce mutually inverse bijections

\[
\{\text{Strict Monoidal Categories } \mathcal{M}\} \simeq \{\text{Strict 2-Categories } \mathcal{C} \text{ with } \text{Ob}(\mathcal{C}) = \{X\}\},
\]
generalizing the identification of Remark 2.1.0.6.

The reader might at this point object that the definition of strict 2-category violates a fundamental principle of category theory: axioms (1) and (2) of Definition 2.2.0.1 require that certain functors are equal. In practice, one often encounters mathematical structures \( \mathcal{C} \)
which do not quite fit in the framework of Definition 2.2.0.1 because the associative law for composition of 1-morphisms in \( C \) holds only up to isomorphism. To address this point, Bénabou introduced a more general type of structure which he called a \textit{bicategory}, which we will refer to here as a \textit{2-category}.

Our goal in this section is to give a brief introduction to the theory of 2-categories. We begin in §2.2.1 by reviewing the definition of a 2-category (Definition 2.2.1.1) and establishing some notational and terminological conventions. Every strict 2-category can be regarded as a 2-category (Example 2.2.1.4), but many of the 2-categories which arise “in nature” fail to be strict: we discuss several examples of this phenomenon in §2.2.2.

To articulate the relationship between 2-categories and strict 2-categories more precisely, it is convenient to view each as the objects of a suitable (ordinary) category. In §2.2.4, we introduce the notion of a \textit{functor} between 2-categories (Definition 2.2.4.5). Roughly speaking, a functor \( F : C \to D \) is an operation which carries objects, 1-morphisms, and 2-morphisms of \( C \) to objects, 1-morphisms, and 2-morphisms of \( D \), which is compatible with the composition laws on \( C \) and \( D \). Here again there are several possible definitions, depending on whether one demands that the compatibility holds strictly (in which case we say that \( F \) is a \textit{strict} functor), up to isomorphism (in which case we say that \( F \) is a \textit{functor}), or up to possible non-invertible 2-morphism (in which case we say that \( F \) is a \textit{lax} functor). We use this notion in §2.2.5 to introduce an (ordinary) category \( 2\text{Cat} \), whose objects are 2-categories and whose morphisms are functors between 2-categories (and consider several other variations on this theme).

The notion of 2-category is more general than the notion of strict 2-category defined above: in general, a 2-category \( C \) need not be strict or even isomorphic (as an object of \( 2\text{Cat} \)) to a strict 2-category \( C' \). However, we will prove in §2.2.7 that every 2-category \( C \) is isomorphic to a \textit{strictly unitary} 2-category \( C' \): that is, a 2-category \( C' \) in which the composition law is strictly unital, but not necessarily strictly associative (Proposition 2.2.7.7). The proof will make use of a certain twisting procedure in the setting of 2-categories (Construction 2.2.6.8), which we will describe in §2.2.6.

**Remark 2.2.0.9.** Let \( C \) be a 2-category. It is generally not possible to find a strict 2-category \( C' \) which is \textit{isomorphic} to \( C \) (as an object of the category \( 2\text{Cat} \) we will introduce in §2.2.5). However, it is always possibly to find a strict 2-category \( C' \) which is \textit{equivalent} to \( C \); we will return to this point in §[?].

### 2.2.1 2-Categories

Let \( C \) be a strict 2-category (Definition 2.2.0.1). Then the composition of 1-morphisms in \( C \) is strictly associative: that is, given a triple of composable 1-morphisms

\[ f : W \to X \quad g : X \to Y \quad h : Y \to Z \]
of $\mathcal{C}$, we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$. Our goal in this section is to introduce the more general notion of (non-strict) 2-category, where we weaken the associativity requirement: rather than demand that the 1-morphisms $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are identical, we instead ask for a specified isomorphism $\alpha_{h,g,f} : h \circ (g \circ f) \cong (h \circ g) \circ f$ in the category $\text{Hom}_\mathcal{C}(W,Z)$. In order to obtain a sensible theory, we must require that these isomorphisms satisfy an analogue of the pentagon identity which appears in Definition 2.1.1.5.

**Definition 2.2.1.1** (Bénabou). A 2-category $\mathcal{C}$ consists of the following data:

- A collection $\text{Ob}(\mathcal{C})$, whose elements we refer to as objects of $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that $X$ is an element of $\text{Ob}(\mathcal{C})$.

- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a category $\text{Hom}_\mathcal{C}(X,Y)$. We refer to objects $f$ of the category $\text{Hom}_\mathcal{C}(X,Y)$ as 1-morphisms from $X$ to $Y$ and write $f : X \to Y$ to indicate that $f$ is a 1-morphism from $X$ to $Y$. Given a pair of 1-morphisms $f, g \in \text{Hom}_\mathcal{C}(X,Y)$, we refer to morphisms from $f$ to $g$ in the category $\text{Hom}_\mathcal{C}(X,Y)$ as 2-morphisms from $f$ to $g$. We will sometimes write $\gamma : f \Rightarrow g$ or $f \gamma = g$ to indicate that $\gamma$ is a 2-morphism from $f$ to $g$.

- For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C})$, a composition functor $\circ : \text{Hom}_\mathcal{C}(Y,Z) \times \text{Hom}_\mathcal{C}(X,Y) \to \text{Hom}_\mathcal{C}(X,Z)$.

- For every object $X \in \text{Ob}(\mathcal{C})$, a 1-morphism $\text{id}_X \in \text{Hom}_\mathcal{C}(X,X)$, which we call the identity 1-morphism from $X$ to itself.

- For every object $X \in \text{Ob}(\mathcal{C})$, an isomorphism $v_X : \text{id}_X \circ \text{id}_X \cong \text{id}_X$ in the category $\text{Hom}_\mathcal{C}(X,X)$. We refer to the 2-morphisms $\{v_X\}_{X \in \text{Ob}(\mathcal{C})}$ as the unit constraints of $\mathcal{C}$.

- For every quadruple of objects $W, X, Y, Z \in \mathcal{C}$, a natural isomorphism $\alpha$ from the functor

$$\text{Hom}_\mathcal{C}(Y,Z) \times \text{Hom}_\mathcal{C}(X,Y) \times \text{Hom}_\mathcal{C}(W,X) \to \text{Hom}_\mathcal{C}(W,Z)$$

$(h, g, f) \mapsto h \circ (g \circ f)$

to the functor

$$\text{Hom}_\mathcal{C}(Y,Z) \times \text{Hom}_\mathcal{C}(X,Y) \times \text{Hom}_\mathcal{C}(W,X) \to \text{Hom}_\mathcal{C}(W,Z)$$

$(h, g, f) \mapsto (h \circ g) \circ f$.

We denote the value of $\alpha$ on a triple $(h, g, f)$ by $\alpha_{h,g,f} : h \circ (g \circ f) \cong (h \circ g) \circ f$. We refer to these isomorphisms as the associativity constraints of $\mathcal{C}$.

These data are required to satisfy the following pair of conditions:
(C) For every pair of objects $X, Y \in \text{Ob}(C)$, the functors
\[
\text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y) \quad f \mapsto f \circ \text{id}_X
\]
\[
\text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y) \quad f \mapsto \text{id}_Y \circ f
\]
are fully faithful.

(P) For every quadruple of composable 1-morphisms

\[
V \xrightarrow{e} W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z
\]

in $C$, the diagram of isomorphisms

\[
\begin{array}{ccc}
  h \circ ((g \circ f) \circ e) & \xrightarrow{\alpha_{h,g,f,e}} & (h \circ (g \circ f)) \circ e \\
  \sim & & \sim \\
  h \circ (g \circ (f \circ e)) & \xrightarrow{\alpha_{h,g,f,e}} & ((h \circ g) \circ f) \circ e
\end{array}
\]

commutes in the category $\text{Hom}_C(V, Z)$.

**Remark 2.2.1.2.** An equivalent formulation of Definition [2.2.1.1] was given by Bénabou in [1]. Beware that Bénabou uses the term *bicategory* for what we call a 2-category.

**Remark 2.2.1.3.** In the situation of Definition [2.2.1.1], we will refer axiom $(P)$ as the *pentagon identity*.

**Example 2.2.1.4** (Strict 2-Categories). Let $C$ be any strict 2-category (in the sense of Definition [2.2.0.1]). Then $C$ can be viewed as a 2-category (in the sense of Definition [2.2.1.1]) by taking the unit and associativity constraints $\nu_X$ and $\alpha_{h,g,f}$ to be identity 2-morphisms in $C$.

**Remark 2.2.1.5.** Let $C$ be a 2-category. Then $C$ can be obtained from an ordinary category (via the construction of Example [2.2.0.6]) if and only if every 2-morphism in $C$ is an identity 2-morphism (note that a 2-category with this property is automatically strict, by virtue of Example [2.2.1.4]).

**Remark 2.2.1.6** (Endomorphism Categories). Let $C$ be a 2-category and let $X$ be an object of $C$. We will denote the category $\text{Hom}_C(X, X)$ by $\text{End}_C(X)$ refer to it as the *endomorphism*
category of $X$. The category $\text{End}_C(X)$ has a monoidal structure, with tensor product given by the composition law

$$\circ : \text{Hom}_C(X, X) \times \text{Hom}_C(X, X) \to \text{Hom}_C(X, X),$$

unit object given by the identity 1-morphism $\text{id}_X$, and the unit and associativity constraints of $\text{End}_C(X)$ given by $v_X$ and the associativity constraints of $C$, respectively.

**Notation 2.2.1.7.** Let $C$ be a 2-category. We will generally follow the convention of denoting objects of $C$ by capital Roman letters, 1-morphisms of $C$ by lowercase Roman letters, and 2-morphisms of $C$ by lowercase Greek letters. However, we will often violate this convention when discussing specific examples. For instance, when studying the (strict) 2-category $\text{Cat}$ of small categories (Example 2.2.0.4), we denote objects using calligraphic letters (such as $C$ and $D$) and 1-morphisms using uppercase Roman letters (such as $F$ and $G$).

**Warning 2.2.1.8.** Let $C$ be a 2-category. If $C$ is strict, then we can extract from $C$ an underlying ordinary category having the same objects and 1-morphisms (Remark 2.2.0.3). However, this operation has no counterpart for a general 2-category $C$: in general, composition of 1-morphisms in $C$ is associative only up to isomorphism.

**Warning 2.2.1.9.** Let $C$ be a 2-category. Then there are two different notions of composition for the 2-morphisms of $C$:

(V) Let $X$ and $Y$ be objects of $C$. Suppose we are given 1-morphisms $f, g, h : X \to Y$ and a pair of 2-morphisms

$$\gamma : f \Rightarrow g \quad \delta : g \Rightarrow h.$$

We can then apply the composition law in the ordinary category $\text{Hom}_C(X, Y)$ to obtain a 2-morphism $f \Rightarrow h$, which we refer to as the vertical composition of $\gamma$ and $\delta$.

(H) Let $X$, $Y$, and $Z$ be objects of $C$. Suppose we are given 2-morphisms $\gamma : f \Rightarrow g$ in the category $\text{Hom}_C(X, Y)$ and $\gamma' : f' \Rightarrow g'$ in the category $\text{Hom}_C(Y, Z)$. Then the image of $(\gamma', \gamma)$ under the composition law

$$\circ : \text{Hom}_C(Y, Z) \times \text{Hom}_C(X, Y) \to \text{Hom}_C(X, Z),$$

is a 2-morphism from $f' \circ f$ to $g' \circ g$, which will refer to as the horizontal composition of $\gamma$ and $\gamma'$.
The terminology is motivated by the following graphical representations of the data described in (V) and (H):

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \gamma \downarrow \delta \\
\downarrow \delta \\
X \xleftarrow{g} Z
\end{array}
\]

To avoid confusion, we will generally denote the vertical composition of 2-morphisms \(\gamma\) and \(\delta\) by \(\delta \gamma\) and the horizontal composition of 2-morphisms \(\gamma\) and \(\gamma'\) by \(\gamma' \circ \gamma\).

**Remark 2.2.1.10.** Let \(\mathcal{C}\) be a 2-category. For each object \(X \in \text{Ob}(\mathcal{C})\), the identity 1-morphism \(\text{id}_X\) and the unit constraint \(\upsilon_X\) are determined (up to unique isomorphism) by the composition law and associativity constraints. More precisely, given any other choice of identity morphism \(\text{id}'_X\) and unit constraint \(\upsilon'_X : \text{id}'_X \circ \text{id}'_X \Rightarrow \text{id}'_X\), there exists a unique invertible 2-morphism \(\gamma : \text{id}_X \Rightarrow \text{id}'_X\) for which the diagram

\[
\begin{array}{ccc}
\text{id}_X \circ \text{id}_X & \xrightarrow{\upsilon_X} & \text{id}_X \\
\downarrow \gamma \circ \gamma & & \downarrow \gamma \\
\text{id}'_X \circ \text{id}'_X & \xrightarrow{\upsilon'_X} & \text{id}'_X
\end{array}
\]

commutes. This follows from Proposition 2.1.2.9 applied to the monoidal category \(\text{End}_{\mathcal{C}}(X)\) of Remark 2.2.1.6.

It is possible to adopt a variant of Definition 2.2.1.1 where we do not require the identity morphisms \(\{\text{id}_X\}_{X \in \text{Ob}(\mathcal{C})}\) (or unit constraints \(\{\upsilon_X\}_{X \in \text{Ob}(\mathcal{C})}\)) to be explicitly specified. This variant is equivalent to Definition 2.2.1.1 for many purposes. However, it is not suitable for our applications: in §2.3, we associate to each 2-category \(\mathcal{C}\) a simplicial set \(N^D_{\bullet}(\mathcal{C})\) called the *Duskin nerve* of \(\mathcal{C}\), whose degeneracy operators depend on the choice of identity morphisms and unit constraints in \(\mathcal{C}\) (though the face operators do not: see Warning 2.3.1.11).

Axiom (C) of Definition 2.2.1.1 requires that, for every pair of objects \(X\) and \(Y\) of a 2-category \(\mathcal{C}\), the functors

\[
\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Y) \quad f \mapsto f \circ \text{id}_X, \text{id}_Y \circ f
\]

are fully faithful. In fact, we can say more: they are canonically isomorphic to the identity functor from \(\text{Hom}_{\mathcal{C}}(X, Y)\) to itself.

**Construction 2.2.1.11 (Left and Right Unit Constraints).** Let \(\mathcal{C}\) be a 2-category. For every 1-morphism \(f : X \to Y\) in \(\mathcal{C}\), we have canonical isomorphisms

\[
\text{id}_Y \circ (\text{id}_Y \circ f) \xrightarrow{\text{id}_Y \circ \text{id}_Y \circ f} (\text{id}_Y \circ \text{id}_Y) \circ f \xrightarrow{\text{id}_Y \circ \text{id}_Y} \text{id}_Y \circ f.
\]
Since composition on the left with \( \text{id}_Y \) is fully faithful, it follows that there is a unique isomorphism \( \lambda_f : \text{id}_Y \circ f \xrightarrow{\sim} f \) for which the diagram

\[
\begin{array}{ccc}
\text{id}_Y \circ (\text{id}_Y \circ f) & \xrightarrow{\alpha_{\text{id}_Y, \text{id}_Y \circ f}} & (\text{id}_Y \circ \text{id}_Y) \circ f \\
\downarrow{\tilde{\lambda}_f} & & \downarrow{\tilde{\text{id}}_Y \circ \text{id}_f} \\
\text{id}_Y \circ f & & \text{id}_Y \circ f
\end{array}
\]

commutes. We will refer to \( \lambda_f \) as the \textit{left unit constraint}. Similarly, there is a unique isomorphism \( \rho_f : f \circ \text{id}_X \xrightarrow{\sim} f \) for which the diagram

\[
\begin{array}{ccc}
f \circ (\text{id}_X \circ \text{id}_X) & \xrightarrow{\alpha_{f, \text{id}_X \circ \text{id}_X}} & (f \circ \text{id}_X) \circ \text{id}_X \\
\downarrow{\tilde{\text{id}}_f \circ \text{id}_X} & & \downarrow{\tilde{\rho}_f \circ \text{id}_X} \\
f \circ \text{id}_X & & f \circ \text{id}_X
\end{array}
\]

commutes; we refer to \( \rho_f \) as the \textit{right unit constraint}.

**Remark 2.2.1.12.** Let \( \mathcal{C} \) be a 2-category and let \( X \) be an object of \( \mathcal{C} \). For every 1-morphism \( f : X \to X \) in \( \mathcal{C} \), the left and right unit constraints

\[
\lambda_f : \text{id}_X \circ f \xrightarrow{\sim} f \quad \rho_f : f \circ \text{id}_X \xrightarrow{\sim} f
\]

of Construction 2.2.1.11 coincide with the left and right unit constraints of Construction 2.1.2.17 applied to the monoidal category \( \text{End}_\mathcal{C}(X) \) of Remark 2.2.1.6.

**Remark 2.2.1.13** (Naturality of Unit Constraints). Let \( \mathcal{C} \) be a 2-category, let \( X \) and \( Y \) be objects of \( \mathcal{C} \), and let \( \gamma : f \Rightarrow g \) be a morphism in the category \( \mathbf{Hom}_\mathcal{C}(X, Y) \). Then the diagram of 2-morphisms

\[
\begin{array}{ccc}
\text{id}_Y \circ f & \xrightarrow{\lambda_f} & f \\
\downarrow{\text{id}_Y \circ \gamma} & & \downarrow{\gamma} \\
\text{id}_Y \circ g & \xrightarrow{\lambda_g} & g
\end{array}
\]

commutes. In other words, the construction \( f \mapsto \lambda_f \) determines a natural isomorphism from the functor

\[
\mathbf{Hom}_\mathcal{C}(X, Y) \to \mathbf{Hom}_\mathcal{C}(X, Y) \quad f \mapsto \text{id}_Y \circ f
\]

to the identity functor. Similarly, the construction \( f \mapsto \rho_f \) determines a natural isomorphism from the functor

\[
\mathbf{Hom}_\mathcal{C}(X, Y) \to \mathbf{Hom}_\mathcal{C}(X, Y) \quad f \mapsto f \circ \text{id}_X
\]

to the identity functor.
We have the following generalization of Proposition 2.1.2.19:

**Proposition 2.2.1.14 (The Triangle Identity).** Let \( \mathcal{C} \) be a 2-category containing a pair of 1-morphisms \( f : X \to Y \) and \( g : Y \to Z \). Then the diagram of 2-morphisms

\[
\begin{array}{c}
\alpha_{g, \text{id}_Y \circ f} \\
\sim \\
\sim \\
\id_g \circ \lambda_f \\
\rho_g \circ \text{id}_f
\end{array}
\]

\[
(g \circ \text{id}_Y \circ f) \sim (g \circ \text{id}_Y) \circ f \sim g \circ f
\]

is commutative.

**Proof.** We have a diagram of isomorphisms

\[
\begin{array}{c}
\alpha \\
\sim \\
\sim \\
\id \\
\rho_g
\end{array}
\]

\[
(g \circ (\text{id}_Y \circ f)) \sim (g \circ (\text{id}_Y \circ \text{id}_Y)) \circ f
\]

Here the outer cycle commutes by the pentagon identity \( (P) \) of Definition 2.2.1.1, the upper rectangle by the functoriality of the associativity constraint, the upper side triangles by the definition of the left and right unit constraints, the quadrilaterals on the lower sides by the functoriality of the associativity constraints, and the lower region by the functoriality of composition. It follows that the middle square is also commutative, which is equivalent to the statement of Proposition 2.2.1.14.

It follows from Proposition 2.2.1.14 that we can recover the unit constraints \( \{ \upsilon_X \}_{X \in \text{Ob}(\mathcal{C})} \) of a 2-category \( \mathcal{C} \) from the left and right unit constraints defined in Construction 2.2.1.11.

**Corollary 2.2.1.15.** Let \( \mathcal{C} \) be a 2-category and let \( X \) be an object of \( \mathcal{C} \). Then the left and right unit constraints

\[
\lambda_{\text{id}_X} : \text{id}_X \circ \text{id}_X \xrightarrow{\sim} \text{id}_X \quad \rho_{\text{id}_X} : \text{id}_X \circ \text{id}_X \xrightarrow{\sim} \text{id}_X
\]
are both equal to the unit constraint $\psi_X : \text{id}_X \circ \text{id}_X \Rightarrow \text{id}_X$.

**Proof.** For any 1-morphism $f : Y \to X$ in $C$, the left unit constraint $\lambda_f$ is characterized by the commutativity of the diagram

$$
\begin{array}{ccc}
\text{id}_X \circ (\text{id}_X \circ f) & \xrightarrow{\alpha_{\text{id}_X \circ \text{id}_X, f}} & (\text{id}_X \circ \text{id}_X) \circ f \\
\sim & & \sim \\
\text{id}_{\text{id}_X \circ \text{id}_X} \circ \lambda_f & & \psi_X \circ \text{id}_f \\
& \downarrow & \\
\text{id}_X \circ f. & & \\
\end{array}
$$

Using Proposition 2.2.1.14 we deduce that $\psi_X \circ \text{id}_f = \rho_{\text{id}_X} \circ \text{id}_f$ as 2-morphisms from $(\text{id}_X \circ \text{id}_X) \circ f$ to $\text{id}_X \circ f$. In other words, the 2-morphisms $\psi_X, \rho_{\text{id}_X} : \text{id}_X \circ \text{id}_X \Rightarrow \text{id}_X$ have the same image under the functor

$$
\text{Hom}_C(X, X) \to \text{Hom}_C(Y, X) \quad g \mapsto g \circ f.
$$

In the special case where $Y = X$ and $f = \text{id}_X$, this functor is fully faithful. It follows that $\psi_X = \rho_{\text{id}_X}$. The equality $\psi_X = \lambda_{\text{id}_X}$ follows by a similar argument. \hfill \Box

We will also need some variants of Proposition 2.2.1.14 (generalizing Exercise 2.1.2.20):

**Proposition 2.2.1.16.** Let $C$ be a 2-category containing a pair of composable 1-morphisms $f : X \to Y$ and $g : Y \to Z$. Then:

1. The associativity constraint $\alpha_{\text{id}_Z, g, f} : \text{id}_Z \circ (g \circ f) \Rightarrow (\text{id}_Z \circ g) \circ f$ is given by the (vertical) composition

$$
\text{id}_Z \circ (g \circ f) \xrightarrow{\lambda_{g \circ f}} g \circ f \xrightarrow{\lambda_{g \circ f}^{-1} \circ \text{id}_f} (\text{id}_Z \circ g) \circ f.
$$

2. The associativity constraint $\alpha_{g, f, \text{id}_X} : g \circ (f \circ \text{id}_X) \Rightarrow (g \circ f) \circ \text{id}_X$ is given by the (vertical) composition

$$
g \circ (f \circ \text{id}_X) \xrightarrow{\text{id}_g \circ \rho_f} g \circ f \xrightarrow{\rho_{g \circ f}^{-1}} (g \circ f) \circ \text{id}_X.
$$

**Proof of Proposition 2.2.1.16.** We will prove (2); the proof of (1) is similar. Set $e = \text{id}_X$, and consider the diagram of isomorphisms

$$
\begin{array}{ccc}
g \circ ((f \circ e) \circ e) & \xrightarrow{\alpha_{g, f \circ e, e}} & (g \circ (f \circ e)) \circ e \\
\downarrow & & \downarrow \\
g \circ (f \circ (e \circ e)) & \xrightarrow{\lambda_e} & g \circ (f \circ e) \xrightarrow{\alpha_{g, f, e}} (g \circ f) \circ e \\
\downarrow & & \downarrow \\
g \circ (f \circ (e \circ e)) & \xrightarrow{\alpha_{g, f, e}} & (g \circ f) \circ (e \circ e) \\
\downarrow & & \downarrow \\
(g \circ f) \circ (e \circ e). & & (g \circ f) \circ (e \circ e).
\end{array}
$$
Here the outer cycle of the diagram commutes by the pentagon identity for $C$, the triangles on the upper left and lower right commute by virtue of Proposition 2.2.1.14, and the upper and lower square diagrams commute by the functoriality of the associativity constraints. It follows that the triangle on the upper right commutes: that is, the identity $\alpha_{g,f,\text{id}_X} = \rho_{g_0f}^{-1}(\text{id}_g \circ \rho_f)$ holds after applying the functor $(\bullet \circ \text{id}_X) : \text{Hom}_C(X,Z) \to \text{Hom}_C(X,Z)$. Since this functor is fully faithful (in fact, it is isomorphic to the identity functor by means of the right unit constraint $\rho$), we conclude that the identity $\alpha_{g,f,\text{id}_X} = \rho_{g_0f}^{-1}(\text{id}_g \circ \rho_f)$ holds in $\text{Hom}_C(X,Z)$ itself.

2.2.2 Examples of 2-Categories

We now collect some examples of 2-categories which arise naturally.

Example 2.2.2.1 (Correspondences). Let $C$ be a category containing a pair of objects $X$ and $Y$. A correspondence from $X$ to $Y$ is an object $M \in C$ together with a pair of morphisms $X \xymatrix{ \leftarrow & M \ar[r]^-g & Y}$ in $C$. The correspondences from $X$ to $Y$ can be regarded as the objects of a category $\mathcal{M}_{X,Y}$, where a morphism from $(M,f,g)$ to $(M',f',g')$ in $\mathcal{M}_{X,Y}$ is given by a morphism $u : M \to M'$ for which the diagram

\[
\begin{array}{ccc}
X & \xymatrix{ M \ar[r]^g & Y } & \ar[l]_f M' \\
\downarrow^u & & \downarrow^{g'} \\
Y & \xymatrix{ M' \ar[r]^{g'} & Y } & \ar[l]_{f'} X
\end{array}
\]

commutes.

Assume now that the category $C$ admits fiber products. We can then construct a 2-category $\text{Corr}(C)$ as follows:

- The objects of $\text{Corr}(C)$ are the objects of $C$.
- For every pair of objects $X, Y \in C$, we take $\text{Hom}_{\text{Corr}(C)}(X,Y)$ to be the category $\mathcal{M}_{X,Y}$; in particular, 1-morphisms from $X$ to $Y$ in the category $\text{Corr}(C)$ can be identified with correspondences from $X$ to $Y$.
- For every triple of objects $X, Y, Z \in C$, the composition law

\[
\circ : \text{Hom}_{\text{Corr}(C)}(Y,Z) \times \text{Hom}_{\text{Corr}(C)}(X,Y) \to \text{Hom}_{\text{Corr}(C)}(X,Z)
\]

is given on objects by the construction $(N,M) \mapsto M \times_Y N$. 

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- For every object \( X \in \mathcal{C} \), the identity 1-morphism from \( X \) to itself in \( \mathcal{C} \) is given by the correspondence \( X \xleftarrow{\text{id}_X} X \xrightarrow{\text{id}_X} X \), and the unit constraint \( \nu_X : X \times_X X \to X \) is the map given by projection onto either factor.

- For every triple of composable 1-morphisms

\[
W \xrightarrow{M} X \xrightarrow{N} Y \xrightarrow{P} Z
\]

in \( \text{Corr}(\mathcal{C}) \), the associativity constraint \( \alpha_{P,N,M} : P \circ (N \circ M) \Rightarrow (P \circ N) \circ M \) is the canonical isomorphism of iterated fiber products \( (M \times_X N) \times_Y P \cong M \times_X (N \times_Y P) \).

We will refer to \( \text{Corr}(\mathcal{C}) \) as the 2-category of correspondences in \( \mathcal{C} \).

**Remark 2.2.2.2.** Let \( \mathcal{C} \) be a category which admits finite limits, and let \( 1 \) denote the final object of \( \mathcal{C} \). Then the endomorphism category \( \text{End}_{\text{Corr}(\mathcal{C})}(1) \) can be identified with the category \( \mathcal{C} \) itself, equipped with the Cartesian monoidal structure of Example 2.1.3.2.

**Example 2.2.2.3 (Bimodules).** We define a 2-category \( \text{Bimod} \) as follows:

- The objects of \( \text{Bimod} \) are associative rings.

- For every pair of associative rings \( A \) and \( B \), we take \( \text{Hom}_{\text{Bimod}}(B, A) \) to be the category whose objects are \( A \)-\( B \) bimodules: that is, abelian groups \( M = A M_B \) equipped with commuting actions of \( A \) on the left and \( B \) on the right.

- For every triple of associative rings \( A, B, \) and \( C \), we take the composition law

\[
\text{Hom}_{\text{Bimod}}(B, A) \times \text{Hom}_{\text{Bimod}}(C, B) \to \text{Hom}_{\text{Bimod}}(C, A)
\]

to be the relative tensor product functor

\[
(M, N) \mapsto M_B \otimes N
\]

- For every associative ring \( A \), we take the identity object of \( \text{Hom}_{\text{Bimod}}(A, A) \) to be the ring \( A \) (regarded as a bimodule over itself) and the unit constraint \( \nu_A : A \otimes_A A \xrightarrow{\sim} A \) is the map given by \( \nu_A(x \otimes y) = xy \).

- For every quadruple of associative rings \( A, B, C, \) and \( D \) equipped with bimodules \( M = A M_B, N = B N_C, \) and \( P = C P_D \), we define the associativity constraint

\[
\alpha_{M,N,P} : M \otimes_B (N \otimes_C P) \xrightarrow{\sim} (M \otimes_B N) \otimes_C P
\]

to be the isomorphism characterized by the identity \( \alpha_{M,N,P}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \).
Example 2.2.2.4 (Delooping a Monoidal Category). Let \( \mathcal{C} \) be a monoidal category. We define a 2-category \( B\mathcal{C} \) as follows:

- The 2-category \( B\mathcal{C} \) has a single object, which we will denote by \( X \).
- The category \( \text{Hom}_{B\mathcal{C}}(X, X) \) is the category \( \mathcal{C} \).
- The composition functor

  \[ \circ : \text{Hom}_{B\mathcal{C}}(X, X) \times \text{Hom}_{B\mathcal{C}}(X, X) \to \text{Hom}_{B\mathcal{C}}(X, X) \]

  is the tensor product functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \).
- The identity morphism \( \text{id}_X \in \text{Hom}_{B\mathcal{C}}(X, X) \) is the unit object \( 1 \in \mathcal{C} \).
- The associativity and unit constraints of \( B\mathcal{C} \) are the associativity and unit constraints for the monoidal structure on \( \mathcal{C} \).

We will refer to the 2-category \( B\mathcal{C} \) as the delooping of \( \mathcal{C} \). Note that \( B\mathcal{C} \) is strict as a 2-category if and only if the monoidal structure on \( \mathcal{C} \) is strict (in which case we recover the delooping construction of Example 2.2.0.8). The construction \( \mathcal{C} \mapsto B\mathcal{C} \) induces a bijection

\[ \{ \text{Monoidal Categories } \mathcal{C} \} \cong \{ \text{2-Categories } \mathcal{E} \text{ with } \text{Ob}(\mathcal{E}) = \{ X \} \} \]

which can be viewed as an equivalence of categories (see Remark 2.2.5.8).

Remark 2.2.2.5. Let \( M \) be a monoid, which we view as a (strict) monoidal category having only identity morphisms. Then the 2-category \( BM \) of Example 2.2.2.4 can be identified with the ordinary category \( BM \) appearing in Remark 2.1.0.6.

2.2.3 Opposite and Conjugate 2-Categories

Recall that every ordinary category \( \mathcal{C} \) has an opposite category \( \mathcal{C}^{\text{op}} \), in which the objects are the same but the order of composition is reversed. In the setting of 2-categories, this operation generalizes in two essentially different ways: we can independently reverse the order of either vertical or horizontal composition. To avoid confusion, we will use different terminology when discussing these two operations.

Construction 2.2.3.1 (The Opposite of a 2-Category). Let \( \mathcal{C} \) be a 2-category. We define a new 2-category \( \mathcal{C}^{\text{op}} \) as follows:

- The objects of \( \mathcal{C}^{\text{op}} \) are the objects of \( \mathcal{C} \). To avoid confusion, for each object \( X \in \mathcal{C} \) we will write \( X^{\text{op}} \) for the corresponding object of \( \mathcal{C}^{\text{op}} \).
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• For every pair of objects \(X, Y \in \mathcal{C}\), we have \(\text{Hom}_{\mathcal{C}}^{\text{op}}(X^{\text{op}}, Y^{\text{op}}) = \text{Hom}_{\mathcal{C}}(Y, X)\). In particular, every 1-morphism \(f : Y \to X\) in the 2-category \(\mathcal{C}\) can be regarded as a 1-morphism from \(X^{\text{op}}\) to \(Y^{\text{op}}\) in the 2-category \(\mathcal{C}^{\text{op}}\), which we will denote by \(f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}\). Similarly, if we are given a pair of 1-morphisms \(f, g : Y \to X\) in the 2-category \(\mathcal{C}\) having the same source and target, then every 2-morphism \(\gamma : f \Rightarrow g\) in \(\mathcal{C}\) determines a 2-morphism from \(f^{\text{op}}\) to \(g^{\text{op}}\) in \(\mathcal{C}^{\text{op}}\), which we will denote by \(\gamma^{\text{op}} : f^{\text{op}} \Rightarrow g^{\text{op}}\).

• For every triple of objects \(X, Y, Z \in \mathcal{C}\), the composition functor \(\circ : \text{Hom}_{\mathcal{C}}^{\text{op}}(Y^{\text{op}}, Z^{\text{op}}) \times \text{Hom}_{\mathcal{C}}^{\text{op}}(X^{\text{op}}, Y^{\text{op}}) \to \text{Hom}_{\mathcal{C}}^{\text{op}}(X^{\text{op}}, Z^{\text{op}})\) for the 2-category \(\mathcal{C}^{\text{op}}\) is given by the the composition functor \(\circ : \text{Hom}_{\mathcal{C}}(Y, X) \times \text{Hom}_{\mathcal{C}}(Z, Y) \to \text{Hom}_{\mathcal{C}}(Z, X)\).

• For every object \(X \in \mathcal{C}\), the identity 1-morphism \(\text{id}_{X}^{\text{op}} \in \text{Hom}_{\mathcal{C}}^{\text{op}}(X^{\text{op}}, X^{\text{op}})\) is given by \(\text{id}_{X}^{\text{op}}\), where \(\text{id}_{X} \in \text{Hom}_{\mathcal{C}}(X, X)\) is the identity 1-morphism associated to \(X\) in the 2-category \(\mathcal{C}\), and the unit constraint \(\upsilon_{X}^{\text{op}}\) is the isomorphism \(\upsilon_{X}^{\text{op}} : \text{id}_{X}^{\text{op}} \circ \text{id}_{X}^{\text{op}} \cong \text{id}_{X}^{\text{op}}\).

• For every triple of composable 1-morphisms
  
  \[ W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \]

  in the 2-category \(\mathcal{C}\), the associativity constraint
  
  \[ \alpha_{f^{\text{op}}, g^{\text{op}}, h^{\text{op}}} : f^{\text{op}} \circ (g^{\text{op}} \circ h^{\text{op}}) \cong (f^{\text{op}} \circ g^{\text{op}}) \circ h^{\text{op}} \]

  in the 2-category \(\mathcal{C}^{\text{op}}\) is given by the inverse \((\alpha_{h,g,f}^{\text{op}})^{-1}\) of the associativity constraint
  
  \[ \alpha_{h,g,f} : h \circ (g \circ f) \cong (h \circ g) \circ f \]

  in the 2-category \(\mathcal{C}\).

We will refer to \(\mathcal{C}^{\text{op}}\) as the opposite of the 2-category \(\mathcal{C}\).

**Example 2.2.3.2.** Let \(\mathcal{C}\) be a category which admits fiber products, and let \(\text{Corr}(\mathcal{C})\) be the 2-category of correspondences in \(\mathcal{C}\) (see Example 2.2.2.1). Then the opposite 2-category \(\text{Corr}(\mathcal{C})^{\text{op}}\) can be identified with \(\text{Corr}(\mathcal{C})\) itself (every correspondence from \(X\) to \(Y\) in \(\mathcal{C}\) can also be viewed as a correspondence from \(Y\) to \(X\)).

**Example 2.2.3.3.** Let \(\mathcal{C}\) be a monoidal category, and let \(B\mathcal{C}\) be the 2-category obtained by delooping \(\mathcal{C}\) (Example 2.2.2.4). Then the opposite 2-category \((B\mathcal{C})^{\text{op}}\) can be identified with \(B(\mathcal{C}^{\text{rev}})\), where \(\mathcal{C}^{\text{rev}}\) denotes the reverse of the monoidal category \(\mathcal{C}\) (Example 2.1.3.5).
Construction 2.2.3.4 (The Conjugate of a 2-Category). Let \( \mathcal{C} \) be a 2-category. We define a new 2-category \( \mathcal{C}^c \) as follows:

- The objects of \( \mathcal{C}^c \) are the objects of \( \mathcal{C} \). To avoid confusion, for each object \( X \in \mathcal{C} \) we will write \( X^c \) for the corresponding object of \( \mathcal{C}^c \).

- For every pair of objects \( X, Y \in \mathcal{C} \), we have \( \text{Hom}_{\mathcal{C}}(X^c, Y^c) = \text{Hom}_{\mathcal{C}}(X, Y)^{\text{op}} \). In particular, every 1-morphism \( f : X \to Y \) in the 2-category \( \mathcal{C} \) can be regarded as a 1-morphism from \( X^c \) to \( Y^c \) in the 2-category \( \mathcal{C}^c \), which we will denote by \( f^c : X^c \to Y^c \).

- Similarly, if we are given a pair of 1-morphisms \( f, g : X \to Y \) in the 2-category \( \mathcal{C} \) having the same source and target, then every 2-morphism \( \gamma : f \Rightarrow g \) in \( \mathcal{C} \) determines a 2-morphism from \( g^c \) to \( f^c \) in \( \mathcal{C}^c \), which we will denote by \( \gamma^c : g^c \Rightarrow f^c \).

- For every triple of objects \( X, Y, Z \in \mathcal{C} \), the composition functor \( \circ : \text{Hom}_{\mathcal{C}}(Y^c, Z^c) \times \text{Hom}_{\mathcal{C}}(X^c, Y^c) \to \text{Hom}_{\mathcal{C}}(X^c, Z^c) \) for the 2-category \( \mathcal{C}^c \) is induced by the composition functor \( \circ : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \to \text{Hom}(X, Z) \).

- For every object \( X \in \mathcal{C} \), the identity 1-morphism \( \text{id}_{X^c} \in \text{Hom}_{\mathcal{C}}(X^c, X^c) \) is given by \( \text{id}^c_X \), where \( \text{id}_X \in \text{Hom}(X, X) \) is the identity 1-morphism associated to \( X \) in the 2-category \( \mathcal{C} \), and the unit constraint \( \upsilon_X^c \) is the isomorphism \( (\upsilon_X^c)^{-1} : \text{id}_{X^c} \circ \text{id}_{X^c} \Rightarrow \text{id}_{X^c} \).

- For every triple of composable 1-morphisms \( g \circ f \) in the 2-category \( \mathcal{C} \), the associativity constraint \( \alpha_{h^c, g^c, f^c} : h^c \circ (g^c \circ f^c) \Rightarrow (h^c \circ g^c) \circ f^c \) in the 2-category \( \mathcal{C}^c \) is given by the inverse \( (\alpha_{h, g, f}^c)^{-1} \) of the associativity constraint \( \alpha_{h, g, f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f \) in the 2-category \( \mathcal{C} \).

We will refer to \( \mathcal{C}^c \) as the conjugate of the 2-category \( \mathcal{C} \).

Example 2.2.3.5. Let \( \mathcal{C} \) be a monoidal category, and let \( \mathcal{B} \mathcal{C} \) be the 2-category obtained by delooping \( \mathcal{C} \) (Example 2.2.2.4). Then the conjugate 2-category \( (\mathcal{B} \mathcal{C})^c \) can be identified with \( \mathcal{B}(\mathcal{C}^{\text{op}}) \), where we endow the opposite category \( \mathcal{C}^{\text{op}} \) with the monoidal structure of Example 2.1.3.4.
Remark 2.2.3.6. Constructions Constructions 2.2.3.1 and 2.2.3.4 are analogous but not identical. At the level of 2-morphisms, passage from a 2-category $\mathcal{C}$ to its opposite $\mathcal{C}^{\text{op}}$ reverses the order of horizontal composition, but preserves the order of vertical composition; passage from $\mathcal{C}$ to its conjugate $\mathcal{C}^{\text{c}}$ preserves the order of horizontal composition and reverses the order of vertical composition. Following the notation of Warning 2.2.1.9, we have

$$\delta^{\text{op}} \gamma^{\text{op}} = (\delta \gamma)^{\text{op}} \quad \gamma^{\text{op}} \circ \gamma'^{\text{op}} = (\gamma' \circ \gamma)^{\text{op}}$$

$$\gamma^{\text{c}} \delta^{\text{c}} = (\delta \gamma)^{\text{c}} \quad \gamma'^{\text{c}} \circ \gamma^{\text{c}} = (\gamma' \circ \gamma)^{\text{c}}.$$

Example 2.2.3.7. Let $\mathcal{C}$ be an ordinary category, which we regard as a 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the opposite 2-category $\mathcal{C}^{\text{op}}$ of Construction 2.2.3.1 coincides with the opposite of $\mathcal{C}$ as an ordinary category (which we can again regard as a 2-category having only identity morphisms). The conjugate 2-category $\mathcal{C}^{\text{c}}$ of Construction 2.2.3.4 can be identified with $\mathcal{C}$ itself.

2.2.4 Functors of 2-Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Roughly speaking, a functor $F : \mathcal{C} \to \mathcal{D}$ should be an operation which carries objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$ to objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$, which is suitably compatible with (horizontal and vertical) composition. Here it is useful to distinguish between different notions of functor, which are differentiated by the degree of compatibility which is assumed.

Definition 2.2.4.1 (Strict Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A strict functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- For every object $X \in \mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
- For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories

$$F_{X,Y} : \text{Hom}_\mathcal{C}(X,Y) \to \text{Hom}_\mathcal{D}(F(X), F(Y)).$$

We will generally abuse notation by writing $F(f)$ for the value of the functor $F_{X,Y}$ on an object $f$ of the category $\text{Hom}_\mathcal{C}(X,Y)$, and $F(\gamma)$ for the value of $F$ on a morphism $\gamma$ in the category $\text{Hom}_\mathcal{C}(X,Y)$.

This data is required to satisfy the following compatibility conditions:

1. For every object $X \in \mathcal{C}$, we have $\text{id}_{F(X)} = F(\text{id}_X)$. 
(2) For every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram of categories

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{C}}(X, Z) \\
\downarrow_{F_{Y,Z} \times F_{X,Y}} & & \downarrow_{F_{X,Z}} \\
\text{Hom}_{\mathcal{D}}(F(Y), F(Z)) \times \text{Hom}_{\mathcal{D}}(F(X), F(Y)) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{D}}(F(X), F(Z))
\end{array}
\]

is strictly commutative.

(3) For every object $X \in \mathcal{C}$, the functor $F_{X,X}$ carries the identity constraint $\nu_X : \text{id}_X \circ \text{id}_X \Rightarrow \text{id}_X$ to the unit constraint $\nu_{F(X)} : \text{id}_{F(X)} \circ \text{id}_{F(X)} \Rightarrow \text{id}_{F(X)}$.

(4) For every composable triple of 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{C}$, we have $F(\alpha_{h,g,f}) = \lambda_{F(f)}F(g)\circ F(f)$. In other words, $F$ carries the associativity constraints of $\mathcal{C}$ to the associativity constraints of $\mathcal{D}$.

Remark 2.2.4.2. In the situation of Definition 2.2.4.1, conditions (3) and (4) are automatically satisfied if the 2-categories $\mathcal{C}$ and $\mathcal{D}$ are strict.

Example 2.2.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be strict 2-categories, which we regard as Cat-enriched categories (Remark 2.2.0.2). Then strict functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.2.4.1) can be identified with Cat-enriched functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.1.7.10).

Exercise 2.2.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a strict functor. Show that, for each morphism $f : X \to Y$ in $\mathcal{C}$, the functor $F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ carries the left and right unit constraints $\lambda_f : \text{id}_Y \circ f \Rightarrow f$ and $\rho_f : f \circ \text{id}_X \Rightarrow f$ to $\lambda_{F(f)}$ and $\rho_{F(f)}$, respectively (see Construction 2.2.1.11).

Note that axiom (2) of Definition 2.2.4.1 implies in particular that for every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, we have an equality $F(g) \circ F(f) = F(g \circ f)$ between objects of the category $\text{Hom}_{\mathcal{D}}(F(X), F(Z))$. In practice, this requirement is often too strong: it is often better to allow a more liberal notion of functor, which is only required to preserve composition up to isomorphism.

Definition 2.2.4.5 (Lax Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A lax functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- For every object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$.
- For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories $F_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$. 
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We will generally abuse notation by writing $F(f)$ for the value of the functor $F_{X,Y}$ on an object $f$ of the category $\text{Hom}_C(X,Y)$, an $F(\gamma)$ for the value of $F$ on a morphism $\gamma$ in the category $\text{Hom}_C(X,Y)$.

- For every object $X \in C$, a 2-morphism $\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X)$ in the 2-category $D$, which we will refer to as the identity constraint.

- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $C$, a 2-morphism $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$, which we will refer to as the composition constraint. We require that, if the objects $X$, $Y$, and $Z$ are fixed, then the construction $(g,f) \mapsto \mu_{g,f}$ is functorial: that is, we can regard $\mu$ as a natural transformation of functors as indicated in the diagram

\[
\begin{array}{ccc}
\text{Hom}_C(Y,Z) \times \text{Hom}_C(X,Y) & \xrightarrow{\circ} & \text{Hom}_C(X,Z) \\
F_{Y,Z} \times F_{X,Y} & \Downarrow \mu & \\
\text{Hom}_D(F(Y),Y(Z)) \times \text{Hom}_D(F(X),F(Y)) & \xrightarrow{\circ} & \text{Hom}_D(F(X),F(Z))
\end{array}
\]

This data is required to be compatible with the unit and associativity constraints of $C$ and $D$ in the following sense:

(a) For every 1-morphism $f : X \rightarrow Y$ in $C$, the left unit constraint $\lambda_{F(f)}$ in $D$ is given by the vertical composition

\[
id_{F(Y)} \circ F(f) \xRightarrow{\epsilon_Y \circ \text{id}_{F(f)}} F(\text{id}_Y) \circ F(f) \xRightarrow{\mu_{\text{id}_Y,f}} F(\text{id}_Y \circ f) \xRightarrow{F(\lambda_f)} F(f).
\]

(b) For every 1-morphism $f : X \rightarrow Y$ in $C$, the right unit constraint $\rho_{F(f)}$ in $D$ is given by the vertical composition

\[
F(f) \circ \text{id}_{F(X)} \xRightarrow{\text{id}_{F(f)} \circ \text{id}_X} F(f) \circ F(\text{id}_X) \xRightarrow{\mu_{f,\text{id}_X}} F(f \circ \text{id}_X) \xRightarrow{F(\rho_f)} F(f).
\]

(c) For every triple of composable 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in the 2-category $C$, we
have a commutative diagram

\[
\begin{array}{ccc}
F(h) \circ (F(g) \circ F(f)) & \xrightarrow{\alpha_{F(h),F(g),F(f)}} & (F(h) \circ F(g)) \circ F(f) \\
\downarrow & & \downarrow \\
F(h) \circ F(g \circ f) & \xrightarrow{F(\alpha_{h,g,f})} & F((h \circ g) \circ f) \\
\downarrow & & \downarrow \\
F(h \circ (g \circ f)) & \xrightarrow{F(\mu_{h,g,f})} & F(h \circ g) \circ F(f) \\
\end{array}
\]

in the category \(\text{Hom}_\mathcal{D}(F(W), F(Z))\).

A functor from \(\mathcal{C}\) to \(\mathcal{D}\) is a lax functor \(F : \mathcal{C} \to \mathcal{D}\) with the property that the identity and composition constraints

\[
\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X) \quad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)
\]

are isomorphisms.

**Warning 2.2.4.6.** The terminology of Definition 2.2.4.5 is not standard. In [1], Bénabou uses the term *morphism* for what we call a lax functor of 2-categories, *homomorphism* for what we call a functor of 2-categories, and *strict homomorphism* for what we call a strict functor of 2-categories. Other authors refer to functors of 2-categories (in the sense of Definition 2.2.4.5) as *weak functors* or *pseudofunctors* (to avoid confusion with the notion of strict functor).

**Remark 2.2.4.7.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be 2-categories and let \(F : \mathcal{C} \to \mathcal{D}\) be a lax functor from \(\mathcal{C}\) to \(\mathcal{D}\). Then, for each object \(X \in \text{Ob}(\mathcal{C})\), we can regard \(F_{X,X} : \text{End}_\mathcal{C}(X) \to \text{End}_\mathcal{D}(F(X))\) as a lax monoidal functor from \(\text{End}_\mathcal{C}(X)\) (endowed with the monoidal structure of Remark 2.2.1.6) to \(\text{End}_\mathcal{D}(F(X))\): the tensor and unit constraints on \(F_{X,X}\) are given by the composition and identity constraints on \(F\), respectively. If \(F\) is a functor, then \(F_{X,X}\) is a monoidal functor.

**Remark 2.2.4.8.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be 2-categories and let \(F : \mathcal{C} \to \mathcal{D}\) be a lax functor from \(\mathcal{C}\) to \(\mathcal{D}\). Then the identity constraints \(\{\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X)\}_{X \in \text{Ob}(\mathcal{C})}\) are uniquely determined by the other data of Definition 2.2.4.5. This follows from Proposition 2.1.5.4, applied to the lax monoidal functor \(F_{X,X} : \text{End}_\mathcal{C}(X) \to \text{End}_\mathcal{D}(F(X))\) of Remark 2.2.4.7.

**Remark 2.2.4.9.** Let \(\mathcal{C}\) be a monoidal category, let \(B\mathcal{C}\) be the 2-category obtained by delooping \(\mathcal{C}\) (Example 2.2.2.4), and let \(X\) denote the unique object of \(B\mathcal{C}\). Let \(\mathcal{D}\) be any...
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2-category, and let $Y$ be an object of $D$. Then the construction of Remark 2.2.4.7 induces bijections

\[
\{\text{Lax Functors } F : B\mathcal{C} \to \mathcal{D} \text{ with } F(X) = Y\} \simeq \{\text{Lax monoidal functors } \mathcal{C} \to \text{End}_\mathcal{D}(Y)\}
\]

\[
\{\text{Functors } F : B\mathcal{C} \to \mathcal{D} \text{ with } F(X) = Y\} \simeq \{\text{Monoidal functors } \mathcal{C} \to \text{End}_\mathcal{D}(Y)\}.
\]

Applying this observation in the case where $D = B\mathcal{C}'$ for some other monoidal category $\mathcal{C}'$, we deduce that (lax) monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$ can be identified with (lax) functors of 2-categories from $B\mathcal{C}$ to $B\mathcal{C}'$.

Example 2.2.4.10. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a strict functor (in the sense of Definition 2.2.4.1). Then we can regard $F$ as a functor from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.2.4.5) by taking the identity and composition constraints

\[
\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X) \quad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)
\]

to be the identity maps (note that in this case, conditions (a), (b), and (c) of Definition 2.2.4.5 reduce to conditions (3) and (4) of Definition 2.2.4.1). Conversely, if $F : \mathcal{C} \to \mathcal{D}$ is a lax functor having the property that each of the identity and composition constraints $\epsilon_X$ and $\mu_{g,f}$ is an identity 2-morphism of $\mathcal{D}$, then we can regard $F$ as a strict 2-functor from $\mathcal{C}$ to $\mathcal{D}$. We therefore have inclusions

\[
\{\text{Strict functors } F : \mathcal{C} \to \mathcal{D}\} \subseteq \{\text{Functors } F : \mathcal{C} \to \mathcal{D}\} \subseteq \{\text{Lax functors } F : \mathcal{C} \to \mathcal{D}\}.
\]

In general, neither of these inclusions is reversible.

Example 2.2.4.11 (Enriched Categories as Lax Functors). Let $S$ be a set, and let $\mathcal{E}_S$ denote the indiscrete category with object set $S$: that is, the objects of $\mathcal{E}_S$ are the elements of $S$, and $\text{Hom}_{\mathcal{E}_S}(X,Y)$ is a singleton for every pair of elements $X,Y \in S$. Regard $\mathcal{E}_S$ as a (strict) 2-category having only identity 2-morphisms (Example 2.2.0.6). Let $\mathcal{C}$ be a monoidal category, and let $B\mathcal{C}$ be its delooping (Example 2.2.2.4). Unwinding the definitions, we see that lax functors $F : \mathcal{E}_S \to B\mathcal{C}$ (in the sense of Definition 2.2.4.5) can be identified with $\mathcal{C}$-enriched categories having object set $S$ (in the sense of Definition 2.1.7.1).

Warning 2.2.4.12. Let $\mathcal{C}$ and $\mathcal{D}$ be strict 2-categories, and let $\mathcal{C}_0$ and $\mathcal{D}_0$ denote their underlying ordinary categories (obtained by ignoring the 2-morphisms of $\mathcal{C}$ and $\mathcal{D}$, respectively). Every strict functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor of ordinary categories $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$. However, if a functor $F : \mathcal{C} \to \mathcal{D}$ is not strict, then it need not give rise to a functor from $\mathcal{C}_0$ to $\mathcal{D}_0$. If $X \overset{f}{\to} Y \overset{g}{\to} Z$ is a composable pair of 1-morphisms in $\mathcal{C}$, then Definition 2.2.4.5 guarantees that the 1-morphisms $F(g) \circ F(f)$ and $F(g \circ f)$ are isomorphic (via the composition constraint $\mu_{g,f}$), but not that they are identical.
Example 2.2.4.13. Let $\mathcal{C}$ be a 2-category and let $\mathcal{D}$ be an ordinary category. Then we regard $\mathcal{D}$ as a 2-category having only identity 2-morphisms (Example 2.2.0.6). It follows that every lax functor $F : \mathcal{C} \to \mathcal{D}$ is automatically a strict functor. Beware that the analogous statement is generally false if the roles of $\mathcal{C}$ and $\mathcal{D}$ are reversed.

Notation 2.2.4.14. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. To supply a lax 2-functor $F : \mathcal{C} \to \mathcal{D}$, one must specify not only the values of $F$ on objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$, but also the identity and composition constraints

$$\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X) \quad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f).$$

In situations where we need to consider more than one lax functor at a time, we will denote these 2-morphisms by $\epsilon^F_X$ and $\mu^F_{g,f}$ (to avoid ambiguity).

Exercise 2.2.4.15. In the situation of Definition 2.2.4.5, show that we can replace (a) and (b) by the following alternative conditions:

- For every object $X \in \mathcal{C}$, the diagram

$$
\begin{array}{ccc}
id_{F(X)} \circ F(\text{id}_X) & \Rightarrow & \text{id}_{F(X)} \\
\downarrow \epsilon_X \circ \epsilon_X & & \downarrow \epsilon_X \\
F(\text{id}_X) \circ F(\text{id}_X) & \Rightarrow & F(\text{id}_X) \\
\downarrow \mu_{\text{id}_X, \text{id}_X} & & \downarrow \mu_{\text{id}_X, \text{id}_X} \\
F(\text{id}_X \circ \text{id}_X) & \Rightarrow & F(\text{id}_X)
\end{array}
$$

commutes (in the endomorphism category $\text{End}_\mathcal{C}(X)$).

- For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, the vertical compositions

$$
\begin{array}{ccc}
id_{F(Y)} \circ F(f) & \Rightarrow & F(\text{id}_Y) \circ F(f) \\
\downarrow \epsilon_Y \circ \epsilon_{F(f)} & & \downarrow \epsilon_Y \\
F(\text{id}_Y) \circ F(f) & \Rightarrow & F(\text{id}_Y \circ f) \\
\downarrow \mu_{\text{id}_Y, f} & & \downarrow \mu_{\text{id}_Y, f} \\
F(f) \circ \text{id}_{F(X)} & \Rightarrow & F(f) \circ F(\text{id}_X)
\end{array}
$$

are monomorphisms in the category $\text{Hom}_\mathcal{D}(F(X), F(Y))$.

See Proposition 2.1.5.13.

For our purposes, it will be useful to consider an intermediate notion of strictness.

Definition 2.2.4.16. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor. We say that $F$ is unitary if, for every object $X \in \mathcal{C}$, the identity constraint $\epsilon_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X)$ is an invertible 2-morphism of $\mathcal{D}$. We say that $F$ is strictly unitary if, for every object $X \in \mathcal{C}$, we have an equality $\text{id}_{F(X)} = F(\text{id}_X)$ and the identity constraint $\epsilon_X$ is the identity 2-morphism from $\text{id}_{F(X)}$ to itself.
Remark 2.2.4.17. Let $C$ and $D$ be 2-categories. Every functor $F : C \to D$ is unitary when viewed as a lax functor from $C$ to $D$. Every strict functor $F : C \to D$ is strictly unitary when viewed as a lax functor from $C$ to $D$.

Remark 2.2.4.18. Let $C$ and $D$ be 2-categories and let $F : C \to D$ be a unitary lax functor. Then one can modify $F$ to produce a strictly unitary lax functor $F' : C \to D$ by the following explicit procedure:

- For every object $X \in C$, we set $F'(X) = F(X)$.
- For every 1-morphism $f : X \to Y$ in $C$ which is not an identity morphism, we set $F'(f) = F(f)$; if $X = Y$ and $f = \text{id}_X$ we instead set $F'(f) = \text{id}_{F(X)}$. In either case, we have an invertible 2-morphism $\varphi_f : F'(f) \Rightarrow F(f)$, given by
  $$\varphi_f = \begin{cases} \epsilon^F_X & \text{if } f = \text{id}_X, \\ \text{id}_{F(f)} & \text{otherwise.} \end{cases}$$
- Let $X$ and $Y$ be objects of $C$, and let $\gamma : f \Rightarrow g$ be a 2-morphism between 1-morphisms $f, g : X \to Y$. We define $F'((\gamma))$ to be the vertical composition $\varphi^{-1}_g F(\gamma) \varphi_f$.
- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $D$, we define the composition constraint $\mu^F_{g,f} : F'(g) \circ F'(f) \Rightarrow F'(g \circ f)$ to be the vertical composition
  $$F'(g) \circ F'(f) \xrightarrow{\varphi_g \circ \varphi_f} F(g) \circ F(f) \xrightarrow{\mu^F_{f,g}} F(g \circ f) \xrightarrow{\varphi^{-1}_{g \circ f}} F'(g \circ f).$$

Consequently, it is generally harmless to assume that a unitary lax functor of 2-categories $F : C \to D$ is strictly unitary.

2.2.5 The Category of 2-Categories

We now show that 2-categories can be regarded as the objects of a category $\mathbf{2Cat}$, in which the morphisms are functors between 2-categories (Definition 2.2.5.5). There are several variants of this construction, depending on what sort of functors we allow.

Construction 2.2.5.1 (Composition of Lax Functors). Let $C$, $D$, and $E$ be 2-categories, and suppose we are given a pair of lax functors $F : C \to D$ and $G : D \to E$. We define a lax functor $GF : C \to E$ as follows:

- On objects, the lax functor $GF$ is given by $(GF)(X) = G(F(X))$. 
• For every pair of objects $X, Y \in \mathcal{C}$, the functor

$$(GF)_{X,Y} : \text{Hom}_\mathcal{C}(X,Y) \to \text{Hom}_\mathcal{E}((GF)(X), (GF)(Y))$$

is given by the composition of functors

$$\text{Hom}_\mathcal{C}(X,Y) \xrightarrow{F_{X,Y}} \text{Hom}_\mathcal{D}(F(X), F(Y)) \xrightarrow{G_{F(X),F(Y)}} \text{Hom}_\mathcal{E}((GF)(X), (GF)(Y)).$$

In other words, the lax functor $GF$ is given on 1-morphisms and 2-morphisms by the formulae

$$(GF)(f) = G(F(f)) \quad (GF)(\gamma) = G(F(\gamma)).$$

• For each object $X \in \mathcal{C}$, the identity constraint $\epsilon_{X}^{GF} : \text{id}_{(GF)(X)} \Rightarrow (GF)(\text{id}_{X})$ is given by the composition

$$\text{id}_{(GF)(X)} \xrightarrow{\epsilon_{F(X)}^{G}} G(\text{id}_{F(X)}) \xrightarrow{G(\epsilon_{X}^{F})} (GF)(\text{id}_{X}).$$

• For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, the composition constraint $\mu_{g,f}^{GF} : (GF)(g) \circ (GF)(f) \Rightarrow (GF)(g \circ f)$ is given by the composition

$$(GF)(g) \circ (GF)(f) \xrightarrow{\mu_{F(g),F(f)}^{G}} G(F(g) \circ F(f)) \xrightarrow{G(\mu_{g,f}^{F})} (GF)(g \circ f).$$

We will refer to $GF$ as the composition of $F$ with $G$, and will sometimes denote it by $G \circ F$.

**Exercise 2.2.5.2.** Check that the composition of lax functors is well-defined. That is, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are lax functors between 2-categories, then the identity and composition constraints $\epsilon_{X}^{GF}$ and $\mu_{g,f}^{GF}$ of Construction 2.2.5.1 are compatible with the unit constraints and associativity constraints of $\mathcal{C}$ and $\mathcal{E}$, as required by Definition 2.2.4.5.

**Remark 2.2.5.3.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be lax functors of 2-categories, and let $GF : \mathcal{C} \to \mathcal{E}$ be their composition. Then:

• If $F$ and $G$ are unitary, then the composition $GF$ is unitary.

• If $F$ and $G$ are functors, then the composition $GF$ is a functor.

• If $F$ and $G$ are strictly unitary, then the composition $GF$ is strictly unitary.

• If $F$ and $G$ are strict functors, then the composition $GF$ is a strict functor.
Example 2.2.5.4. Let $\mathcal{C}$ be a 2-category. We let $\text{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ be the strict functor which carries every object, 1-morphism, and 2-morphism of $\mathcal{C}$ to itself. We will refer to $\text{id}_\mathcal{C}$ as the identity functor on $\mathcal{C}$. Note that it is both a left and right unit for the composition of lax functors given in Construction 2.2.5.1.

Definition 2.2.5.5. We let $\text{2Cat}_{\text{Lax}}$ denote the ordinary category whose objects are (small) 2-categories and whose morphisms are lax functors between 2-categories (Definition 2.2.4.5), with composition given by Construction 2.2.5.1 and identity morphisms given by Example 2.2.5.4.

We define (non-full) subcategories

$$2\text{Cat}_{\text{Str}} \subseteq 2\text{Cat} \subseteq 2\text{Cat}_{\text{Lax}} \supseteq 2\text{Cat}_{\text{ULax}}$$

- The objects of $2\text{Cat}$ are 2-categories, and the morphisms of $2\text{Cat}$ are functors.
- The objects of $2\text{Cat}_{\text{Str}}$ are strict 2-categories, and the morphisms of $2\text{Cat}_{\text{Str}}$ are strict functors.
- The objects of $2\text{Cat}_{\text{ULax}}$ are 2-categories, and the morphisms of $2\text{Cat}_{\text{ULax}}$ are strictly unitary lax functors.

We will refer to $2\text{Cat}$ as the category of 2-categories, and to $2\text{Cat}_{\text{Str}}$ as the category of strict 2-categories.

Remark 2.2.5.6. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then the collection $\text{Hom}_{2\text{Cat}}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ can be identified with the set of objects of a certain 2-category $\text{Fun}(\mathcal{C}, \mathcal{D})$, called the 2-category of functors from $\mathcal{C}$ to $\mathcal{D}$. We will return to this point in more detail in §[?].

Example 2.2.5.7. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories, which we regard as 2-categories having only identity 2-morphisms (see Example 2.2.0.6). Then every lax functor of 2-categories from $\mathcal{C}$ to $\mathcal{D}$ is automatically strict (Example 2.2.4.13), and can be identified with a functor from $\mathcal{C}$ to $\mathcal{D}$ in the usual sense. In other words, we can view Example 2.2.0.6 as supplying fully faithful embeddings (of ordinary categories)

$$\text{Cat} \hookrightarrow 2\text{Cat}_{\text{Str}} \quad \text{Cat} \hookrightarrow 2\text{Cat} \quad \text{Cat} \hookrightarrow 2\text{Cat}_{\text{Lax}} \quad \text{Cat} \hookrightarrow 2\text{Cat}_{\text{ULax}}.$$ 

Remark 2.2.5.8. Let $\text{MonCat}$ denote the ordinary category whose objects are monoidal categories and whose morphisms are monoidal functors (that is, the underlying category of the strict 2-category $\text{MonCat}$ of Example 2.2.0.5). Then the construction $\mathcal{C} \mapsto B\mathcal{C}$ determines a fully faithful embedding from $\text{MonCat}$ to the category $2\text{Cat}$ of Definition 2.2.5.5 which fits into a pullback diagram

$$\text{MonCat} \ar[r] \ar[d] & 2\text{Cat} \ar[d]_{\text{Ob}(\mathcal{C})} \ar[r] & \text{Set};$$

$$\{*\} \ar[r] \ar[d] & \text{Ob}(\mathcal{C}) \ar[d] \ar[r] & \text{Set};$$
here $* = \{X\}$ denotes a set containing a single fixed object $X$. Similarly, the ordinary category of monoidal categories and lax monoidal functors can be regarded as a full subcategory of $2\text{Cat}_{\text{Lax}}$.

**Remark 2.2.5.9** (Functors on Opposite 2-Categories). Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $\mathcal{C}^{\text{op}}$ and $\mathcal{D}^{\text{op}}$ denote their opposites (Construction [2.2.3.1]). Then every lax functor $F : \mathcal{C} \to \mathcal{D}$ induces a lax functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$, given explicitly by the formulae

$$F^{\text{op}}(X^{\text{op}}) = F(X)^{\text{op}} \quad F^{\text{op}}(f^{\text{op}}) = F(f)^{\text{op}} \quad F^{\text{op}}(\gamma^{\text{op}}) = F(\gamma)^{\text{op}}$$

$$\epsilon_{X^{\text{op}}} = (\epsilon_X)^{\text{op}} \quad \mu_{g^{\text{op}},f^{\text{op}}} = (\mu_{g,f})^{\text{op}}.$$ 

In this case, $F$ is a functor if and only if $F^{\text{op}}$ is a functor, and a strict functor if and only if $F^{\text{op}}$ is a strict functor. This operation is compatible with composition, and therefore induces equivalences of categories

$$2\text{Cat}_{\text{Str}} \simeq 2\text{Cat}_{\text{Str}} \quad 2\text{Cat} \simeq 2\text{Cat} \quad 2\text{Cat}_{\text{Lax}} \simeq 2\text{Cat}_{\text{Lax}} \quad 2\text{Cat}_{\text{ULax}} \simeq 2\text{Cat}_{\text{ULax}}.$$ 

**Remark 2.2.5.10** (Functors on Conjugate 2-Categories). Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $\mathcal{C}^c$ and $\mathcal{D}^c$ denote their conjugates (Construction [2.2.3.4]). Then every functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F^c : \mathcal{C}^c \to \mathcal{D}^c$, given explicitly by the formulae

$$F^c(X^c) = F(X)^c \quad F^c(f^c) = F(f)^c \quad F^c(\gamma^c) = F(\gamma)^c$$

$$\epsilon_{X^c} = (\epsilon_X^{-1})^c \quad \mu_{g^c,f^c} = (\mu_{g,f}^{-1})^c.$$ 

In this case, the functor $F$ is strict if and only if $F^c$ is strict. This operation is compatible with composition, and therefore induces equivalences of categories

$$2\text{Cat}_{\text{Str}} \simeq 2\text{Cat}_{\text{Str}} \quad 2\text{Cat} \simeq 2\text{Cat}$$

**Warning 2.2.5.11.** The construction of Remark [2.2.5.10] requires that the identity and composition constraints of $F$ are invertible, and therefore does not extend to lax functors between 2-categories. In general, one cannot identify lax functors from $\mathcal{C}$ to $\mathcal{D}$ with lax functors from $\mathcal{C}^c$ to $\mathcal{D}^c$: the definition of lax functor is asymmetrical with respect to vertical composition.

Let $F : \mathcal{C} \to \mathcal{D}$ be a (lax) functor between 2-categories. According to Example [2.2.4.10] $F$ is strict if and only if the identity and composition constraints

$$\epsilon_X : \text{id}_F(X) \Rightarrow F(\text{id}_X) \quad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

are identity 2-morphisms in $\mathcal{D}$. In [2.3.1] it will be useful to consider a weaker version of this condition, where we require strict compatibility with the formation of identity morphisms but not with respect to composition in general.
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2.2.6 Isomorphisms of 2-Categories

We now study isomorphisms in the setting of 2-categories.

**Definition 2.2.6.1.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be 2-categories. We will say that a functor \( F : \mathcal{C} \to \mathcal{D} \) is an *isomorphism* if it is an isomorphism in the category \( 2\text{Cat} \) of Definition 2.2.5.5. That is, \( F \) is an isomorphism if there exists a functor \( G : \mathcal{D} \to \mathcal{C} \) such that \( GF = \text{id}_\mathcal{C} \) and \( FG = \text{id}_\mathcal{C} \). We say that 2-categories \( \mathcal{C} \) and \( \mathcal{D} \) are *isomorphic* if there exists an isomorphism from \( \mathcal{C} \) to \( \mathcal{D} \).

**Remark 2.2.6.2.** Let \( F : \mathcal{C} \to \mathcal{D} \) be an isomorphism of 2-categories, and let \( G : \mathcal{D} \to \mathcal{C} \) be the inverse isomorphism. Then:

- The functor \( F \) is strictly unitary if and only if \( G \) is strictly unitary. In this case, we say that \( F \) is a *strictly unitary isomorphism*.
- The functor \( F \) is strict if and only if \( G \) is strict. In this case, we say that \( F \) is a *strict isomorphism*.

We say that 2-categories \( \mathcal{C} \) and \( \mathcal{D} \) are *strictly isomorphic* if there is a strict isomorphism from \( \mathcal{C} \) to \( \mathcal{D} \).

**Warning 2.2.6.3.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be 2-categories which are strictly isomorphic. Then \( \mathcal{C} \) is strict if and only if \( \mathcal{D} \) is strict. If we assume only that \( \mathcal{C} \) and \( \mathcal{D} \) are isomorphic (rather than strictly isomorphic), then we cannot draw the same conclusion. In other words, the condition that a 2-category \( \mathcal{C} \) is strict is invariant under *strict* isomorphism, but not under isomorphism.

**Warning 2.2.6.4.** The notions of isomorphism and strict isomorphism of 2-categories are somewhat artificial. As in classical category theory, there is notion of equivalence of 2-categories (Definition [?]) which is more general than isomorphism and more appropriate for describing what it means for 2-categories to be “the same.”

**Remark 2.2.6.5.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor of 2-categories. Then \( F \) is an isomorphism (in the sense of Definition 2.2.6.1) if and only if it satisfies the following conditions:

- The functor \( F \) induces a bijection from the set of objects of \( \mathcal{C} \) to the set of objects of \( \mathcal{D} \).
- For every pair of objects \( X, Y \in \mathcal{C} \), the functor \( F \) induces an isomorphism of categories \( \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{D}(F(X), F(Y)) \).

One might be tempted to consider a more liberal version of Definition 2.2.6.1 working with lax functors rather than functors. However, the resulting notion of isomorphism turns out to be the same.
Proposition 2.2.6.6. Let $C$ and $D$ be 2-categories, and let $F : C \to D$ be a lax functor which is an isomorphism in the category $2\text{Cat}_{\text{Lax}}$. Then $F$ is a functor.

Proof. We will show that, for every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $C$, the composition constraint $\mu_{g,f}^F : F(g) \circ F(f) \Rightarrow F(g \circ f)$ is an isomorphism (in the ordinary category $\text{Hom}_D(F(X), F(Z))$); the analogous statement for the identity constraints $\epsilon^F_X : \text{id}_{F(X)} \Rightarrow F(\text{id}_X)$ follows by a similar (but easier) argument.

Let $G : D \to C$ be a lax functor which is an inverse of $F$ in the category $2\text{Cat}_{\text{Lax}}$. For any pair of composable 1-morphisms $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ in the 2-category $D$, the composition constraint $\mu_{g',f'}^{G\circ F}$ for the lax functor $F \circ G$ is given by the vertical composition

$$(F \circ G)(g') \circ (F \circ G)(f') \xrightarrow{\mu_{g',f'}^{G\circ F}} F(G(g') \circ G(f')) \xrightarrow{F(\mu_{g',f'}^G)} (F \circ G)(g' \circ f').$$

Since $F \circ G$ coincides with $\text{id}_D$ as a lax functor, this composition is the identity 2-morphism from $g' \circ f'$ to itself. In particular, we see that $F(\mu_{g',f'}^G)$ has a right inverse in the category $\text{Hom}_D(X', Z')$. It follows that $\mu_{g',f'}^{G\circ F} = G(F(\mu_{g',f'}^G))$ has a right inverse in the category $\text{Hom}_C(G(X'), G(Z'))$.

Applying the same argument with the roles of $F$ and $G$ reversed, we see that the composition constraint $\mu_{g,f}^{G\circ F} = \text{id}_{g\circ f}$ factors as a vertical composition

$$(G \circ F)(g) \circ (G \circ F)(f) \xrightarrow{\mu_{g,f}^{F\circ G}} G(F(g) \circ F(f)) \xrightarrow{G(\mu_{g,f}^F)} (G \circ F)(g \circ f).$$

In particular, this shows that $\mu_{F(g),F(f)}^G$ has a left inverse (in the category $\text{Hom}_C(X, Z)$). Applying the preceding argument in the case $g' = F(g)$ and $f' = F(f)$, we see that $\mu_{F(g),F(f)}^G$ also has a right inverse. It follows that $\mu_{F(g),F(f)}^G$ is an isomorphism in the category $\text{Hom}_C(X, Z)$. Since $G(\mu_{g,f}^F)$ is a left inverse of $\mu_{F(g),F(f)}^G$, it must also be an isomorphism. It follows that $F(G(\mu_{g,f}^F)) = \mu_{g,f}^F$ is an isomorphism in the category $\text{Hom}_D(F(X), F(Z))$, as desired. \qed

We now construct some examples of non-strict isomorphisms of 2-categories.

Notation 2.2.6.7. Let $C$ be a 2-category. A twisting cochain for $C$ is a datum which assigns, to every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a 1-morphism $(g \circ f) : X \to Z$ and an invertible 2-morphism $\mu_{g,f} : g \circ f \simeq g \circ f$. In this case, we will (slightly) abuse notation by identifying the twisting cochain with the collection of 2-morphisms $\{\mu_{g,f}\}$.

Construction 2.2.6.8. Let $C$ be a 2-category equipped with a twisting cochain

$$\{\mu_{g,f}\} = \{\mu_{g,f} : (g \circ f) \Rightarrow (g \circ f)\}.$$

We define a new 2-category $C'$ as follows:
• The objects of $C'$ are the objects of $C$.

• For every pair of objects $X, Y \in C$, we define $\text{Hom}_c'(X, Y)$ to be the category $\text{Hom}_c(X, Y)$. In particular, we can identify 1-morphisms of $C'$ with 1-morphisms of $C$, 2-morphisms of $C'$ with 2-morphisms of $C$, and the vertical composition of 2-morphisms in $C'$ with the vertical composition of 2-morphisms in $C$.

• For every object $X \in C$, the identity 1-morphism from $X$ to itself in the 2-category $C'$ is the same as the identity morphism from $X$ to itself in the 2-category $C$.

• For every triple of objects $X, Y, Z \in C$, the composition functor $\text{Hom}_c'(Y, Z) \times \text{Hom}_c'(X, Y) \to \text{Hom}_c'(X, Z)$ is given on objects by $(g, f) \mapsto g \circ' f$ and on morphisms by the construction

$$((\delta : g \Rightarrow g', \gamma : f \Rightarrow f') \mapsto \mu_{g', f'}(\delta \circ \gamma)\mu_{g, f}^{-1}.$$ 

• For every object $X \in C$, the unit constraint $\nu'_X : \text{id}_X \circ' \text{id}_X \Rightarrow \text{id}_X$ for the 2-category $C'$ is given by the composition

$$\text{id}_X \circ' \text{id}_X \underbrace{\mu_{\text{id}_X, \text{id}_X}^{-1}}_{\text{id}_X} \text{id}_X \circ \text{id}_X \underbrace{\nu_X}_{\text{id}_X} \Rightarrow \text{id}_X.$$ 

• For every triple of composable 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ of $C$, the associativity constraint of $C'$ is given by the composition

$$h \circ' (g \circ f) \underbrace{\mu_{h, g \circ f}^{-1}}_{\text{id}_h \circ \mu_{g, f}^{-1}} \Rightarrow h \circ (g \circ f) \underbrace{\alpha_{h, g \circ f}}_{\text{id}_h \circ \mu_{g, f}^{-1}} \Rightarrow h \circ (g \circ f) \underbrace{\mu_{h, g \circ f}^{-1}}_{\text{id}_h \circ \mu_{g, f}^{-1}} \Rightarrow (h \circ g) \circ f \underbrace{\alpha_{h, g \circ f}}_{\text{id}_h \circ \mu_{g, f}^{-1}} \Rightarrow (h \circ g) \circ f \underbrace{\mu_{h, g \circ f}^{-1}}_{\text{id}_h \circ \mu_{g, f}^{-1}} \Rightarrow (h \circ g) \circ' f.$$ 

We will refer to $C'$ as the twist of $C$ with respect to $\{\mu_{g, f}\}$.

**Exercise 2.2.6.9.** Let $C$ be a 2-category equipped with a twisting cochain $\{\mu_{g, f}\}$. Show that the 2-category $C'$ of Construction 2.2.6.8 is well-defined. Moreover, there is a strictly unitary isomorphism of 2-categories $C \to C'$ which carries each object, 1-morphism, and 2-morphism of $C$ to itself, where the composition constraints are given by $\{\mu_{g, f}\}$.
Exercise 2.2.6.10. Let \( F : C \to D \) be a strictly unitary isomorphism of 2-categories. Show that there is a unique twisting cochain \( \{ \mu_{g,f} \} \) on the 2-category \( C \) such that \( F \) factors as a composition \( C \xrightarrow{G} C' \xrightarrow{H} D \), where \( G \) is the strictly unitary isomorphism of Exercise 2.2.6.9 and \( H \) is a strict isomorphism of 2-categories. In other words, the notion of twisting cochain (in the sense of Notation 2.2.6.7) measures the difference between strictly unitary isomorphisms and strict isomorphisms in the setting of 2-categories.

Remark 2.2.6.11. It is possible to consider a generalization of the twisting procedure of Construction 2.2.6.8 in which one modifies not only the composition law for 1-morphisms of \( C \), but also the choice of identity 1-morphisms of \( C \). Since we will not need this generalization, we leave the details to the reader.

Example 2.2.6.12. Let \( G \) be a group with identity element \( 1 \in G \), let \( \Gamma \) be an abelian group on which \( G \) acts by automorphisms, let \( \alpha : G \times G \times G \to \Gamma \) be a 3-cocycle, let \( C \) be the monoidal category of Example 2.1.3.3, and let \( B \mathcal{C} \) be the 2-category obtained by delooping \( C \) (Example 2.2.2.4). A twisting cochain for the 2-category \( B \mathcal{C} \) (in the sense of Notation 2.2.6.7) can be identified with a map of sets

\[
\mu : G \times G \to \Gamma \quad (g, f) \mapsto \mu_{g,f}.
\]

Let \( (B \mathcal{C})' \) be the 2-category denote the twist of \( B \mathcal{C} \) with respect to \( \mu \). Unwinding the definitions, we see that \( (B \mathcal{C})' \) is obtained by delooping the same category \( \mathcal{C} \) with respect to a different monoidal structure: namely, the monoidal structure supplied by the 3-cocycle \( \alpha' : G \times G \times G \to \Gamma \) given by the formula

\[
\alpha'_{h,g,f} = \alpha_{h,g,f} - h(\mu_{g,f}) + \mu_{h,g,f} - \mu_{h,gf} + \mu_{h,g}.
\]

We can summarize the situation as follows:

- To every 3-cocycle \( \alpha : G \times G \times G \to \Gamma \), we can associate a 2-category \( B \mathcal{C} \) in which the 1-morphisms are the elements of \( G \), the 2-morphisms are the elements of \( \Gamma \), and the associativity constraint is given by \( \alpha \).

- If \( \alpha, \alpha' : G \times G \times G \to \Gamma \) are cohomologous 3-cocycles on \( G \) with values in \( \Gamma \), then the associated 2-categories \( \mathcal{C} \) and \( \mathcal{C}' \) are isomorphic (though not necessarily strictly isomorphic). More precisely, every choice of 2-cocycle \( \mu : G \times G \to \Gamma \) satisfying \( \alpha = \alpha' + \partial(\mu) \) determines a strictly unitary isomorphism from \( \mathcal{C} \) to \( \mathcal{C}' \). Here \( \partial \) denotes the boundary operator from 2-cochains to 3-cocycles, given concretely by the formula

\[
(\partial \mu)_{h,g,f} = h(\mu_{g,f}) - \mu_{h,g,f} + \mu_{h,gf} - \mu_{h,g}.
\]

Example 2.2.6.13. The 2-categories Bimod and Corr(\( \mathcal{C} \)) of Examples 2.2.2.3 and 2.2.2.1 both depend on certain auxiliary choices:
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Let $A$, $B$, and $C$ be associative rings, and suppose we are given a pair of bimodules $M = A_M B$ and $N = B_N C$. Then we can regard $M$ and $N$ as 1-morphisms in the 2-category Bimod, whose composition is defined to be the relative tensor product $M \otimes_B N$. This tensor product is well-defined up to (unique) isomorphism: it is universal among abelian groups $P$ which are equipped with a $B$-bilinear map $M \times N \to P$. However, it is possible to give many different constructions of an abelian group with this universal property, each of which gives a (slightly) different composition law for the 1-morphisms in the 2-category Bimod.

Let $\mathcal{C}$ be a category which admits fiber products, and suppose we are given a pair of correspondences

$$X \leftarrow M \to Y \quad Y \leftarrow N \to Z$$

in $\mathcal{C}$. Then $M$ and $N$ can be regarded as 1-morphisms in the 2-category $\text{Corr}(\mathcal{C})$, whose composition is given by the fiber product $M \times_Y N$ (regarded as a correspondence from $X$ to $Z$). This fiber product is well-defined up to (unique) isomorphism as an object of $\mathcal{C}$, but there is generally no way to choose a preferred representative of its isomorphism class. Consequently, different choices of fiber product lead to (slightly) different definitions for the composition of 1-morphisms in the 2-category $\text{Corr}(\mathcal{C})$.

By making a different choice of conventions in these examples, one can obtain 2-categories $\text{Bimod}'$ and $\text{Corr}'(\mathcal{C})$ having the same objects, 1-morphisms, and 2-morphisms as the 2-categories Bimod and Corr($\mathcal{C}$), but different composition laws for 1-morphisms. In this case, the 2-categories Bimod' and Corr'($\mathcal{C}$) can be obtained from Bimod and Corr($\mathcal{C}$) (respectively) by the twisting procedure of Construction 2.2.6.8. In particular, the resulting 2-categories Bimod' and Corr'($\mathcal{C}$) are isomorphic (though not necessarily strictly isomorphic) to the 2-categories Bimod and Corr($\mathcal{C}$), respectively.

2.2.7 Strictly Unitary 2-Categories

We now introduce a special class of 2-categories.

**Definition 2.2.7.1.** Let $\mathcal{C}$ be a 2-category. We will say that $\mathcal{C}$ is strictly unitary if, for each 1-morphism $f : X \to Y$ in $\mathcal{C}$, the left and right unit constraints

$$\lambda_f : \text{id}_Y \circ f \xRightarrow{\sim} f \quad \rho_f : f \circ \text{id}_X \xRightarrow{\sim} f$$

are identity 2-morphisms of $\mathcal{C}$.

**Proposition 2.2.7.2.** Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is strictly unitary if and only if the following conditions are satisfied:

(a) For each 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$.  


(b) For each object $X$ of $C$, the unit constraint $\nu_X : \text{id}_X \circ \text{id}_X \Rightarrow \text{id}_X$ is the identity morphism from $\text{id}_X \circ \text{id}_X = \text{id}_X$ to itself.

(c) For every 1-morphism $f : X \to Y$ in $C$, the associativity constraints $\alpha_{\text{id}_Y, \text{id}_Y, f}$ and $\alpha_{f, \text{id}_X, \text{id}_X}$ are equal to the identity (as 2-morphisms from $f$ to itself).

Proof. If $C$ is strictly unitary, then (a) is clear and (b) follows from Corollary 2.2.1.15. Assume that (a) and (b) are satisfied. For any 1-morphism $f : X \to Y$ in $C$, the left unit constraint $\lambda_f$ is characterized by the commutativity of the diagram

\[
\begin{array}{ccc}
\text{id}_Y \circ (\text{id}_Y \circ f) & \overset{\alpha_{\text{id}_Y, \text{id}_Y, f}}{\Rightarrow} & (\text{id}_Y \circ \text{id}_Y) \circ f \\
\downarrow{\text{id}_{\text{id}_Y}} & & \downarrow{\nu_Y \circ \text{id}_f} \\
\text{id}_Y \circ f, & & \text{id}_Y \circ f,
\end{array}
\]

and is therefore the identity 2-morphism if and only if $\alpha_{\text{id}_Y, \text{id}_Y, f}$ is an identity 2-morphism (from $f$ to itself). Similarly, the right unit constraint $\rho_f$ is an identity 2-morphism if and only if $\alpha_{f, \text{id}_X, \text{id}_X}$ is an identity 2-morphism in $C$. □

Remark 2.2.7.3. Let $C$ be a strictly unitary 2-category. Then $C$ satisfies the following stronger versions of conditions (a) and (c) of Proposition 2.2.7.2:

(a') For every pair of objects $X, Y \in C$, the functors

\[
\text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y) \quad f \mapsto \text{id}_Y \circ f
\]

\[
\text{Hom}_C(X, Y) \to \text{Hom}_C(X, Y) \quad f \mapsto f \circ \text{id}_X
\]

are equal to the identity.

(c') For every pair of 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $C$, the associativity constraints $\alpha_{g,f,\text{id}_X}$, $\alpha_{g,\text{id}_Y,f}$, and $\alpha_{\text{id}_Z, g, f}$ are equal to the identity (as 2-morphisms from $g \circ f$ to itself).

Here (a') follows from the naturality of the left and right unit constraints (Remark 2.2.1.13), and (c') follows from Propositions 2.2.1.14 and 2.2.1.16.

Example 2.2.7.4. Let $G$ be a group with identity element $1 \in G$, let $\Gamma$ be an abelian group on which $G$ acts by automorphisms, let $\alpha : G \times G \times G \to \Gamma$ be a 3-cocycle, let $C$ be the monoidal category of Example 2.1.3.3, and let $B C$ be the 2-category obtained by delooping $C$ (Example 2.2.2.4). The following conditions are equivalent:

- The 3-cocycle $\alpha$ is normalized: that is, it satisfies the equations

  \[ \alpha_{x,y,1} = \alpha_{x,1,y} = \alpha_{x,y,1} = 0 \]

  for every pair of elements $x, y \in G$. 


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- The 2-category $B C$ is strictly unitary, in the sense of Definition 2.2.7.1.

**Remark 2.2.7.5.** Let $C$ and $D$ be strictly unitary 2-categories (Definition 2.2.7.1). Then a strictly unitary lax functor $F : C \to D$ is given by the following data:

- For each object $X \in C$, an object $F(X) \in D$.
- For every pair of objects $X, Y \in C$, a functor of ordinary categories $F_{X,Y} : \text{Hom}_C(X,Y) \to \text{Hom}_D(F(X), F(Y))$.
- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $C$, a composition constraint $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$, depending functorially on $f$ and $g$.

This data must be required to satisfy axiom (c) of Definition 2.2.4.5, together with the identities $F(id_X) = id_{F(X)}$ for each object $X \in C$ and $\mu_{id_Y,f} = id_{F(f)} = \mu_{f,id_X}$ for each 1-morphism $f : X \to Y$ of $C$.

**Remark 2.2.7.6.** Let $C$ be a strictly unitary 2-category, let $\{\mu_{g,f}\}$ be a twisting cochain for $C$ (see Notation 2.2.6.7), and let $C'$ denote the twist of $C'$ with respect to $\{\mu_{g,f}\}$ (Construction 2.2.6.8). The following conditions are equivalent:

1. The 2-category $C'$ is strictly unitary.
2. For every 1-morphism $f : X \to Y$ in $C$, both $\mu_{f,id_X}$ and $\mu_{id_Y,f}$ are identity 2-morphisms (from $f \circ id_X = f = id_Y \circ f$ to itself).

If these conditions are satisfied, we will say that the twisting cochain $\{\mu_{g,f}\}$ is normalized.

It is generally harmless to assume that a 2-category $C$ is strictly unitary, by virtue of the following:

**Proposition 2.2.7.7.** Let $C$ be a 2-category. Then there exists a strictly unitary isomorphism $C \simeq C'$, where $C'$ is a strictly unitary 2-category.

*Proof.* Let $\mu = \{\mu_{g,f}\}$ be the twisting cochain on $C$ given by the formula

$$\mu_{g,f} = \begin{cases} 
\lambda_f : g \circ f \Rightarrow f & \text{if } g = id_Y \\
\rho_g : g \circ f \Rightarrow g & \text{if } f = id_Y \\
id_{g\circ f} : g \circ f \Rightarrow g \circ f & \text{otherwise.}
\end{cases}$$

Note that this prescription is consistent, since $\lambda_f = v_X = \rho_g$ in the special case where $f = g = id_X$ (Corollary 2.2.1.15). Let $C'$ be the twist of $C$ with respect to the cocycle $\{\mu_{g,f}\}$ (Construction 2.2.6.8). Then $C'$ is a strictly unitary 2-category (in the sense of Definition 2.2.7.1), and Exercise 2.2.6.9 supplies a strictly unitary isomorphism of 2-categories $C \simeq C'$. □
Remark 2.2.7.8. Let $2\text{Cat}_{\text{ULax}}'$ denote the subcategory of $2\text{Cat}_{\text{Lax}}$ (and full subcategory of $2\text{Cat}_{\text{ULax}}$) whose objects are strictly unitary 2-categories and whose morphisms are strictly unitary lax functors. It follows from Proposition 2.2.7.7 that the inclusion $2\text{Cat}_{\text{ULax}}' \rightarrow 2\text{Cat}_{\text{ULax}}$ is an equivalence of categories.

Remark 2.2.7.9. Let $G$ be a group and let $\Gamma$ be an abelian group with an action of $G$. When applied to the 2-categories described in Example 2.2.7.4, Proposition 2.2.7.7 reduces to the assertion that every 3-cocycle $\alpha : G \times G \times G \rightarrow \Gamma$ is cohomologous to a normalized 3-cocycle $\alpha' : G \times G \times G \rightarrow \Gamma$.

2.3 The Duskin Nerve of a 2-Category

In §1.3, we defined an $\infty$-category to be a simplicial set $X_\bullet$ which satisfies the weak Kan extension condition. Beware that this terminology is potentially misleading. Roughly speaking, an $\infty$-category (in the sense of Definition 1.3.0.1) should be viewed as a higher category $\mathcal{C}$ with the property that every $k$-morphism in $\mathcal{C}$ is invertible for $k \geq 2$. The framework of weak Kan complexes does not capture the entirety of higher category theory, or even the entirety of the theory of 2-categories (as described in §2.2). To address this point, it is convenient to restrict our attention to a special class of 2-categories.

Definition 2.3.0.1. A $(2,1)$-category is a 2-category $\mathcal{C}$ with the property that every 2-morphism in $\mathcal{C}$ is invertible.

Remark 2.3.0.2. The terminology of Definition 2.3.0.1 fits into a general paradigm. Given $0 \leq m \leq n \leq \infty$, let us informally use the term $(n,m)$-category to refer to an $n$-category $\mathcal{C}$ having the property that every $k$-morphism of $\mathcal{C}$ is invertible for $k > m$. Following this convention, the $\infty$-categories of Definition 1.3.0.1 should really be called $(\infty,1)$-categories.

Remark 2.3.0.3. Let $\mathcal{C}$ be a $(2,1)$-category. Then every lax functor of 2-categories $F : \mathcal{D} \rightarrow \mathcal{C}$ is automatically a functor. Consequently, there is no need to distinguish between functors and lax functors when working in the setting of $(2,1)$-categories.

Our goal in this section is to show that the theory of $\infty$-categories can be viewed as a generalization of the theory of $(2,1)$-categories. Recall that, to every category $\mathcal{C}$, one can associate a simplicial set $N_\bullet(\mathcal{C})$ called the nerve of $\mathcal{C}$ (Construction 1.2.1.1). We proved in Chapter 1 that $\mathcal{C} \mapsto N_\bullet(\mathcal{C})$ determines a fully faithful embedding from the category $\text{Cat}$ of small categories to the category $\text{Set}_\Delta$ of simplicial sets (Proposition 1.2.2.1), and that every simplicial set of the form $N_\bullet(\mathcal{C})$ is an $\infty$-category (Example 1.3.0.4). The construction $\mathcal{C} \mapsto N_\bullet(\mathcal{C})$ has a generalization to the setting of 2-categories. In §2.3.1, we associate to each 2-category $\mathcal{C}$ a simplicial set $N_\bullet^D(\mathcal{C})$ called the Duskin nerve of $\mathcal{C}$ (introduced by Duskin and Street; see [11] and [37]). This construction has the following features (both established by Duskin in [11]):
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- If \( \mathcal{C} \) is a \((2,1)\)-category, then the Duskin nerve \( N^\bullet_\bullet(\mathcal{C}) \) is an \( \infty \)-category (Theorem 2.3.2.1). We prove this in \( \S 2.3.2 \) as a consequence of a more general result which applies to the Duskin nerve of any 2-category (Theorem 2.3.2.5), whose proof we defer to \( \S 2.3.3 \).

- Let \( \mathcal{C} \) and \( \mathcal{D} \) be 2-categories. In \( \S 2.3.4 \) we show that passage to the Duskin nerve induces a bijection

\[
\{ \text{Strictly unitary lax functors } F : \mathcal{C} \to \mathcal{D} \} \\
\sim \\
\{ \text{Maps of simplicial sets } N^\bullet_\bullet(\mathcal{C}) \to N^\bullet_\bullet(\mathcal{D}) \};
\]

see Theorem 2.3.4.1. In other words, the formation of Duskin nerves induces a fully faithful embedding from the category \( 2\text{Cat}_{ULax} \) of Definition 2.2.5.5 to the category of simplicial sets.

By virtue of Theorem 2.3.4.1, it is mostly harmless to abuse terminology by identifying a 2-category \( \mathcal{C} \) with the simplicial set \( N^\bullet_\bullet(\mathcal{C}) \) (each can be recovered from the other, up to canonical isomorphism). Theorem 2.3.2.1 then asserts that, under this identification, every \((2,1)\)-category can be regarded as an \( \infty \)-category (see Remark 2.3.4.2 for a more precise statement).

The remainder of this section is devoted to giving a more concrete description of the Duskin nerve in two examples:

- Let \( \mathcal{E} \) be a category which admits fiber products, let \( \text{Corr}(\mathcal{E}) \) be the 2-category of correspondences in \( \mathcal{E} \) (Example 2.2.2.1). In \( \S 2.3.5 \) we give an explicit description of the simplicial set \( N^\bullet_\bullet(\text{Corr}(\mathcal{E})) \): roughly speaking, \( n \)-simplices of \( N^\bullet_\bullet(\text{Corr}(\mathcal{E})) \) can be identified with diagrams in \( \mathcal{E} \) indexed by the “pyramid” \( \{(i, j) \in [n] \times [n]^\text{op} : i \leq j\} \) (see Remark 2.3.5.9). We prove this by establishing a universal property of the 2-category \( \text{Corr}(\mathcal{E}) \) related to the formation of twisted arrow categories (Theorem 2.3.5.7).

- Let \( \mathcal{C} \) be a strict 2-category. In \( \S 2.3.6 \), we show that \( n \)-simplices of the Duskin nerve \( N^\bullet_\bullet(\mathcal{C}) \) can be identified with \textit{strict} functors \( \text{Path}^c_{[2]}[n] \to \mathcal{C} \) (Corollary 2.3.6.8). Here \( \text{Path}^c_{[2]}[n] \) denotes a certain 2-categorical variant of the path category introduced in \( \S 1.2.6 \) which will play an important role in our discussion of the homotopy coherent nerve of a simplicial category (see \( \S 2.4.3 \)).

2.3.1 The Duskin Nerve

In \( \S 1.2 \) we associated to each category \( \mathcal{C} \) a simplicial set \( N^\bullet_\bullet(\mathcal{C}) \), called the \textit{nerve of \( \mathcal{C} \)}. This construction has a natural generalization to the setting of 2-categories.
Construction 2.3.1.1 (The Duskin Nerve). Let $n$ be a nonnegative integer and let $[n]$ denote the linearly ordered set $\{0 < 1 < 2 < \cdots < n\}$. We will regard $[n]$ as a category, hence also as a 2-category having only identity 2-morphisms (Example 2.2.0.6). For any 2-category $\mathcal{C}$, we let $N^D_n(\mathcal{C})$ denote the set of all strictly unitary lax functors from $[n]$ to $\mathcal{C}$ (Definition 2.2.4.16). The construction $[n] \mapsto N^D_n(\mathcal{C})$ determines a simplicial set, given as a functor by the composition

$$\Delta^{\text{op}} \hookrightarrow \text{Cat}^{\text{op}} \hookrightarrow 2\text{Cat}_{\text{ULax}}^{\text{op}} \xrightarrow{\text{Hom}_{2\text{Cat}_{\text{ULax}}}(-,\mathcal{C})} \text{Set}.$$ 

We will denote this simplicial set by $N^D_\bullet(\mathcal{C})$ and refer to it as the Duskin nerve of the 2-category $\mathcal{C}$.

Remark 2.3.1.2. In the setting of strict 2-categories, the Duskin nerve $\mathcal{C} \mapsto N^D_\bullet(\mathcal{C})$ was introduced by Street in [37]. The generalization to arbitrary 2-categories was given by Duskin in [11].

Example 2.3.1.3. Let $\mathcal{C}$ be an ordinary category, viewed as a 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the Duskin nerve $N^D_\bullet(\mathcal{C})$ can be identified with the nerve $N_\bullet(\mathcal{C})$ of $\mathcal{C}$ as an ordinary category (Construction 1.2.1.1).

Remark 2.3.1.4. Let $\mathcal{C}$ be a 2-category and let $\mathcal{C}^{\text{op}}$ denote the opposite 2-category (see Construction 2.2.3.1). Then we have a canonical isomorphism of simplicial sets $N^D_\bullet(\mathcal{C}^{\text{op}}) \simeq N^D_\bullet(\mathcal{C})^{\text{op}}$, where $N^D_\bullet(\mathcal{C})^{\text{op}}$ denotes the opposite of the simplicial set $N^D_\bullet(\mathcal{C})$ (see Notation 1.3.2.1).

Warning 2.3.1.5. Let $\mathcal{C}$ be a 2-category and let $\mathcal{C}^c$ be the conjugate of $\mathcal{C}$, obtained by reversing vertical composition (Construction 2.2.3.4). There is no simple relationship between Duskin nerves of $\mathcal{C}$ and $\mathcal{C}^c$ (since the operation $\mathcal{C} \mapsto \mathcal{C}^c$ is not functorial with respect to lax functors; see Warning 2.2.5.11).

Remark 2.3.1.6 (Functoriality). The construction $\mathcal{C} \mapsto N^D_\bullet(\mathcal{C})$ determines a functor from the category $2\text{Cat}_{\text{ULax}}$ of small 2-categories (with morphisms given by strictly unitary lax functors) to the category $\text{Set}_\Delta$ of simplicial sets. This functor fits into the general paradigm of Variant 1.1.7.6: it arises from a cosimplicial object of the category $2\text{Cat}_{\text{ULax}}$, given by the inclusion $\Delta \hookrightarrow \text{Cat} \hookrightarrow 2\text{Cat}_{\text{ULax}}$. Beware that, unlike the usual nerve functor $N_\bullet : \text{Cat} \to \text{Set}_\Delta$, the Duskin nerve $N^D_\bullet : 2\text{Cat}_{\text{ULax}} \to \text{Set}_\Delta$ does not admit a left adjoint: Proposition 1.1.8.22 does not apply, because the category $2\text{Cat}_{\text{ULax}}$ does not admit small colimits (one can address this problem by restricting to strict 2-categories: we will return to this point in §2.3.6).

Remark 2.3.1.7. Let $\mathcal{C}$ be a 2-category, let $\{\mu_{g,f}\}$ be a twisting cochain for $\mathcal{C}$ (Notation 2.2.6.7), and let $\mathcal{C}'$ be the twist of $\mathcal{C}$ with respect to $\{\mu_{g,f}\}$ (Construction 2.2.6.8). Then the
2.3. THE DUSKIN NERVE OF A 2-CATEGORY

2.3.1.17 and Remark 2.3.5.9

Remark 2.3.1.8. Let us make Construction 2.3.1.1 more explicit. Fix a 2-category \( \mathcal{C} \). Unwinding the definitions, we see that an element of \( N_n^D(\mathcal{C}) \) consists of the following data:

(0) A collection of objects \( \{X_i\}_{0 \leq i \leq n} \) of the 2-category \( \mathcal{C} \).

(1) A collection of 1-morphisms \( \{f_{j,i} : X_i \to X_j\}_{0 \leq i, j \leq n} \) in the 2-category \( \mathcal{C} \).

(2) A collection of 2-morphisms \( \{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i, j, k \leq n} \) in the 2-category \( \mathcal{C} \).

These data are required to satisfy the following conditions:

(a) For \( 0 \leq i \leq n \), the 1-morphism \( f_{i,i} : X_i \to X_i \) is the identity 1-morphism \( \text{id}_{X_i} \).

(b) For \( 0 \leq i \leq j \leq n \), the 2-morphisms

\[
\mu_{j,j,i} : f_{j,j} \circ f_{j,i} \Rightarrow f_{j,i} \quad \mu_{j,i,i} : f_{j,i} \circ f_{i,i} \Rightarrow f_{j,i}
\]

are the left unit constraints \( \lambda_{f_{j,i}} \) and the right unit constraints \( \rho_{f_{j,i}} \), respectively.

(c) For \( 0 \leq i \leq j \leq k \leq \ell \leq n \), we have a commutative diagram

In the category \( \text{Hom}_\mathcal{C}(X_i, X_\ell) \).

In the description of Remark 2.3.1.8 it is possible to be more efficient by eliminating some of the “redundant” information.

Proposition 2.3.1.9. Let \( \mathcal{C} \) be a 2-category and let \( n \) be a nonnegative integer. Suppose we are given the following data:
(0) A collection of objects \( \{X_i\}_{0 \leq i \leq n} \) of the 2-category \( \mathcal{C} \).

(1') A collection of 1-morphisms \( \{f_{j,i} : X_i \to X_j\}_{0 \leq i < j \leq n} \) in the 2-category \( \mathcal{C} \).

(2') A collection of 2-morphisms \( \{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i < j < k \leq n} \) in the 2-category \( \mathcal{C} \).

This data can be extended uniquely to an \( n \)-simplex of the Duskin nerve \( \mathsf{ND}(\mathcal{C}) \) (as described in Remark 2.3.1.8) if and only if the following condition is satisfied:

(c') For \( 0 \leq i < j < k < \ell \leq n \), we have a commutative diagram

\[
\begin{array}{ccc}
    f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) & \xrightarrow{\alpha_{f_{\ell,k} \circ f_{k,j} \circ f_{j,i}}} & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \\
    \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
    f_{\ell,k} \circ f_{k,i} & \xrightarrow{\mu_{\ell,k,j} \circ \text{id}_{f_{j,i}}} & f_{\ell,j} \circ f_{j,i} \\
\end{array}
\]

in the category \( \mathsf{Hom}_\mathcal{C}(X_i, X_\ell) \).

**Proof.** We wish to show that there is a unique way to choose 1-morphisms \( f_{j,i} : X_i \to X_j \) for \( i = j \) and 2-morphisms \( \mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i} \) for \( i = j \leq k \) and \( i \leq j = k \) so that conditions (a), (b), and (c) of Remark 2.3.1.8 are satisfied. The uniqueness is clear: to satisfy condition (a), we must have \( f_{i,i} = \text{id}_{X_i} \) for \( 0 \leq i \leq n \), and to satisfy condition (b) we must have \( \mu_{k,j,i} = \rho_{f_{j,i}} \) when \( i = j \) and \( \mu_{k,j,i} = \lambda_{f_{k,j}} \) when \( j = k \). To complete the proof, it will suffice to verify the following:

(I) The prescription above is consistent. That is, when \( i = j = k \), we have \( \rho_{f_{j,i}} = \lambda_{f_{k,j}} \) (as morphisms of the category \( \mathsf{Hom}_\mathcal{C}(X_i, X_k) \)).

(II) The prescription above satisfies condition (c) of Remark 2.3.1.8. That is, the diagram

\[
\begin{array}{ccc}
    f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) & \xrightarrow{\alpha_{f_{\ell,k} \circ f_{k,j} \circ f_{j,i}}} & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \\
    \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
    f_{\ell,k} \circ f_{k,i} & \xrightarrow{\mu_{\ell,k,j} \circ \text{id}_{f_{j,i}}} & f_{\ell,j} \circ f_{j,i} \\
\end{array}
\]

commutes in the special cases \( 0 \leq i = j \leq k \leq \ell \leq n \), \( 0 \leq i \leq j = k \leq \ell \leq n \), and \( 0 \leq i < j \leq k = \ell \leq n \).
Assertion (I) follows from Corollary 2.2.1.15. Assertion (II) follows from the triangle identity in \( \mathcal{C} \) in the case \( j = k \), and from Proposition 2.2.1.16 in the cases \( i = j \) and \( k = \ell \).

**Corollary 2.3.1.10.** Let \( \mathcal{C} \) be a 2-category. Then the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) is 3-coskeletal (Definition [?]). In other words, if \( S \) is a simplicial set, then any map from the 3-skeleton \( \text{sk}_3(S) \to \mathbb{N}^D \mathcal{C} \) extends uniquely to a map \( S \to \mathbb{N}^D \mathcal{C} \).

**Warning 2.3.1.11.** Let \( \mathcal{C} \) be a 2-category. By virtue of Proposition 2.3.1.9, we can identify \( n \)-simplices of the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) with triples

\[
(\{X_i\}_{0 \leq i \leq n}, \{f_{j,i}\}_{0 \leq i < j \leq n}, \{\mu_{k,j,i}\}_{0 \leq i < j < k \leq n})
\]

satisfying condition (\( c' \)) of Proposition 2.3.1.9. This gives a description of \( \mathbb{N}^D_n \mathcal{C} \) which makes no reference to the identity 1-morphisms of \( \mathcal{C} \) or the left and right unit constraints of \( \mathcal{C} \). The resulting identification is functorial with respect to injective maps of linearly ordered sets \( [m] \to [n] \). In other words, we can construct the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) as a semisimplicial set (see Variant 1.1.1.6) without knowing the left and right unit constraints of \( \mathcal{C} \). However, the left and right unit constraints of \( \mathcal{C} \) are needed to define the degeneracy operators on the simplicial set \( \mathbb{N}^D_n \mathcal{C} \).

**Remark 2.3.1.12.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be 2-categories and let \( F : \mathcal{C} \to \mathcal{D} \) be a lax functor. If \( F \) is strictly unitary, then composition with \( F \) induces a map of simplicial sets \( \mathbb{N}^D \mathcal{C} \to \mathbb{N}^D \mathcal{D} \). However, even without the assumption that \( F \) is strictly unitary, one can use the description of Proposition 2.3.1.9 to obtain a collection of maps \( \mathbb{N}^D_n \mathcal{C} \to \mathbb{N}^D_n \mathcal{D} \) which are compatible with the face operators on the simplicial sets \( \mathbb{N}^D_n \mathcal{C} \) and \( \mathbb{N}^D_n \mathcal{D} \) (though not necessarily with the degeneracy operators). In other words, if we regard the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) as a semisimplicial set, then it is functorial with respect to all (lax) functors between 2-categories.

**Example 2.3.1.13** (Vertices of the Duskin Nerve). Let \( \mathcal{C} \) be a 2-category. Using Proposition 2.3.1.9, we can identify vertices of the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) with objects of the 2-category \( \mathcal{C} \).

**Example 2.3.1.14** (Edges of the Duskin Nerve). Let \( \mathcal{C} \) be a 2-category. Using Proposition 2.3.1.9, we can identify edges of the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) with 1-morphisms \( f : X \to Y \) of the 2-category \( \mathcal{C} \). Under this identification, the face and degeneracy operators

\[
d_0, d_1 : \mathbb{N}^D_1 \mathcal{C} \to \mathbb{N}^D_0 \mathcal{C} \quad s_0 : \mathbb{N}^D_0 \mathcal{C} \to \mathbb{N}^D_1 \mathcal{C}
\]

are given by \( d_0(f : X \to Y) = Y, d_1(f : X \to Y) = X \), and \( s_0(X) = \text{id}_X \).

**Example 2.3.1.15** (2-Simplices of the Duskin Nerve). Let \( \mathcal{C} \) be a 2-category. Using Proposition 2.3.1.9, we see that a 2-simplex \( \sigma \) of the Duskin nerve \( \mathbb{N}^D \mathcal{C} \) can be identified with the following data:
• A triple of objects $X, Y, Z \in \mathcal{C}$.

• A triple of 1-morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ in the 2-category $\mathcal{C}$ (corresponding to the facts $d_2(\sigma)$, $d_0(\sigma)$, and $d_1(\sigma)$, respectively).

• A 2-morphism $\mu : g \circ f \Rightarrow h$, which we depict as a diagram

$$
\begin{array}{ccc}
X & \overset{f}{\rightarrow} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \overset{\mu}{\rightarrow} & Z
\end{array}
$$

Example 2.3.1.16 (3-Simplices of the Duskin Nerve). Let $\mathcal{C}$ be a 2-category. Using Proposition 2.3.1.9, we see that a map of simplicial sets $\partial \Delta^3 \to \mathcal{N}^D_{\bullet}(\mathcal{C})$ can be identified with the following data:

• A collection of objects $\{X_i\}_{0 \leq i \leq 3}$ of the 2-category $\mathcal{C}$.

• A collection of 1-morphisms $\{f_{j,i} : X_i \to X_j\}_{0 \leq i < j \leq 3}$.

• A quadruple of 2-morphisms

$$
\begin{align*}
\mu_{2,1,0} & : f_{2,1} \circ f_{1,0} \Rightarrow f_{2,0} & \mu_{3,2,1} & : f_{3,2} \circ f_{2,1} \Rightarrow f_{3,1} \\
\mu_{3,1,0} & : f_{3,1} \circ f_{1,0} \Rightarrow f_{3,0} & \mu_{3,2,0} & : f_{3,2} \circ f_{2,0} \Rightarrow f_{3,0}.
\end{align*}
$$

This data can be conveniently visualized as a pair of diagrams:
representing “front” and “back” perspectives of the boundary of a 3-simplex. A 3-simplex of the Duskin nerve $N^D_\bullet(C)$ can be identified with a map $\partial \Delta^3 \to N^D_\bullet(C)$ as above which satisfies an additional compatibility condition: namely, the commutativity of the diagram

\[
\begin{array}{ccc}
    \text{id}_{f_{3,2}} \circ (f_{2,1} \circ f_{1,0}) & \overset{\alpha_{f_{3,2},f_{2,1},f_{1,0}}}{\longrightarrow} & (f_{3,2} \circ f_{2,1}) \circ f_{1,0} \\
    f_{3,2} \circ f_{2,0} & \overset{\mu_{3,2,0}}{\longrightarrow} & f_{3,0} \\
    f_{3,2} \circ f_{2,0} & \overset{\mu_{3,2,0}}{\longrightarrow} & f_{3,0} \\
\end{array}
\]

in the ordinary category $\text{Hom}_C(X_0, X_3)$.

**Example 2.3.1.17** (The Duskin Nerve of Bimod). Let Bimod denote the 2-category of Example 2.2.2.3. Then an $n$-simplex of the Duskin nerve $N^D_\bullet(\text{Bimod})$ can be identified with a collection of abelian groups $\{A_{j,i}\}_{0 \leq i < j \leq n}$ equipped with unit elements $e_i \in A_{i,i}$ and bilinear multiplication maps $\cdot : A_{k,j} \times A_{j,i} \to A_{k,i}$ satisfying the identities $e_j \cdot x = x = x \cdot e_i$ for $x \in A_{j,i}$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for $x \in A_{\ell,k}, y = A_{k,j},$ and $z \in A_{j,i}$ (where $0 \leq i < j < k \leq \ell \leq n$). In this case, the multiplication equips each $A_{i,i}$ with the structure of an associative ring (which is an object of the 2-category Bimod), each $A_{j,i}$ with the structure of an $A_{j,j}$-$A_{i,i}$ bimodule (which is a 1-morphism in the 2-category Bimod). For $0 \leq i < j < k \leq n$, the bilinear map $A_{k,j} \times A_{j,i} \to A_{k,i}$ can be identified with a map of bimodules $\mu_{k,j,i} : A_{k,j} \otimes A_{j,i} A_{j,i} \to A_{k,i}$, which we can regard as a 2-morphism in the category Bimod.

**Example 2.3.1.18** (The Classifying Simplicial Set of a Monoidal Category). Let $\mathcal{C}$ be a monoidal category (Definition 2.1.2.10) and let $B\mathcal{C}$ denote the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.4). We will denote the Duskin nerve of $B\mathcal{C}$ by $B_\bullet \mathcal{C}$ and refer to it as the classifying simplicial set of $\mathcal{C}$. By virtue of Proposition 2.3.1.9, we can identify $n$-simplices of the simplicial set $B_\bullet \mathcal{C}$ with pairs

\[
(\{C_{j,i}\}_{0 \leq i < j \leq n}, \{\mu_{k,j,i}\}_{0 \leq i < j < k \leq n})
\]

where each $C_{j,i}$ is an object of $\mathcal{C}$ and each $\mu_{k,j,i}$ is a morphism from $C_{k,j} \otimes C_{j,i}$ to $C_{k,i}$, satisfying the following coherence condition:
For $0 \leq i < j < k < \ell \leq n$, the diagram

\[
\begin{array}{c}
C_{\ell,k} \otimes (C_{k,j} \otimes C_{j,i}) \\
id_{C_{\ell,k}} \otimes \mu_{k,j,i}
\end{array}
\xrightarrow{\alpha_{C_{\ell,k},C_{k,j},C_{j,i}}}
\begin{array}{c}
(C_{\ell,k} \otimes C_{k,j}) \otimes C_{j,i} \\
\mu_{\ell,j,i} \otimes id_{C_{j,i}}
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
C_{\ell,j} \otimes C_{j,i} \\
\mu_{\ell,j,i}
\end{array}
\begin{array}{c}
\xrightarrow{\alpha_{C_{\ell,j},C_{j,i}}}
\end{array}
\begin{array}{c}
C_{\ell,i}
\end{array}
\]

is commutative.

**Remark 2.3.1.19.** Let $G$ be a monoid, regarded as a monoidal category having only identity morphisms. Then the classifying simplicial set $B \bullet G$ of Example 2.3.1.18 agrees (up to canonical isomorphism) with the simplicial set $B \bullet G$ given by the Milnor construction, described in Example 1.2.4.3.

### 2.3.2 From 2-Categories to ∞-Categories

We now use Construction 2.3.1.1 to connect the theory of 2-categories (in the sense of Definition 2.2.1.1) to the theory of ∞-categories (in the sense of Definition 1.3.0.1).

**Theorem 2.3.2.1 (Duskin [11]).** Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is (2,1)-category if and only if the Duskin nerve $N_\bullet^D(\mathcal{C})$ is an ∞-category.

**Example 2.3.2.2.** Let $\mathcal{C}$ be a monoidal category and suppose that every morphism in $\mathcal{C}$ is an isomorphism. Then the classifying simplicial set $B \bullet \mathcal{C}$ of Example 2.3.1.18 is an ∞-category.

We will deduce Theorem 2.3.2.1 from a more general statement (Theorem 2.3.2.5), which gives a filling criterion for inner horns in the Duskin nerve $N_\bullet^D(\mathcal{C})$ for an arbitrary 2-category $\mathcal{C}$. First, we need a bit of terminology.

**Definition 2.3.2.3.** Let $X_\bullet$ be a simplicial set. We will say that a 2-simplex $\sigma$ of $X_\bullet$ is thin if it satisfies the following condition:

\[ (\ast) \text{ Let } n \geq 3, \text{ let } 0 < i < n, \text{ and let } \tau \text{ denote the 2-simplex of } \Lambda^n_i \text{ given by the map } \]

\[ [2] \simeq \{i - 1, i, i + 1\} \subseteq [n]. \]

Then any map of simplicial sets $f_0 : \Lambda^n_i \rightarrow X_\bullet$ satisfying $f_0(\tau) = \sigma$ can be extended to an $n$-simplex of $X_\bullet$. 

\[ [2] \]
Example 2.3.2.4. Let $X_\bullet$ be a simplicial set. If $X_\bullet$ is an $\infty$-category (in the sense of Definition 1.3.0.1), then every 2-simplex of $X_\bullet$ is thin. Conversely, if every 2-simplex of $X_\bullet$ is thin, then $X_\bullet$ is an $\infty$-category if and only if every map of simplicial sets $f_0 : \Lambda^2_1 \to X_\bullet$ can be extended to a 2-simplex of $X_\bullet$.

We will deduce Theorem 2.3.2.1 from the following result, whose proof will be given in §2.3.3:

Theorem 2.3.2.5. Let $C$ be a 2-category and let $\sigma$ be a 2-simplex of the Duskin nerve $N^D_\bullet(C)$, corresponding to a diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow^{f} & \searrow^{\gamma} & \downarrow^{g} \\
\downarrow^{h} & & \\
Z & \rightarrow & \\
\end{array}
\]

(see Example 2.3.1.15). Then $\sigma$ is thin if and only if $\gamma : g \circ f \Rightarrow h$ is an isomorphism in the category $\text{Hom}_C(X, Z)$.

Proof of Theorem 2.3.2.1 from Theorem 2.3.2.5. Let $C$ be a 2-category. If the Duskin nerve $N^D_\bullet(C)$ is an $\infty$-category, then every 2-simplex of $N^D_\bullet(C)$ is thin (Example 2.3.2.4), so that every 2-morphism in $C$ is invertible by virtue of Theorem 2.3.2.5. Conversely, if $C$ is a $(2,1)$-category, then every 2-simplex of $N^D_\bullet(C)$ is thin (Theorem 2.3.2.5). Consequently, to show that $N^D_\bullet(C)$ is an $\infty$-category, it will suffice to show that every map of simplicial sets $u_0 : \Lambda^2_1 \to N^D_\bullet(C)$ can be extended to a 2-simplex of $N^D_\bullet(C)$. Note that we can identify $u_0$ with a composable pair of 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $C$. To extend this to a 2-simplex of $N^D_\bullet(C)$, it suffices to choose a 1-morphism $h : X \to Z$ and a 2-morphism $\gamma : g \circ f \Rightarrow h$. This is always possible: for example, we can take $h = g \circ f$ and $\gamma$ to be the identity 2-morphism. □

Remark 2.3.2.6. Let $C$ be a $(2,1)$-category, so that the Duskin nerve $N^D_\bullet(C)$ is an $\infty$-category. Then:

- Objects of the $\infty$-category $N^D_\bullet(C)$ can be identified with objects of the 2-category $C$.
- If $X$ and $Y$ are objects of $C$, then morphisms from $X$ to $Y$ in the $\infty$-category $N^D_\bullet(C)$ can be identified with 1-morphisms from $X$ to $Y$ in the 2-category $C$.
- If $f, g : X \to Y$ are 1-morphisms in $C$ having the same domain and codomain, then $f$ and $g$ are homotopic when regarded as morphisms of the $\infty$-category $N^D_\bullet(C)$ (Definition 1.3.3.1) if and only if they are isomorphic when viewed as objects of the groupoid $\text{Hom}_C(X, Y)$. More precisely, vertical composition with the left unit
constraint $\lambda_f : \text{id}_Y \circ f \Rightarrow f$ induces a bijection

$$\{\text{Isomorphisms from } f \text{ to } g \text{ in the groupoid } \text{Hom}_C(X,Y)\} \sim \{\text{Homotopies from } f \text{ to } g \text{ in the } \infty\text{-category } N^\bullet_D(C)\}.$$

Let us now collect some other consequences of Theorem 2.3.2.5.

**Corollary 2.3.2.7.** Let $C$ be a 2-category. Then every degenerate 2-simplex of the Duskin nerve $N^\bullet_D(C)$ is thin.

*Proof.* Combine Theorem 2.3.2.5 with the observation that, for every 1-morphism $f : X \to Y$ of $C$, the left and right unit constraints $\lambda_f : \text{id}_Y \circ f \Rightarrow f \quad \rho_f : f \circ \text{id}_X \Rightarrow f$ are isomorphisms (in the category $\text{Hom}_C(X,Y)$).

**Corollary 2.3.2.8.** Let $C$ and $D$ be 2-categories and let $F : C \to D$ be a strictly unitary lax functor. Then $F$ is a functor if and only if the induced map of simplicial sets $N^\bullet_D(C) \Rightarrow N^\bullet_D(D)$ carries thin 2-simplices of $N^\bullet_D(C)$ to thin 2-simplices of $N^\bullet_D(D)$.

*Proof.* Let $\sigma$ be a 2-simplex of $N^\bullet_D(C)$, corresponding to a diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow f & \Downarrow \gamma & \downarrow h \\ X & \xrightarrow{h} & Z \end{array}$$

in $C$. Let $\sigma'$ denote the image of $\sigma$ in $N^\bullet_D(D)$, corresponding to the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(g)} & F(Z) \\ \downarrow F(f) & \Downarrow F(\gamma) & \downarrow F(h) \\ F(X) & \xrightarrow{F(h)} & F(Z) \end{array}$$

where $\gamma'$ is given by the (vertical) composition

$$F(g) \circ F(f) \xRightarrow{\mu_{g,f}} F(g \circ f) \xRightarrow{F(\gamma)} F(h).$$

Since $\sigma$ is thin, the 2-morphism $\gamma$ is an isomorphism (Theorem 2.3.2.5). It follows that $\sigma'$ is thin if and only if $\mu_{g,f}$ is an isomorphism. In particular, the strictly unitary lax functor $F$ preserves thin 2-simplices if and only if $\mu_{g,f}$ is an isomorphism for every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $C$: that is, if and only if $F$ is a functor.

\[\square\]
Warning 2.3.2.9. Let \( \mathcal{C} \) be a 2-category. Let us say that a 2-simplex \( \sigma \) of the Duskin nerve \( N^D_\bullet(\mathcal{C}) \) is \textit{special} if it corresponds to a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow h & & \downarrow g \\
& \searrow \gamma & \\
& & Z,
\end{array}
\]

where \( h = g \circ f \) and \( \gamma = \text{id}_{g \circ f} \). Arguing as in the proof of Corollary 2.3.2.8, we see that a strictly unitary lax functor \( F : \mathcal{C} \to \mathcal{D} \) is \textit{strict} if and only if it carries special 2-simplices of \( N^D_\bullet(\mathcal{C}) \) to special 2-simplices of \( N^D_\bullet(\mathcal{D}) \). Beware, however, that the special 2-simplices of \( N^D_\bullet(\mathcal{C}) \) and \( N^D_\bullet(\mathcal{D}) \) do not have an \textit{intrinsic} description in terms of the simplicial sets \( N^D_\bullet(\mathcal{C}) \) and \( N^D_\bullet(\mathcal{D}) \) themselves. In particular, it is possible to have an isomorphism of simplicial sets \( N^D_\bullet(\mathcal{C}) \simeq N^D_\bullet(\mathcal{D}) \) which does not preserve special 2-simplices (corresponding to an isomorphism of 2-categories which is strictly unitary but not strict).

In general, passage from a 2-category \( \mathcal{C} \) to its Duskin nerve \( N^D_\bullet(\mathcal{C}) \) involves a slight loss of information. From the simplicial set \( N^D_\bullet(\mathcal{C}) \), we can recover the objects of \( \mathcal{C} \) (these can be identified with vertices of \( N^D_\bullet(\mathcal{C}) \)) and the collection of 1-morphisms \( f : X \to Y \) from an object \( X \) to an object \( Y \) (these can be identified with edges of \( N^D_\bullet(\mathcal{C}) \) having source \( X \) and target \( Y \)). However, the composition \( g \circ f \) of a pair of composable 1-morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \) cannot be recovered from the structure of \( N^D_\bullet(\mathcal{C}) \) as an abstract simplicial set. The best we can do is to ask for a thin 2-simplex \( \sigma \) of \( N^D_\bullet(\mathcal{C}) \) satisfying \( d_0(\sigma) = g \) and \( d_2(\sigma) = f \). Such a simplex can be viewed as “witnessing” the presence of an isomorphism of the edge \( h = d_1(\sigma) \) with the composition \( g \circ f \). Put another way, the abstract simplicial set \( N^D_\bullet(\mathcal{C}) \) contains enough information to reconstruct the composition \( g \circ f \) up to (unique) isomorphism, but not enough information to select a canonical representative of its isomorphism class. This can be viewed as a feature, rather than a bug: the Duskin nerve \( N^D_\bullet(\mathcal{C}) \) often admits a more invariant description than the 2-category \( \mathcal{C} \) itself (since the information lost by passing from \( \mathcal{C} \) to \( N^D_\bullet(\mathcal{C}) \) depends on choices that one would prefer not make in the first place; see Remark 2.3.1.7).

If \( \mathcal{C} \) is a 2-category which contains non-invertible 2-morphisms, then the Duskin nerve \( N^D_\bullet(\mathcal{C}) \) is not an \( \infty \)-category. However, we can extract an \( \infty \)-category by applying the Duskin nerve to a smaller 2-category.

Construction 2.3.2.10. Let \( \mathcal{C} \) be a 2-category. We define a new 2-category \( \text{Core}_1(\mathcal{C}) \) as follows:

- The objects of \( \text{Core}_1(\mathcal{C}) \) are the objects of \( \mathcal{C} \).
- For every pair of objects \( X, Y \in \mathcal{C} \), the category \( \text{Hom}_{\text{Core}_1(\mathcal{C})}(X, Y) \) is the core \( \text{Hom}_\mathcal{C}(X, Y) \simeq \text{of the category} \text{Hom}_\mathcal{C}(X, Y) \) (see Construction 1.2.4.4).
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- The composition law, associativity constraints, and unit constraints of \(\text{Core}_1(C)\) are given by restricting the composition law, associativity constraints, and unit constraints of \(C\).

Then \(\text{Core}_1(C)\) is a \((2,1)\)-category which we will refer to as the 1-core of \(C\).

More informally: for any 2-category \(C\), the \((2,1)\)-category \(\text{Core}_1(C)\) is obtained by discarding the non-invertible 2-morphisms of \(C\).

**Remark 2.3.2.11** (The Universal Property of the \(\text{Core}_1(C)\)). Let \(C\) be a 2-category. Then the 1-core \(\text{Core}_1(C)\) is characterized (up to isomorphism) by the following properties:

- The 1-core \(\text{Core}_1(C)\) is a \((2,1)\)-category.

- For every \((2,1)\)-category \(D\), every functor \(F : D \to C\) factors (uniquely) through \(\text{Core}_1(C)\).

**Warning 2.3.2.12.** In the situation of Remark 2.3.2.11, it is not true that a lax functor \(F : D \to C\) factors through the 1-core \(\text{Core}_1(C)\) (even when \(D\) is a \((2,1)\)-category): any lax functor which admits such a factorization is automatically a functor, by virtue of Remark 2.3.0.3.

**Remark 2.3.2.13.** Let \(C\) be a 2-category and let \(\text{Core}_1(C)\) denote its 1-core. Then the Duskin nerve \(\mathbb{N}^D_\bullet(\text{Core}_1(C))\) is an \(\infty\)-category (Theorem 2.3.2.1). Unwinding the definitions, we see that \(\mathbb{N}^D_\bullet(\text{Core}_1(C))\) can be identified with the largest simplicial subset \(X_\bullet\) of \(\mathbb{N}^D_\bullet(C)\) having the property that each 2-simplex of \(X_\bullet\) is thin when regarded as a 2-simplex of \(\mathbb{N}^D_\bullet(C)\) (so that an \(n\)-simplex \(\sigma \in \mathbb{N}^D_n(C)\) belongs to \(\mathbb{N}^D_\bullet(\text{Core}_1(C))\) if and only if, for every map \(\Delta^2 \to \Delta^n\), the composition \(\Delta^2 \to \Delta^n \xrightarrow{\sigma} \mathbb{N}^D_\bullet(C)\) is thin).

### 2.3.3 Thin 2-Simplices of a Duskin Nerve

Let \(C\) be a 2-category and let \(\sigma\) be a 2-simplex of the Duskin nerve \(\mathbb{N}^D_\bullet(C)\), corresponding to a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow^{\gamma} & & \\
X & \xrightarrow{h} & Z \\
& \xrightarrow{f} & \\
\end{array}
\]

Our goal is to prove Theorem 2.3.2.5 which asserts that \(\sigma\) is thin (in the sense of Definition 2.3.2.3) if and only if the 2-morphism \(\gamma : g \circ f \Rightarrow h\) is invertible. This follows from Propositions 2.3.3.1 and Proposition 2.3.3.2 below.
Proposition 2.3.3.1. Let \( C \) be a 2-category, let \( n \geq 3 \), and let \( u : \Lambda^n_\ell \to \mathbf{N}_\bullet^D(C) \) be a map of simplicial sets for some \( 0 < \ell < n \). Let \( \sigma \) denote the 2-simplex of \( \mathbf{N}_\bullet^D(C) \) obtained by composing \( u \) with the map \( \Delta^2 \to \Lambda^n_\ell \) given by the map of linearly ordered sets \( [2] = \{ \ell - 1, \ell, \ell + 1 \} \subseteq [n] \), corresponding to a diagram

\[
\begin{array}{ccc}
X_\ell & \xrightarrow{\gamma} & X_{\ell+1} \\
\downarrow & & \downarrow \\
X_{\ell-1} & \rightarrow & X_{\ell+1}
\end{array}
\]

in the 2-category \( C \). If \( \gamma \) is invertible, then \( u \) extends uniquely to an \( n \)-simplex of \( \mathbf{N}_\bullet^D(C) \).

Proof. Using Examples 2.3.1.13 and 2.3.1.14, we see that the restriction of \( u \) to the 1-skeleton of \( \Lambda^n_\ell \) is given by a collection of objects \( \{ X_i \}_{0 \leq i \leq n} \) of \( C \), together with 1-morphisms \( \{ f_{ji} : X_i \to X_j \}_{0 \leq i < j \leq n} \). For \( n \geq 5 \), the horn \( \Lambda^n_\ell \) contains the 3-skeleton of \( \Delta^n \), so the existence and uniqueness of the desired extension is automatic by virtue of Corollary 2.3.1.10 (in particular, we do not need to assume that \( 0 < \ell < n \) or that \( \gamma \) is invertible). We now treat the case \( n = 3 \). We will assume that \( \ell = 1 \) (the case \( \ell = 2 \) follows by symmetry), so that we can use Example 2.3.1.15 to identify \( u \) with a triple of 2-morphisms

\[
\mu_{210} : f_{21} \circ f_{10} \Rightarrow f_{20} \quad \mu_{310} : f_{31} \circ f_{10} \Rightarrow f_{30} \quad \mu_{321} : f_{32} \circ f_{21} \Rightarrow f_{31}.
\]

Using the description of 3-simplices of \( \mathbf{N}_\bullet^D(C) \) supplied by Example 2.3.1.16, we see an extension of \( u \) to a 3-simplex of the Duskin nerve \( \mathbf{N}_\bullet^D(C) \) can be identified with a 2-morphism \( \mu_{320} : f_{32} \circ f_{20} \Rightarrow f_{30} \) satisfying the equation

\[
\mu_{320}( \text{id}_{f_{32}} \circ \mu_{210} ) = \mu_{310}( \mu_{321} \circ \text{id}_{f_{10}} ) \alpha_{f_{32}, f_{21}, f_{10}}.
\]

Our assumption guarantees that \( \gamma = \mu_{210} \) is an isomorphism; it follows that the preceding equation has a unique solution, given by

\[
\mu_{320} = \mu_{310}( \mu_{321} \circ \text{id}_{f_{10}} ) \alpha_{f_{32}, f_{21}, f_{10}}( \text{id}_{f_{32}} \circ \mu_{210}^{-1} ).
\]

We now treat the case \( n = 4 \). For simplicity, we will assume that \( \ell = 2 \) (the cases \( \ell = 1 \) and \( \ell = 3 \) follow by a similar argument). To simplify the notation in what follows, we will denote the composition of a pair of 1-morphisms of \( C \) by \( hg \), rather than \( h \circ g \). Note that the horn \( \Lambda^n_\ell \) contains the 2-skeleton of \( \Delta^n \), so the morphism \( u \) can be identified with a collection of 2-morphisms \( \mu_{kji} : f_{kj} \circ f_{ji} \Rightarrow f_{ki} \). Using Example 2.3.1.16, we note that the extension of \( u \) to a 4-simplex of \( \mathbf{N}_\bullet^D(C) \) is automatically unique, and exists if and only if the outer cycle
commutes in the diagram

Here the unlabelled 2-morphisms are induced by the associativity constraints of $\mathcal{C}$. This follows from a diagram chase, since $\mu_{321} = \gamma$ is an isomorphism and each of the inner cycles of the diagram commutes (the 4-cycles commute by functoriality, the central 5-cycle commutes by the pentagon identity in $\mathcal{C}$, and the remaining 5-cycles commute by virtue of our assumption that $u$ is defined on the 0th, 1st, 3rd, and 4th face of the simplex $\Delta^4$).

**Proposition 2.3.3.2.** Let $\mathcal{C}$ be a 2-category and let $\sigma$ be a 2-simplex of the Duskin nerve $N^\bullet_{\mathcal{D}}(\mathcal{C})$, corresponding to a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\gamma} & & \downarrow{g} \\
\end{array}
\]

in the 2-category $\mathcal{C}$. Assume that the following condition is satisfied:

(*) Let $n \in \{3, 4\}$ and let $u : \Delta^n_1 \rightarrow N^\bullet_{\mathcal{D}}(\mathcal{C})$ be a map of simplicial sets such that $u|_{\Delta^2} = \sigma$; here we identify $\Delta^2$ with a simplicial subset of $\Delta^n_1 \subseteq \Delta^n$ via the inclusion map $[2] \hookrightarrow [n]$.

Then $u$ extends to an $n$-simplex of $N^\bullet_{\mathcal{D}}(\mathcal{C})$.

Then $\gamma$ is invertible.

**Proof.** Without loss of generality, we may assume that $\mathcal{C}$ is strictly unitary (Proposition 2.2.7.7). Applying (*) in the case $n = 3$, we can extend $\sigma$ to a 3-simplex of $N^\bullet_{\mathcal{D}}(\mathcal{C})$ which is
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represented by the pair of diagrams

\[ \begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \\
\downarrow{h} \quad \downarrow{\gamma} \quad \downarrow{\delta} \quad \downarrow{id_Z} \\
Z \end{array} \quad \begin{array}{c}
X \xrightarrow{id_g \circ f} Y \xrightarrow{g \circ f} Z \\
\downarrow{\mu_3,0} \quad \downarrow{\mu_4,3,0} \quad \downarrow{\mu_4,3,0} \quad \downarrow{id_{id_Z}} \\
Z.
\end{array} \]

It follows that \( \gamma \) admits a left inverse, given by the vertical composition \( \delta : h \Rightarrow g \circ f \). To show that this composition is also a right inverse, we apply \((\ast)\) in the case \( n = 4 \) to construct a 4-simplex \( \tau \) of \( N^D_\bullet(C) \) whose two-dimensional faces correspond to the 2-morphisms

\[
\begin{align*}
\mu_{2,1,0} &= \mu_{4,1,0} = \gamma & \mu_{3,1,0} &= \id_{g \circ f} & \mu_{3,2,0} &= \delta & \mu_{4,2,0} &= \id_h \\
\mu_{4,3,0} &= \gamma & \mu_{3,2,1} &= \mu_{4,2,1} &= \mu_{4,3,1} &= \id_g & \mu_{4,3,2} &= \id_{id_Z}.
\end{align*}
\]

The 3-simplex \( d_1(\tau) \) then witnesses the identity

\[
\mu_{4,2,0}(\mu_{4,3,2} \circ \id_h) = \mu_{4,3,0}(\id_{id_Z} \circ \mu_{3,2,0}),
\]

which shows that \( \delta \) is also a right inverse to \( \gamma \).

\[\square\]

2.3.4 Recovering a 2-Category from its Duskin Nerve

In §1.2.2 we proved that the nerve functor

\[ N_\bullet : \text{Cat} \to \text{Set}_\Delta \]

is fully faithful. This result generalizes to the setting of 2-categories:

\[\boxed{\text{Theorem 2.3.4.1 (Duskin [11]).}}\]

Let \( \mathcal{C} \) and \( \mathcal{D} \) be 2-categories. Then passage to the Duskin nerve induces a bijection

\[ \{\text{Strictly unitary lax functors } \mathcal{C} \to \mathcal{D}\} \to \{\text{Maps of simplicial sets } N^D_\bullet(\mathcal{C}) \to N^D_\bullet(\mathcal{D})\}. \]

In other words, the Duskin nerve functor \( N^D_\bullet : 2\text{Cat}_{\text{Lax}} \to \text{Set}_\Delta \) is fully faithful.
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Remark 2.3.4.2. Combining Theorem 2.3.4.1, Theorem 2.3.2.1, and Remark 2.3.0.3, we see that the construction $\mathcal{C} \mapsto \mathbf{N}^D_{\bullet}(\mathcal{C})$ determines a fully faithful embedding from the ordinary category of $(2,1)$-categories (where morphisms are strictly unitary functors in the sense of Definition 2.2.4.16) to the ordinary category of $\infty$-categories (where morphisms are functors in the sense of Definition 1.4.0.1).

Remark 2.3.4.3. In [11], Duskin proves a stronger version of Theorem 2.3.4.1, which also identifies the essential image of the functor $\mathbf{N}^D_{\bullet}: 2\text{Cat}_{\text{ULax}} \rightarrow \text{Set}_\Delta$.

Example 2.3.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. We say that a lax monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is strictly unitary if the unit $\epsilon: 1_D \rightarrow F(1_C)$ is an identity morphism of $\mathcal{D}$. It follows from Theorem 2.3.4.1 and Remark 2.2.4.9 that the formation of classifying simplicial sets induces a bijection

$$\{\text{Strictly unitary lax monoidal functors } F: \mathcal{C} \rightarrow \mathcal{D}\} \xrightarrow{\sim} \{\text{Maps of simplicial sets } B_\bullet \mathcal{C} \rightarrow B_\bullet \mathcal{D}\}.$$

Corollary 2.3.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then passage to the Duskin nerve induces a bijection

$$\{\text{Strictly unitary functors } \mathcal{C} \rightarrow \mathcal{D}\} \xrightarrow{} \{\text{Maps } \mathbf{N}^D_{\bullet}(\mathcal{C}) \rightarrow \mathbf{N}^D_{\bullet}(\mathcal{D}) \text{ preserving thin 2-simplices}\}.$$

Proof. Combine Theorem 2.3.4.1 with Corollary 2.3.2.8.

Proof of Theorem 2.3.4.1. By virtue of Proposition 2.2.7.7, we may assume without loss of generality that the 2-categories $\mathcal{C}$ and $\mathcal{D}$ are strictly unitary (this assumption will simplify some of the notation in what follows). Let $U: \mathbf{N}^D_{\bullet}(\mathcal{C}) \rightarrow \mathbf{N}^D_{\bullet}(\mathcal{D})$ be a map of simplicial sets. Then:

- Each object $X$ of $\mathcal{C}$ can be identified with a vertex of the Duskin nerve $\mathbf{N}^D_{\bullet}(\mathcal{C})$ (Example 2.3.1.13), whose image under $U$ is a vertex of the Duskin nerve $\mathbf{N}^D_{\bullet}(\mathcal{D})$. This vertex can be identified with an object of $\mathcal{D}$, which we denote by $U_0(X)$.

- Each 1-morphism $f: X \rightarrow Y$ of $\mathcal{C}$ can be identified with an edge of the Duskin nerve $\mathbf{N}^D_{\bullet}(\mathcal{C})$ (Example 2.3.1.14), whose image under $U$ is an edge of the Duskin nerve $\mathbf{N}^D_{\bullet}(\mathcal{D})$. This edge can be identified with a 1-morphism of $\mathcal{D}$, which we will denote by $U_1(f): U_0(X) \rightarrow U_0(Y)$. 


• Let \( f : X \to Y, \ g : Y \to Z, \) and \( h : X \to Z \) be 1-morphisms of \( \mathcal{C} \), and let \( \gamma : g \circ f \Rightarrow h \) be a 2-morphism of \( \mathcal{C} \). The 2-morphism \( \gamma \) determines a 2-simplex of the Duskin nerve \( N^\bullet_D(\mathcal{C}) \) (Example \ref{duskinej}). The image of this 2-simplex under \( U \) is a 2-simplex of the Duskin nerve \( N^\bullet_D(\mathcal{D}) \), which we can identify with a 2-morphism \( U_2(\gamma) : U_1(g) \circ U_1(f) \Rightarrow U_1(h) \) in \( \mathcal{D} \). Beware that this notation is slightly abusive: the 2-morphism \( U_2(\gamma) \) is a priori dependent not only on \( \gamma \), but also on the factorization of the source of \( \gamma \) as a composition \( g \circ f \).

Let \( F : \mathcal{C} \to \mathcal{D} \) be a strictly unitary lax functor. Unwinding the definitions, we see that the induced map of simplicial sets \( N^\bullet_D(F) : N^\bullet_D(\mathcal{C}) \to N^\bullet_D(\mathcal{D}) \) coincides with \( U \) if and only if the following conditions are satisfied:

(0) For every object \( X \in \mathcal{C} \), we have \( F(X) = U_0(X) \) (as objects of \( \mathcal{D} \)).

(1) For every 1-morphism \( f : X \to Y \) in \( \mathcal{C} \), we have \( F(f) = U_1(f) \) (as 1-morphisms from \( F(X) = U_0(X) \) to \( F(Y) = U_0(Y) \) in \( \mathcal{D} \)).

(2) For every triple of 1-morphisms \( f : X \to Y, \ g : Y \to Z, \) and \( h : X \to Z \) in \( \mathcal{C} \) and every 2-morphism \( \gamma : g \circ f \Rightarrow h \), the 2-morphism \( U_2(\gamma) : U_1(g) \circ U_1(f) \Rightarrow U_1(h) \) of \( \mathcal{D} \) is given by the (vertical) composition

\[
U_1(g) \circ U_1(f) = F(g) \circ F(f) \xrightarrow{\mu_{g,f}} F(g \circ f) \xrightarrow{F(\gamma)} F(h) = U_1(h),
\]

Let us note two special cases of condition (2). Taking \( h = g \circ f \) and \( \gamma : g \circ f \Rightarrow h \) to be the identity 2-morphism, we obtain the following:

\((2_0)\) For every pair of composable 1-morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \) of \( \mathcal{C} \), the composition constraint \( \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f) \) coincides with the 2-morphism \( U_2(\text{id}_Y f) \).

Taking \( g \) to be the identity morphism \( \text{id}_Y : Y \to Y \) and invoking our assumption that \( \mathcal{C} \) and \( \mathcal{D} \) are strictly unitary, we also obtain:

\((2_1)\) For every pair of 1-morphisms \( f, h : X \to Y \) in \( \mathcal{C} \) and every 2-morphism \( \gamma : f \Rightarrow h \), we have

\[
U_2(\gamma) = F(\gamma)\mu_{\text{id}_Y f} = F(\gamma)
\]

(here the second identity follows from Remark \ref{rem2}, since the 2-categories \( \mathcal{C} \) and \( \mathcal{D} \) are strictly unitary).

We wish to show that there is a unique strictly unitary lax functor \( F : \mathcal{C} \to \mathcal{D} \) satisfying conditions (0), (1), and (2). The uniqueness is clear: by virtue of the analysis above, the functor \( F \) must be given on objects, 1-morphisms, and 2-morphisms of \( \mathcal{C} \) by the formulae

\[
F(X) = U_0(X) \quad F(f) = U_1(f) \quad F(\gamma) = U_2(\gamma)
\]
(where, in the third formula, we identify the domain of each 2-morphism $\gamma : f \Rightarrow h$ in $\text{Hom}_C(X,Y)$ with the composition $\text{id}_Y \circ f$, and the composition constraint $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ must be given by $\mu_{g,f} = U_2(\text{id}_{gof})$. To complete the proof, it will suffice to show that these formulae supply a well-defined lax functor $F : C \Rightarrow D$, and that $F$ satisfies condition (2) above (note that $F$ satisfies conditions (0) and (1) by construction).

We first show that $F$ satisfies condition (2). Suppose we are given a triple of 1-morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$, together with a 2-morphism $\gamma : g \circ f \Rightarrow h$ in the 2-category $C$. Consider the map $\partial\Delta^3 \to N^\bullet_D(C)$ represented by the pair of diagrams

(see Example 2.3.1.16). Using the identity $\alpha_{\text{id}_Z,g,f} = \text{id}_{gof}$ (Remark 2.2.7.3), we see that these diagrams satisfy the compatibility condition of Example 2.3.1.16 and can therefore be regarded as a 3-simplex of $N^\bullet_D(C)$. Applying the map of simplicial sets $U$, we deduce that the diagrams

...
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determine a 3-simplex of $N_{\bullet}^{D}(D)$: that is, we have a commutative diagram

$$
\begin{array}{c}
\text{id}_{F(Z)} \circ (F(g) \circ F(f)) \\
\text{id}_{Z} \circ F(g \circ f)
\end{array}
\xrightarrow{\alpha_{id_{F(Z)}, F(g), F(f)}}
\begin{array}{c}
(id_{F(Z)} \circ F(g)) \circ F(f) \\
F(g) \circ F(f)
\end{array}
\xleftarrow{\mu_{g,f}}
\begin{array}{c}
\text{id}_{F(Z)} \circ (F(g) \circ F(f)) \\
F(g) \circ F(f)
\end{array}
\xrightarrow{id}
\begin{array}{c}
\text{id}_{Z} \circ F(g \circ f) \\
F(h).
\end{array}
$$

By virtue of Remark 2.2.7.3, we see that this is equivalent to the identity $U_{2}(\gamma) = F(\gamma)\mu_{g,f}$ asserted by (2).

Note that from condition (2), we can deduce that $F$ satisfies the dual of condition (2): that is, for every 2-morphism $\gamma : g \Rightarrow h$ in $\text{Hom}_{C}(X,Y)$, we have $F(\gamma) = U_{2}(\gamma)$, where the right hand side is computed by regarding $\gamma$ as a 2-morphism with domain $g \circ \text{id}_{X}$. It follows that the construction of $F$ from $U$ is invariant under the operation of replacing $C$ and $D$ by the opposite 2-categories $C^{\text{op}}$ and $D^{\text{op}}$ (this will be useful in what follows, since it reduces the number of identities that we need to check).

We now show that, for every pair of objects $X, Y \in C$, the construction of $F$ on 1-morphisms and 2-morphisms determines a functor $\text{Hom}_{C}(X,Y) \rightarrow \text{Hom}_{D}(F(X),F(Y))$. For this, we must establish the following:

1. For each 1-morphism $f : X \rightarrow Y$ in $C$, we have $F(id_{f}) = \text{id}_{F(f)}$ (as 2-morphisms from $F(f)$ to itself in $D$). By definition, this is equivalent to the identity $U_{2}(id_{f}) = \text{id}_{F(f)}$, which follows from the compatibility of the map $U : N_{\bullet}^{D}(C) \rightarrow N_{\bullet}^{D}(D)$ with the degeneracy operators

$$s_{1} : N_{1}^{D}(C) \Rightarrow N_{2}^{D}(C) \quad s_{1} : N_{1}^{D}(D) \Rightarrow N_{2}^{D}(D).$$

2. For every triple of 1-morphisms $f, g, h : X \Rightarrow Y$ in $C$ and every pair of 2-morphisms $\gamma : f \Rightarrow g$, $\delta : g \Rightarrow h$, we have $F(\delta \gamma) = F(\delta)F(\gamma)$. To prove this, consider the map $\partial \Delta^{3} \rightarrow N_{\bullet}^{D}(C)$ represented by the pair of diagrams

$$
\begin{array}{c}
X \\
\xrightarrow{f} Y
\end{array}
\xrightarrow{\gamma}
\begin{array}{c}
Y \\
\xrightarrow{\text{id}_{Y}} Y
\end{array}
\xleftarrow{\delta}
\begin{array}{c}
X \\
\xleftarrow{h} Y
\end{array}
$$
(see Example 2.3.1.16). It follows from Remark 2.2.7.3 that the associativity constraint \( \alpha_{\text{id}_Y, \text{id}_Y, f} \) is the identity, so that the diagrams above satisfy the compatibility condition of Example 2.3.1.16 and therefore determine a 3-simplex of \( N^\bullet_D(C) \). Applying the map of simplicial sets \( U \), we deduce that there exists a 3-simplex of the Duskin nerve \( N^\bullet_D \) whose boundary is given by the diagrams

Using the criterion of Example 2.3.1.16, we see that this is equivalent to the identity \( F(\delta \gamma) = F(\delta) F(\gamma) \).

We now show that, for every triple of objects \( X, Y, Z \in C \), the composition constraints \( \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f) \) depends functorially on \( f \in \text{Hom}_C(X,Y) \) and \( g \in \text{Hom}_C(Y,Z) \). We will argue that for fixed \( f \), the construction \( g \mapsto \mu_{g,f} \) is functorial; functoriality in \( g \) will then follow by symmetry. Suppose we are given a 2-morphism \( \gamma : g \Rightarrow h \) in \( C \); we wish to show that the diagram \( \tau \):
commutes in the category $\text{Hom}_D(F(X), F(Z))$. To prove this, we consider the map $\partial \Delta^3 \to \mathbf{N}^D_\bullet(C)$ represented by the pair of diagrams

\[ Y \ar{r}{g} & Z \ar{d}{\gamma \circ \text{id}_f} \ar{r}{\text{id}_Z} & Z \\
X \ar{r}{f} \ar{ur}{\text{id}_{g \circ f}} & Y \ar{ur}{g \circ f} \ar{ur}{\text{id}_{g \circ f}} \ar{r}{\text{id}_Z} & Z \ar{ur}{\gamma \circ \text{id}_f} \ar{ur}{\text{id}_Z} \\
X \ar{r}{f} \ar{ur}{\text{id}_{h \circ f}} & Y \ar{ur}{g} \ar{ur}{h} \ar{ur}{\text{id}_Z} \ar{ur}{\gamma} \ar{ur}{h \circ f} \ar{r}{\text{id}_Z} & Z \ar{ur}{\gamma \circ \text{id}_f} \ar{ur}{\text{id}_Z} \\
X \ar{r}{h} \ar{ur}{\text{id}_X} & Y \ar{ur}{g} \ar{ur}{h} \ar{ur}{\text{id}_Z} \ar{ur}{\gamma} \ar{ur}{h \circ f} \ar{r}{\text{id}_Z} & Z \ar{ur}{\gamma \circ \text{id}_f} \ar{ur}{\text{id}_Z} \\
\]

Using the identity $\alpha_{\text{id}_Z, g, f} = \text{id}_{g \circ f}$ supplied by Remark 2.2.7.3 we see that this diagram defines a 3-simplex of $\mathbf{N}^D_\bullet(C)$. Applying the map of simplicial sets $U$, we deduce that there is a 3-simplex of $\mathbf{N}^D_\bullet(D)$ whose boundary is represented by the pair of diagrams

\[ F(Y) \ar{r}{F(g)} & F(Z) \ar{d}{\text{id}_{F(Z)}} \ar{r}{\text{id}_{F(Z)}} & F(Z) \\
F(f) \ar{ur}{F(\mu_{g, f})} \ar{ur}{F(g \circ f)} \ar{ur}{F(\gamma \circ \text{id}_f)} & F(h) \ar{d}{\text{id}_{F(Z)}} \ar{d}{\text{id}_{F(Z)}} \ar{d}{\text{id}_{F(Z)}} & F(Z) \\
F(X) \ar{r}{F(h \circ f)} & F(h) \ar{ur}{F(\mu_{h, f})} \ar{ur}{F(f)} \ar{ur}{F(\gamma)} \ar{ur}{F(h \circ f)} \ar{r}{F(h \circ f)} & F(Z). \\
\]

This translates to the commutativity of the diagram

\[ \text{id}_{F(Z)} \circ F(g) \circ F(f) \ar{rr}{\alpha_{\text{id}_{F(Z)}, F(g), F(f)}} & & (\text{id}_{F(Z)} \circ F(g)) \circ F(f) \ar{rr}{\text{id}_{F(Z)} \circ F(g \circ f)} & & F(h \circ f), \]

\[ F(\gamma \circ \text{id}_f) \ar{rr}{\mu_{h, f}} & & \text{id}_{F(Z)} \circ F(g \circ f) \ar{rr}{\text{id}_{F(Z)} \circ F(g \circ f)} & & F(h \circ f). \]
which (again by virtue of Remark 2.2.7.3) is equivalent to the commutativity of the diagram \( \tau \).

To complete the proof, it will suffice to show that \( F \) and \( \mu \) satisfy conditions (a), (b), and (c) of Definition 2.2.4.5. Condition (a) is immediate from the construction, and (b) follows by symmetry. To verify (c), suppose we are given a triple of composable 1-morphisms \( W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \) in the 2-category \( \mathcal{C} \). Consider the 3-simplex of \( \mathbf{N}^\bullet_\mathcal{D}(\mathcal{C}) \) represented by the pair of diagrams

\[
\begin{array}{c}
\begin{tikzcd}
W & X \arrow{r}{g} & Z \\
& Y \arrow{u}{f} \arrow{ul}{\id_{gof}} \arrow{ur}{(hog)of}
\end{tikzcd}
\end{array}
\]

Applying \( U \), we obtain a 3-simplex of \( \mathbf{N}^\bullet_\mathcal{D}(\mathcal{D}) \) represented by the pair of diagrams

\[
\begin{array}{c}
\begin{tikzcd}
F(W) & F(X) \arrow{r}{F(g)} & F(Z) \\
& F(Y) \arrow{ul}{F(f)} \arrow{ur}{F(h)} \arrow{ul}{F((hog)of)}
\end{tikzcd}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzcd}
F(W) & F(X) \arrow{r}{F(g)} & F(Y) \\
& F(Z) \arrow{ul}{F(f)} \arrow{ur}{F(h)} \arrow{ul}{F((hog)of)}
\end{tikzcd}
\end{array}
\]
which is equivalent to the commutativity of the pentagon appearing in the diagram

\[
\begin{array}{ccc}
F(h) \circ (F(g) \circ F(f)) & \xrightarrow{\alpha_{F(h),F(g),F(f)}} & (F(h) \circ F(g)) \circ F(f) \\
\downarrow \text{id}_{F(h)} \circ \mu_{g,f} & & \downarrow \mu_{h,g} \circ \text{id}_{F(f)} \\
F(h) \circ (g \circ f) & \xrightarrow{U_2(\alpha_{h,g,f})} & F(h \circ g) \circ F(f) \\
\downarrow \mu_{h,g \circ f} & & \downarrow \mu_{h,g} \\
F((h \circ g) \circ f) & \xrightarrow{F(\alpha_{h,g,f})} & F((h \circ g) \circ f)
\end{array}
\]

in the category \(\text{Hom}_D(F(W), F(Z))\). Since the triangle on the lower left commutes by virtue of (2), it follows that the outer cycle of the diagram commutes, as desired.

\(\Box\)

### 2.3.5 Twisted Arrows and the Nerve of \(\text{Corr}(\mathcal{C})^c\)

Let \(\mathcal{E}\) be a category which admits fiber products and let \(\text{Corr}(\mathcal{E})\) denote the 2-category of correspondences in \(\mathcal{E}\) (Example 2.2.2.1). Our goal in this section is to give an explicit description of the Duskin nerve of the conjugate 2-category \(\text{Corr}(\mathcal{E})^c\) (Corollary 2.3.5.8 and Remark 2.3.5.9). We will obtain this description by formulating a universal property of \(\text{Corr}(\mathcal{E})^c\) as an object of the category \(2\text{Cat}_{\text{ULax}}\). First, we need a brief digression.

**Construction 2.3.5.1 (The Twisted Arrow Category).** Let \(\mathcal{C}\) be a category. We define a new category \(\text{Tw}(\mathcal{C})\) as follows:

- An object of \(\text{Tw}(\mathcal{C})\) is a morphism \(f : C \to D\) in \(\mathcal{C}\).

- Let \(f : C \to D\) and \(f' : C' \to D'\) be objects of \(\text{Tw}(\mathcal{C})\). A morphism from \(f\) to \(f'\) in \(\text{Tw}(\mathcal{C})\) is a pair of morphisms \(u : C \to C', v : D' \to D\) in \(\mathcal{C}\) satisfying \(f = v \circ f' \circ u\), so that we have a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow u & & \downarrow v \\
C' & \xrightarrow{f'} & D'
\end{array}
\]

- Let \(f : C \to D\), \(f' : C' \to D'\), and \(f'' : C'' \to D''\) be objects of \(\text{Tw}(\mathcal{C})\). If \((u, v)\) is a morphism from \(f\) to \(f'\) in \(\text{Tw}(\mathcal{C})\) and \((u', v')\) is a morphism from \(f'\) to \(f''\) in \(\mathcal{C}\), then the composition \((u', v') \circ (u, v)\) in \(\text{Tw}(\mathcal{C})\) is the pair \((u' \circ u, v \circ v')\).
We will refer to $\text{Tw}(\mathcal{C})$ as the *twisted arrow category* of $\mathcal{C}$.

**Remark 2.3.5.2.** Let $[1] = \{0 < 1\}$ denote a linearly ordered set with two elements. For any category $\mathcal{C}$, we can identify morphisms of $\mathcal{C}$ with functors $F : [1] \to \mathcal{C}$. The collection of such functors can be organized into a category $\text{Fun}([1], \mathcal{C})$, which we refer to as the *arrow category* of $\mathcal{C}$. The arrow category $\text{Fun}([1], \mathcal{C})$ has the same objects as the twisted arrow category $\text{Tw}(\mathcal{C})$. However, the morphisms are different: if $f : C \to D$ and $f' : C' \to D'$ are morphisms of $\mathcal{C}$, then morphisms from $f$ to $f'$ in $\text{Fun}([1], \mathcal{C})$ can be identified with commutative diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow & & \downarrow \\
C' & \xrightarrow{f'} & D'.
\end{array}
$$

**Remark 2.3.5.3.** Let $\mathcal{C}$ be a category and let $\text{Tw}(\mathcal{C})$ denote its twisted arrow category. There is an evident forgetful functor $\text{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\text{op}}$, given on objects by the construction $(f : C \to D) \mapsto (C, D)$.

**Example 2.3.5.4.** Let $Q$ be a partially ordered set, which we regard as a category. Then the twisted arrow category $\text{Tw}(Q)$ can be identified (via the forgetful functor of Remark 2.3.5.3) with the partially ordered set

$$
\{(p, q) \in Q \times Q^{\text{op}} : p \leq q\} \subseteq Q \times Q^{\text{op}}.
$$

**Remark 2.3.5.5** (Functoriality). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of categories. Then $F$ induces a functor of twisted arrow categories $\text{Tw}(F) : \text{Tw}(\mathcal{C}) \to \text{Tw}(\mathcal{D})$. This construction is compatible with composition, and can therefore be regarded as a functor $\text{Tw} : \text{Cat} \to \text{Cat}$ from the category of (small) categories to itself.

**Construction 2.3.5.6.** Let $\mathcal{C}$ and $\mathcal{E}$ be categories, where $\mathcal{E}$ admits fiber products, and let $F : \text{Tw}(\mathcal{C}) \to \mathcal{E}$ be a functor. We define a strictly unitary lax functor $F^+ : \mathcal{C} \to \text{Corr}(\mathcal{E})^c$ as follows:

- For each $C \in \mathcal{C}$, we define $F^+(C) = F(\text{id}_C)$; here we regard the identity morphism $\text{id}_C : C \to C$ as an object of the twisted arrow category $\text{Tw}(\mathcal{C})$.

- For each morphism $f : C \to D$ in $\mathcal{C}$, we define $F^+(f)$ to be the 1-morphism of $\text{Corr}(\mathcal{E})^c$ given by the correspondence

$$
F(\text{id}_C) \xleftarrow{F(\text{id}_C; f)} F(f) \xrightarrow{F(f; \text{id}_D)} F(\text{id}_D);
$$

this determines the values of $F^+$ on 2-morphisms, since every 2-morphism in $\mathcal{C}$ is an identity 2-morphism.
For every pair of composable morphisms \( C \xrightarrow{f} D \xrightarrow{g} E \), the composition constraint \( \mu_{g,f} : F^+(g) \circ F^+(f) \Rightarrow F^+(g \circ f) \) is the 2-morphism of \( \text{Corr}(\mathcal{C})^c \) corresponding to the map \( F(g \circ f) \to F(f) \times_{F(id_D)} F(g) \) classified by the commutative diagram

\[
\begin{array}{ccc}
F(g \circ f) & \xrightarrow{F(id_{C},g)} & F(g) \\
\downarrow F(id_{C},g) & & \downarrow F(id_{D}) \\
F(f) & \xleftarrow{F(id_{D},g)} & F(g)
\end{array}
\]

in the category \( \mathcal{E} \).

**Theorem 2.3.5.7.** Let \( \mathcal{C} \) and \( \mathcal{E} \) be categories where \( \mathcal{E} \) admits fiber products. Then Construction 2.3.5.6 induces a bijection of sets

\[
\{ \text{Functors } F : \text{Tw} \to \mathcal{E} \} \xrightarrow{\sim} \{ \text{Strictly Unitary Lax Functors } F^+ : \mathcal{C} \to \text{Corr}(\mathcal{E})^c \}.
\]

**Corollary 2.3.5.8.** Let \( \mathcal{E} \) be a category which admits fiber products and let \( \text{Corr}(\mathcal{E}) \) denote the 2-category of correspondences in \( \mathcal{E} \) (Example 2.2.2.1). Then the simplicial set \( N^D_{\bullet}(\text{Corr}(\mathcal{E})^c) \) is given concretely by the formula

\[
([n] \in \Delta) \mapsto \{ \text{Functors } \text{Tw}([n]) \to \mathcal{E} \}.
\]

**Remark 2.3.5.9.** Let \( \mathcal{E} \) be a category which admits fiber products. Combining Corollary 2.3.5.8 with Example 2.3.5.4, we see that \( n \)-simplices of the Duskin nerve \( N^D_{\bullet}(\text{Corr}(\mathcal{E})^c) \) can be identified with diagrams

\[
\{(i,j) \in [n] \times [n]^{op} : i \leq j \} \to \mathcal{E},
\]

which we can represent graphically as

\[
\begin{array}{ccc}
X_{0,n} & \xleftarrow{X_{0,0}} & X_{1,1} & \cdots & \xleftarrow{X_{n-1,n-1}} & X_{n,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
X_{0,1} & \xleftarrow{X_{0,0}} & X_{1,1} & \cdots & \xleftarrow{X_{n-1,n-1}} & X_{n,n} \\
X_{0,0} & & & & & \\
\end{array}
\]

In §[?], we will use this description to extend the definition of \( \text{Corr}(\mathcal{E}) \) to the case where \( \mathcal{E} \) is an \( \infty \)-category.
Example 2.3.5.10. Let \( \mathcal{E} \) be a category which admits fiber products. Then 2-simplices \( \sigma \) of the Duskin nerve \( N^D_\bullet(\text{Corr}(\mathcal{E})^c) \) can be identified with commutative diagrams

\[
\begin{array}{ccc}
X_{0,2} & \rightarrow & X_{1,2} \\
\downarrow & & \downarrow \\
X_{0,1} & \rightarrow & X_{1,1} \\
\downarrow & & \downarrow \\
X_{0,0} & \rightarrow & X_{1,2}
\end{array}
\]

in the category \( \mathcal{E} \). Such a diagram corresponds to a thin 2-simplex of \( N^D_\bullet(\text{Corr}(\mathcal{E})^c) \) (in the sense of Definition 2.3.2.3) if and only if the square appearing in the diagram is a pullback: that is, it induces an isomorphism \( X_{0,2} \rightarrow X_{0,1} \times_{X_{1,1}} X_{1,2} \).

Proof of Theorem 2.3.5.7. Let \( \mathcal{C} \) and \( \mathcal{E} \) be categories, where \( \mathcal{E} \) admits fiber products, and let \( G : \mathcal{C} \rightarrow \text{Corr}(\mathcal{E})^c \) be a strictly unitary lax functor of 2-categories. For every morphism \( f : C \rightarrow D \) in the category \( \mathcal{C} \), we can identify \( G(f) \) with a correspondence from \( G(C) \) to \( G(D) \) in the category \( \mathcal{E} \), given by a diagram we will denote by \( G(C) \xleftarrow{u(f)} M(f) \xrightarrow{v(f)} G(D) \). Our assumption that \( G \) is strictly unitary guarantees the following:

\(*)\) For each object \( C \in \mathcal{C} \), the object \( M(\text{id}_C) \) is equal to \( G(C) \), and the maps \( u(\text{id}_C) : M(\text{id}_C) \rightarrow G(C) \) and \( v(\text{id}_C) : M(\text{id}_C) \rightarrow G(C) \) are the identity morphisms from \( G(C) \) to itself in the category \( \mathcal{E} \).

For every pair of composable 1-morphisms \( C \xrightarrow{f} D \xrightarrow{g} E \), the composition constraint \( \mu_{g,f} \) for the lax functor \( G \) can be identified with a morphism from \( M(g \circ f) \) to the fiber product \( M(f) \times_{G(D)} M(g) \) in the category \( \mathcal{E} \), or equivalently with a pair of morphisms

\[
p(g,f) : M(g \circ f) \rightarrow M(f) \quad q(g,f) : M(g \circ f) \rightarrow M(g)
\]

satisfying \( v(f) \circ p(g,f) = u(g) \circ q(g,f) \). The axioms for a lax functor (Definition 2.2.4.5) then translate to the following additional conditions:

\( a \) For every 1-morphism \( f : C \rightarrow D \) in the category \( \mathcal{C} \), \( p(\text{id}_D, f) \) is the identity morphism from \( M(f) \) to itself.

\( b \) For every 1-morphism \( f : C \rightarrow D \) in the category \( \mathcal{C} \), \( q(f, \text{id}_C) \) is the identity morphism from \( M(f) \) to itself.

\( c \) For every composable triple of 1-morphisms \( W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \) in the category \( \mathcal{C} \), we have

\[
p(h \circ g, f) = p(g,f) \circ p(h,g \circ f) \quad q(h,g \circ f) = q(h,g) \circ q(h \circ g, f)
\]
\[
q(g,f) \circ p(h,g \circ f) = p(h,g) \circ q(h \circ g, f).
\]
2.3. THE DUSKIN NERVE OF A 2-CATEGORY

We wish to show that there exists a unique functor of ordinary categories $F : \text{Tw}(\mathcal{C}) \to \mathcal{E}$ such that $G = F^+$, where $F^+$ is the lax functor associated to $F$ by Construction 2.3.5.6. For this condition to be satisfied, the functor $F$ must have the following properties:

(0) For each object $C \in \mathcal{C}$, we have $F(\text{id}_C) = G(C)$ (this guarantees that $G$ and $F^+$ coincide on objects).

(1) For each morphism $f : C \to D$ in $\mathcal{C}$ (regarded as an object of $\text{Tw}(\mathcal{C})$), we have $F(f) = M(f)$, and the morphisms $u(f)$ and $v(f)$ are given by $F(\text{id}_C, f)$ and $F(f, \text{id}_D)$, respectively (this guarantees that $G$ and $F^+$ coincide on 1-morphisms, and therefore also on 2-morphisms).

(2) For every pair of composable 1-morphisms $C \xrightarrow{f} D \xrightarrow{g} E$, the morphisms $p(g, f) : M(g \circ f) \to M(f)$ and $q(g, f) : M(g \circ f) \to M(g)$ are given by $F(\text{id}_C, g) : F(g \circ f) \to F(f)$ and $F(f, \text{id}_E) : F(g \circ f) \to F(g)$, respectively (this guarantees that the composition constraints on $G$ and $F^+$ coincide).

Note that the value of $F$ on each object of $\text{Tw}(\mathcal{C})$ is determined by condition (1). Moreover, if $(u, v)$ is a morphism from $f : C \to D$ to $f' : C' \to D'$ in the category $\text{Tw}(\mathcal{C})$, then condition (2) guarantees that $F(u, v)$ must be equal to the composition

$$F(f) = M(v \circ f' \circ u) \xrightarrow{p(v, f' \circ u)} M(f' \circ u) \xrightarrow{q(f', u)} M(f') = F(f').$$

This proves the uniqueness of the functor $F$.

To prove existence, we define $F$ on objects $f$ of $\text{Tw}(\mathcal{C})$ by the formula $F(f) = M(f)$, and on morphisms $(u, v) : f \to f'$ by the formula $F(u, v) = q(f', u) \circ p(v, f' \circ u)$. For any morphism $f : C \to D$ in $\mathcal{C}$, we can use (a) and (b) to compute

$$F(\text{id}_C, \text{id}_D) = q(f, \text{id}_C) \circ p(\text{id}_D, f \circ \text{id}_C) = \text{id}_{M(f)} \circ \text{id}_{M(f)} = \text{id}_{F(f)},$$

so that $F$ carries identity morphisms in $\text{Tw}(\mathcal{C})$ to identity morphisms in $\mathcal{E}$. To complete the proof that $F$ is a functor, we note that for every pair of composable morphisms

$$(f : C \to D) \xrightarrow{(u, v)} (f' : C' \to D') \xrightarrow{(u'', v'')} (f'' : C'' \to D'')$$

in $\text{Tw}(\mathcal{C})$, the identities given in (c) allow us to compute

$$F(u', v') \circ F(u, v) = q(f'', u') \circ p(v', f''u') \circ q(v' f''u', u) \circ p(v, v' f''u'u)$$
$$= q(f'', u') \circ q(f''u', u) \circ p(v', f''u'u) \circ p(v, v' f''u'u)$$
$$= q(f'', u'u) \circ p(v'v, f''u'u)$$
$$= F(u' \circ u, v \circ v').$$
We now complete the proof by showing that the functor $F$ satisfies conditions (0), (1), and (2). Condition (0) is an immediate consequence of $\ast$. To prove (2), we note that for any pair of composable morphisms $C \xrightarrow{f} D \xrightarrow{g} E$, identities $(a)$ and $(b)$ yield equalities

$$F(\text{id}_C, g) = q(f, \text{id}_C) \circ p(g, f \circ \text{id}_C) = p(g, f) \quad F(f, \text{id}_E) = q(g, f) \circ p(\text{id}_E, g \circ f) = q(g, f).$$

For (1), we note that if $f : C \to D$ is a morphism in $\mathcal{C}$, then we have

$$F(\text{id}_C, f) = q(\text{id}_C, \text{id}_C) \circ p(f, \text{id}_C \circ \text{id}_C) \quad = \quad \text{id}_{\mathcal{G}(C)} \circ p(f, \text{id}_C) \quad = \quad v(f) \circ p(f, \text{id}_C) \quad = \quad u(f) \circ q(f, \text{id}_C) \quad = \quad u(f)$$

and a similar calculation yields $F(f, \text{id}_D) = v(f)$.

We close this section by describing a variant of Theorem 2.3.5.7 which will be useful in §5.2.1 (see Example 5.2.1.11).

**Construction 2.3.5.11.** Let $\mathcal{C}$ be a category, let $\mathcal{D}$ be a 2-category containing objects $X$ and $Y$, and let $F : \text{Tw}(\mathcal{C})^{\text{op}} \to \text{Hom}_\mathcal{D}(X, Y)$ be a functor. We define a strictly unitary lax functor $U_F : [1] \times \mathcal{C} \to \mathcal{D}$ as follows:

1. The lax functor $U_F$ is given on objects by $U_F(0, A) = X$ and $U_F(1, A) = Y$ for each $A \in \mathcal{C}$.

2. Let $f : A \to B$ be a morphism in the category $\mathcal{C}$, which we also regard as an object of the twisted arrow category $\text{Tw}(\mathcal{C})$. For $0 \leq i \leq j \leq 1$, we let $f_{ji}$ denote the corresponding morphism from $(i, A)$ to $(j, B)$ in the product category $[1] \times \mathcal{C}$. Then the lax functor $U_F$ is given on 1-morphisms by the formula

$$U_F(f_{ji}) = \begin{cases} 
\text{id}_X & \text{if } i = j = 0 \\
\text{id}_Y & \text{if } i = j = 1 \\
F(f) & \text{if } 0 < i < j = 1.
\end{cases}$$

3. Let $f : A \to B$ and $g : B \to C$ be composable morphisms in the category $\mathcal{C}$, and let $0 \leq i \leq j \leq k \leq 1$. Then the composition constraint $\mu_{g_{kj}, f_{ji}}$ for the lax functor $U_F$ is given as follows:

- If $i = j = k = 0$, then $\mu_{g_{kj}, f_{ji}}$ is the unit constraint $\upsilon_X : \text{id}_X \circ \text{id}_X \xrightarrow{\sim} \text{id}_X$ of the 2-category $\mathcal{D}$. 


• If \( i = 0 \) and \( j = k = 1 \), then \( \mu^{g_{kj},f_{ji}} \) is given by the composition

\[
\text{id}_Y \circ F(f) \xrightarrow{\lambda_{F(f)}} F(f) \xrightarrow{F(id_A,g)} F(g \circ f),
\]

where \( \lambda_{F(f)} \) is the left unit constraint of Construction 2.2.11, and we regard the pair \((id_A, g)\) as an element of \( \text{Hom}_{\text{Tw}(C)}(g \circ f, f) \).

• If \( i = j = 0 \) and \( k = 1 \), then \( \mu^{g_{kj},f_{ji}} \) is given by the composition

\[
F(g) \circ \text{id}_X \xrightarrow{\rho_{F(g)}} F(g) \xrightarrow{F(f, \text{id}_C)} F(g \circ f),
\]

where \( \rho_{F(g)} \) is the right unit constraint of Construction 2.2.11, and we regard the pair \((f, \text{id}_C)\) as an element of \( \text{Hom}_{\text{Tw}(C)}(g \circ f, g) \).

• If \( i = j = k = 1 \), then \( \mu^{g_{kj},f_{ji}} \) is equal to the unit constraint \( \nu_Y : \text{id}_Y \circ \text{id}_Y \sim \text{id}_Y \) of the 2-category \( D \).

01GA Exercise 2.3.5.12. Show that Construction 2.3.5.11 is well-defined. That is, given a functor \( F : \text{Tw}(C)^{\text{op}} \to \text{Hom}_D(X,Y) \) as in Construction 2.3.5.11, show that there is a unique strictly unitary lax functor \( U_F \) satisfying properties (1), (2), and (3) of Construction 2.3.5.11.

01GB Theorem 2.3.5.13. Let \( C \) be a category and let \( D \) be a 2-category containing objects \( X \) and \( Y \). Then the assignment \( F \mapsto U_F \) of Construction 2.3.5.11 induces a monomorphism of sets

\[
\{ \text{Functors } F : \text{Tw}(C)^{\text{op}} \to \text{Hom}_D(X,Y) \} \rightarrow \{ \text{Strictly unitary lax functors } U : [1] \times C \to D \}.
\]

The image of this monomorphism consists of those strictly unitary lax functors \( U : [1] \times C \to D \) having the property that \( U|_{\{0\} \times C} \) and \( U|_{\{1\} \times C} \) are the constant functors taking the values \( X \) and \( Y \), respectively.

01GC Corollary 2.3.5.14. Let \( C \) be a category, let \( D \) be a 2-category containing objects \( X \) and \( Y \), and let \( N^D_\bullet(D) \) denote the Duskin nerve of \( D \). Then Construction 2.3.5.11 determines a bijection

\[
\{ \text{Functors } F : \text{Tw}(C)^{\text{op}} \to \text{Hom}_D(X,Y) \} \rightarrow \{ \text{Morphisms } N_\bullet(C) \to \{(X,Y)\} \times_{\text{Fun}(\partial \Delta^1, N^D_\bullet(D))} \text{Fun}(\Delta^1, N^D_\bullet(D)) \}.
\]
Proof. By definition, a morphism from $N_•(C)$ to $\text{Hom}_{N_•(\mathcal{D})}(X,Y)$ can be identified with a morphism of simplicial sets

$$U : N_•([1] \times C) \simeq \Delta^1 \times N_•(C) \to N_•(\mathcal{D})$$

such that $U|_{N_•([0] \times C)}$ and $U|_{N_•([1] \times D)}$ are the constant maps taking the values $X$ and $Y$, respectively. The desired result now follows by combining Theorems 2.3.5.13 and 2.3.4.1.

Proof of Theorem 2.3.5.13. Let $\mathcal{C}$ be an ordinary category, let $\mathcal{D}$ be a 2-category containing objects $X$ and $Y$, and let $U : [1] \times \mathcal{C} \to \mathcal{D}$ be a strictly unitary lax functor having the property that $U|_{[0] \times C}$ and $U|_{[1] \times C}$ are the constant functors taking the values $X$ and $Y$, respectively. We wish to show that there exists a unique functor of ordinary categories $F : \text{Tw}(\mathcal{C})^{\text{op}} \to \text{Hom}_{\mathcal{D}}(X,Y)$ such that $U$ is equal to the strictly unitary lax functor $U_F$ given by Construction 2.3.5.11. To prove this, we may assume without loss of generality that the 2-category $\mathcal{D}$ is strictly unitary (Proposition 2.2.7.7). Given a morphism $f : A \to B$ in the category $\mathcal{C}$ and a pair of integers $0 \leq i \leq j \leq 1$, we write $f_{ji} : (i, A) \to (j, B)$ for the corresponding morphism in the product category $[1] \times \mathcal{C}$. Unwinding the definitions, we see that the identity $U = U_F$ imposes the following requirements on the functor $F$:

1. Let $f : A \to B$ be a morphism in the category $\mathcal{C}$, which we identify with both with an object in the twisted arrow category $\text{Tw}(\mathcal{C})$. Then $F(f)$ is equal to $U(f_{10}) \in \text{Hom}_{\mathcal{D}}(X,Y)$.

2. Let $f : A \to B$ be a morphism in the category $\mathcal{C}$, and regard the pairs $(\text{id}_A, g)$ and $(f, \text{id}_C)$ element of $\text{Hom}_{\text{Tw}(\mathcal{C})}(g \circ f, f)$ and $\text{Hom}_{\text{Tw}(\mathcal{C})}(g \circ f, g)$, respectively. Then $F(\text{id}_A, g)$ and $F(f, \text{id}_C)$ are equal to the composition contraints $\mu_{g_{11},f_{10}}$ and $\mu_{g_{10},f_{00}}$ for the lax functor $U$, respectively.

3. Let $f : A \to B$ and $g : B \to C$ be composable morphisms in the category $\mathcal{C}$, and regard the pairs $(\text{id}_A, g)$ and $(f, \text{id}_C)$ element of $\text{Hom}_{\text{Tw}(\mathcal{C})}(g \circ f, f)$ and $\text{Hom}_{\text{Tw}(\mathcal{C})}(g \circ f, g)$, respectively. Then $F(\text{id}_A, g)$ and $F(f, \text{id}_C)$ are equal to the composition contraints $\mu_{g_{11},f_{10}}$ and $\mu_{g_{10},f_{00}}$ for the lax functor $U$, respectively.

We now establish the uniqueness of the functor $F$. The value of $F$ on objects is determined by condition (2). If $f : A \to B$ and $f' : A' \to B'$ are objects of the twisted arrow category $\text{Tw}(\mathcal{C})$, then an element of $\text{Hom}_{\text{Tw}(\mathcal{C})}(f', f)$ can be identified with a pair $(u, v)$ where $u \in \text{Hom}_\mathcal{C}(A', A)$ and $v \in \text{Hom}_\mathcal{C}(B, B')$ satisfy $f' = v \circ f \circ u$. In this case, $(u, v)$ is factors as a composition $(\text{id}_{A'}, v) \circ (u, \text{id}_B)$, so condition (3) guarantees the identity

$$F(u, v) = F(u, \text{id}_B') \circ F(\text{id}_A, v) = \mu_{(v \circ f)_{01}, u_{00}} \circ \mu_{v_{11}, f_{10}}.$$

This proves the uniqueness of $F$ on morphisms.

To prove existence, we define the functor $F$ on objects $f \in \text{Tw}(\mathcal{C})$ by setting $F(f) = U(f_{10})$, and on morphisms $(u, v) \in \text{Hom}_{\text{Tw}(\mathcal{C})}(f', f)$ by the formula

$$F(u, v) = \mu_{f_{10} \circ v_{00}, u_{00}} \circ \mu_{v_{11}, f_{10}}.$$
Note that this prescription automatically satisfies condition (2). Since $U$ is a strictly unitary functor between strictly unitary 2-categories, its composition constraints $\mu_{g,h}$ are the identity whenever either $g$ or $h$ is an identity morphism (Remark 2.2.7.5), which shows that $F$ satisfies condition (3) and carries identity morphisms to identity morphisms. We will complete the proof by showing that $F$ is compatible with composition. Let $f : A \to B$, $f' : A' \to B'$, and $f'' : A'' \to B''$ be morphisms in the twisted arrow category $\text{Tw}(\mathcal{C})$, and suppose we are given morphisms $(u, v) \in \text{Hom}_{\text{Tw}(\mathcal{C})}(f', f)$ and $(u', v') \in \text{Hom}_{\text{Tw}(\mathcal{C})}(f'', f')$. We wish to prove an equality $F(u \circ u', v' \circ v) = F(u', v') \circ F(u, v)$ of morphisms from $F(f)$ to $F(f'')$ in the category $\text{Hom}_\mathcal{D}(X,Y)$. Unwinding the definitions, this is equivalent to the commutativity of the outer cycle of the diagram

\[
\begin{array}{ccc}
F(vfu) & \xrightarrow{\mu_{vfu}} & \xrightarrow{\mu_{v'fu}} F(v'fu) \\
\downarrow{\mu_{(vf)01,00}} & & \downarrow{\mu_{(v'fu)10}} \\
F(vf) & \xrightarrow{\mu_{vf11,01}} & F(v'vf) \\
\downarrow{\mu_{(vf)01,00}} & & \downarrow{\mu_{(v'vf)10,00}} \\
F(f) & \xrightarrow{\mu_{(vf'u)11,01}} & F(v'vfu) \\
\downarrow{\mu_{(v'fu)10,00}} & & \downarrow{\mu_{(v'vfu)10,00}} \\
F(f) & \xrightarrow{\mu_{(vf'u)11,01}} & F(v'vfu) \\
\end{array}
\]

in the category $\text{Hom}_\mathcal{D}(X,Y)$. In fact, the entire diagram commutes. The commutativity of the upper square follows by applying property (c) of Definition 2.2.4.5 to the composable triple of morphisms

\[
(0, A') \xrightarrow{u'00} (0, A) \xrightarrow{(vf)01} (1, B') \xrightarrow{v'11} (1, B'')
\]

in the product category $[1] \times \mathcal{C}$. The commutativity of the lower left triangle follows by applying property (c) to the composable triple of morphisms

\[
(0, A) \xrightarrow{f10} (1, B) \xrightarrow{v11} (1, B') \xrightarrow{v'11} (1, B'')
\]

and noting that the composition constraint $\mu_{v'11,v11}$ is equal to the identity (by virtue of our assumption that the lax functor $U|_{\{1\} \times \mathcal{C}}$ is constant). Similarly, the commutativity of the lower right triangle follows by applying (c) to the composable triple of morphisms

\[
(0, A'') \xrightarrow{u''00} (0, A') \xrightarrow{u00} (0, A) \xrightarrow{(v'vf)10} (1, B'')
\]

and noting that the composition constraint $\mu_{u00,u'00}$ is equal to the identity (by virtue of our assumption that the lax functor $U|_{\{0\} \times \mathcal{C}}$ is constant).
2.3.6 The Duskin Nerve of a Strict 2-Category

Let $C$ be a strict 2-category (Definition 2.2.0.1). Then we can regard $C$ as a 2-category (in which the associativity and unit constraints are identity morphisms), and form the Duskin nerve $N^D(C)$ by applying Construction 2.3.1.1. However, the Duskin nerve of a strict 2-category admits a more direct description, which can be formulated entirely in terms of strict 2-categories (and strict functors between them). The proof is based on a construction which will play an important role in §2.4.3.

Construction 2.3.6.1 (The Path 2-Category of a Partially Ordered Set). Let $(Q, \leq)$ be a partially ordered set. We define a strict 2-category $\text{Path}_{(2)}[Q]$ as follows:

- The objects of $\text{Path}_{(2)}[Q]$ are the elements of $Q$.
- Given elements $x, y \in Q$, we let $\text{Hom}_{\text{Path}_{(2)}[Q]}(x, y)$ denote the partially ordered set of all finite linearly ordered subsets

$$S = \{ x = x_0 < x_1 < \cdots < x_n = y \} \subseteq Q$$

having least element $x$ and greatest element $y$ (ordered by inclusion). We regard the partially ordered set $\text{Hom}_{\text{Path}_{(2)}[Q]}(x, y)$ as a category, having a unique morphism $S \Rightarrow T$ when $S$ is contained in $T$.

- For every element $x \in Q$, the identity 1-morphism $\text{id}_x \in \text{Hom}_{\text{Path}_{(2)}[Q]}(x, x)$ is given by the singleton $\{ x \}$ (regarded as a linearly ordered subset of $Q$, having greatest and least element $x$).

- For every triple of objects $x, y, z \in Q$, the composition functor

$$\circ : \text{Hom}_{\text{Path}_{(2)}[Q]}(y, z) \times \text{Hom}_{\text{Path}_{(2)}[Q]}(x, y) \to \text{Hom}_{\text{Path}_{(2)}[Q]}(x, z)$$

is given on objects by the construction $(S, T) \mapsto S \cup T$.

We will refer to $\text{Path}_{(2)}[Q]$ as the path 2-category of $Q$.

Remark 2.3.6.2 (Comparison with the Path Category). Let $(Q, \leq)$ be a partially ordered set. We let $\text{Path}[Q]$ denote the underlying category of the strict 2-category $\text{Path}_{(2)}[Q]$. The category $\text{Path}[Q]$ can be described concretely as follows:

- The objects of $\text{Path}[Q]$ are the elements of $Q$.
- If $x$ and $y$ are elements of $Q$, then a morphism from $x$ to $y$ in $\text{Path}[Q]$ is given by a finite linearly ordered subset

$$S = \{ x = x_0 < x_1 < x_2 < \cdots < x_n = y \} \subseteq Q$$

having least element $x$ and largest element $y$. 
2.3. THE DUSKIN NERVE OF A 2-CATEGORY

Note that Path\([Q]\) can also be realized as the path category of a directed graph \(\text{Gr}(Q)\) (as defined in Construction 1.2.6.1). Here \(\text{Gr}(Q)\) denotes the underlying directed graph of the category \(Q\), given concretely by

\[
\text{Vert}(\text{Gr}(Q)) = Q \quad \text{Edge}(\text{Gr}(Q)) = \{(x, y) \in Q : x < y\}
\]

where we regard each ordered pair \((x, y) \in \text{Edge}(\text{Gr}(Q))\) as an edge with source \(s(x, y) = x\) and target \(t(x, y) = y\).

\[\text{Remark 2.3.6.3.}\]

Let \((Q, \leq)\) be a partially ordered set, which we regard as a category (having a unique morphism from \(x\) to \(y\) when \(x \leq y\)). Note that, for every pair of elements \(x, y \in Q\), the category \(\text{Hom}_{\text{Path}_Q(2)}(x, y)\) is empty unless \(x \leq y\). It follows that there is a unique (strict) functor \(\text{Path}_Q(2) \to Q\) which is the identity on objects.

\[\text{Notation 2.3.6.4.}\]

Let \((Q, \leq)\) be a partially ordered set. We let \(\text{Path}^c_{\text{Path}_Q(2)}[Q]\) denote the conjugate of the path 2-category \(\text{Path}_Q(2)[Q]\) (Construction 2.2.3.4). In the special case where \(Q = [n] = \{0 < 1 < \cdots < n\}\), we will denote \(\text{Path}_Q(2)[Q]\) and \(\text{Path}^c_{\text{Path}_Q(2)}[n]\) by \(\text{Path}^c_2[n]\) and \(\text{Path}^c_{\text{Path}_Q(2)}[n]\), respectively.

\[\text{Construction 2.3.6.5.}\]

Let \((Q, \leq)\) be a partially ordered set, which we regard as a category having a unique morphism \(e_{y,x}\) for every pair of elements \(x, y \in Q\) with \(x \leq y\). We define a strictly unitary lax functor \(T_Q : Q \to \text{Path}^c_{\text{Path}_Q(2)}[Q]\) as follows:

- On objects, the lax functor \(T_Q\) is given by \(T_Q(x) = x\).
- On 1-morphisms, the lax functor \(T_Q\) is given by \(T_Q(e_{y,x}) = \{x, y\} \in \text{Hom}_{\text{Path}^c_{\text{Path}_Q(2)}[Q]}(x, y)\) whenever \(x \leq y\) in \(Q\).
- For every triple of elements \(x, y, z \in Q\) satisfying \(x \leq y \leq z\), the composition constraint \(\mu_{z,y,x} : T_Q(e_{z,y}) \circ T_Q(e_{y,z}) \Rightarrow T_Q(e_{z,x})\) is the 2-morphism of \(\text{Path}^c_{\text{Path}_Q(2)}[Q]\) corresponding to the inclusion of linearly ordered sets

\[
T_Q(e_{z,x}) = \{x, z\} \subseteq \{x, y, z\} = \{y, z\} \cup \{x, y\} = T_Q(e_{y,x}) \circ T_Q(e_{z,y}).
\]

\[\text{Remark 2.3.6.6.}\]

Let \((Q, \leq)\) be a partially ordered set, let \(T_Q : Q \to \text{Path}^c_{\text{Path}_Q(2)}[Q]\) be the lax functor of Construction 2.3.6.5, and let \(F : \text{Path}^c_{\text{Path}_Q(2)}[Q] \to Q^c = Q\) be (the conjugate of) the functor of Remark 2.3.6.3 (so that \(F\) is the identity on objects). Then the composition

\[
Q \xrightarrow{T_Q} \text{Path}^c_{\text{Path}_Q(2)}[Q] \xrightarrow{F} Q
\]

is the identity functor from \(Q\) to itself. Beware that the composition

\[
\text{Path}^c_{\text{Path}_Q(2)}[Q] \xrightarrow{F} Q \xrightarrow{T_Q} \text{Path}^c_{\text{Path}_Q(2)}[Q]
\]
is not the identity (as a lax functor from $\text{Path}_c^{(2)}[Q]$ to itself). This composition carries each object of $\text{Path}_c^{(2)}[Q]$ to itself, but is given on 1-morphism by the construction $\{x_0 < x_1 < \cdots < x_n\} \mapsto \{x_0 < x_n\}$.

The 2-category $\text{Path}^{(2)}[Q]$ of Construction 2.3.6.1 is characterized by the following universal property:

**Theorem 2.3.6.7.** Let $Q$ be a partially ordered set and let $T_Q : Q \to \text{Path}_c^{(2)}[Q]$ be the lax functor of Construction 2.3.6.5. For every strict 2-category $C$, composition with $T_Q$ induces a bijection

$$\{\text{Strict functors } F^+ : \text{Path}_c^{(2)}[Q] \to C\} \leftrightarrow \{\text{Strictly unitary lax functors } F : Q \to C\}.$$

Before giving the proof of Theorem 2.3.6.7, let us note one of its consequences. The construction $[n] \mapsto \text{Path}_c^{(2)}[n]$ determines a functor from the simplex category $\Delta$ of Definition 1.1.1.2 to the (ordinary) category $2\text{Cat}_{\text{Str}}$ of strict 2-categories (Definition 2.2.5.5). We will view this functor as a cosimplicial object of $2\text{Cat}_{\text{Str}}$ which we denote by $\text{Path}_c^{(2)}[\bullet]$. Applying the construction of Variant 1.1.7.6, we obtain a functor $\text{Sing}\_\bullet^{\text{Path}_c^{(2)}} : 2\text{Cat}_{\text{Str}} \to \text{Set}_\Delta$, which carries each strict 2-category $C$ to the simplicial set $[n] \mapsto \text{Hom}_{2\text{Cat}_{\text{Str}}}([\text{Path}_c^{(2)}[n], C])$. Using Theorem 2.3.6.7, we can identify this construction with the Duskin nerve functor $\text{N}_D \bullet : 2\text{Cat}_{\text{Str}} \to \text{Set}_\Delta$. In particular, we have the following:

**Corollary 2.3.6.8.** For every strict 2-category $C$, there is a canonical isomorphism of simplicial sets

$$\text{Sing}\_\bullet^{\text{Path}_c^{(2)}}(C) \cong \text{N}_D \bullet(C),$$

given on $n$-simplices by composition with the lax functor $T_{[n]} : [n] \to \text{Path}[n]^c$ of Construction 2.3.6.5. In other words, the Duskin nerve $\text{N}_D \bullet(C)$ is given by

$$\text{N}_D \bullet(C) \cong \{\text{Strict functors } \text{Path}_c^{(2)}[n] \to C\}.$$
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Proof of Theorem 2.3.6.7. Let $C$ be a strict 2-category, let $Q$ be a partially ordered set, and let $F : Q \to C$ be a strictly unitary lax functor. We wish to show that $F$ factors uniquely as a composition

$$Q \xrightarrow{T_Q} \text{Path}^c_{(2)}[Q] \xrightarrow{F^+} C,$$

where $T_Q$ is the strictly unitary lax functor of Construction 2.3.6.5 and $F^+$ is a strict functor from $\text{Path}^c_{(2)}[Q]$ to $C$.

For every pair of elements $x, y \in Q$ satisfying $x \leq y$, we let $e_{y,x} : x \to y$ denote the unique morphism from $x$ to $y$ in the category $Q$, and for every triple $x, y, z \in Q$ satisfying $x \leq y \leq z$, we let $\mu_{z,y,x} : F(e_{z,y}) \circ F(e_{y,x}) \Rightarrow F(e_{z,x})$ denote the composition constraint for the lax monoidal functor $F$. Unwinding the definitions, we see that a strict functor $F^+ : \text{Path}^c_{(2)}[Q] \to C$ satisfies $F^+ \circ T_Q = F$ if and only if the following conditions are satisfied:

(0) For every element $x \in Q$, we have $F^+(x) = F(x)$ (as objects of the 2-category $C$).

(1) For every pair of elements $x, y \in Q$ satisfying $x \leq y$, we have $F^+(\{x, y\}) = F(e_{y,x})$ (as 1-morphisms from $F(x)$ to $F(y)$ in the strict 2-category $C$).

(2) For every triple of elements $x, y, z \in Q$ satisfying $x \leq y \leq z$, the functor $F^+$ carries the inclusion $\{x, z\} \subseteq \{x, y, z\}$ (regarded as a 2-morphism from $\{y, z\} \circ \{x, y\}$ to $\{x, z\}$ in the strict 2-category $\text{Path}^c_{(2)}[Q]$) to $F(\mu_{z,y,x})$ (regarded as a 2-morphism from $F(e_{z,y}) \circ F(e_{y,x})$ to $F(e_{z,x})$ in the strict 2-category $C$).

Note that, since we are requiring $F^+$ to be a strict functor, we can replace (1) by the following stronger condition:

(1’) For every nonempty finite linearly ordered subset $S = \{x_0 < x_1 < \cdots < x_n\} \subseteq Q$, the functor $F^+$ carries $S$ (regarded as a 1-morphism from $x_0$ to $x_n$ in the strict 2-category $\text{Path}^c_{(2)}[Q]$) to the composition $F(e_{x_n,x_{n-1}}) \circ \cdots \circ F(e_{x_1,x_0})$ (regarded as a 1-morphism from $F(x_0)$ to $F(x_n)$ in the strict 2-category $C$). In what follows, we will denote this composition by $F(S)$.

Let $S = \{x_0 < x_1 < \cdots < x_n\}$ be a nonempty finite linearly ordered subset of $Q$. For each $0 \leq i \leq j \leq n$, set $f_{j,i} = F(e_{x_j,x_i})$, which we regard as a 1-morphism from $F(x_i)$ to $F(x_j)$ in the 2-category $C$. Let $x_i$ be an element of $S$ which is neither the largest nor the smallest (so that $0 < i < n$). In this case, we let $\gamma_{S,x_i} : F(S) \Rightarrow F(S \setminus \{x_i\})$ denote the 2-morphism of $C$ given by the horizontal composition

$$\gamma_{S,x_i} = \text{id}_{f_{n,n-1}} \circ \cdots \circ \text{id}_{f_{i+2,i+1}} \circ F(\mu_{x_{i+1},x_i,x_{i-1}}) \circ \text{id}_{f_{i-1,i-2}} \circ \cdots \circ \text{id}_{f_{1,0}}.$$

More generally, given a sequence of distinct elements $s_1, s_2, \cdots, s_m \in S \setminus \{x_0, x_n\}$, we let $\gamma_{S,s_1,\ldots,s_m} : F(S) \Rightarrow F(S \setminus \{s_1, \ldots, s_m\})$ denote the 2-morphism of $C$ given by the vertical
composition
\[ F(S) \xrightarrow{\gamma_{S,s}} F(S \setminus \{s_1\}) \xrightarrow{\gamma_{S \setminus \{s_1\},s_2}} F(S \setminus \{s_1,s_2\}) \Rightarrow \cdots \Rightarrow F(S \setminus \{s_1,\ldots,s_m\}). \]
Since the strict functor \( F^+ \) is required to be compatible with vertical and horizontal composition, we can replace (2) by the following stronger condition:

(2') Let \( S = \{x_0 < x_1 < \cdots < x_n\} \) be a nonempty finite linearly ordered subset of \( Q \).
Then, for every sequence of distinct elements \( s_1, \ldots, s_m \in S \setminus \{x_0, x_n\} \), the functor \( F^+ \) carries the inclusion \( S \setminus \{s_1, \ldots, s_m\} \subseteq S \) (regarded as a 2-morphism from \( S \) to \( S \setminus \{s_1, \ldots, s_m\} \) in the strict 2-category \( \text{Path}^{\leq}_2(Q) \)) to the 2-morphism \( \gamma_{S,s_1,\ldots,s_m} \) (regarded as a 2-morphism from \( F(S) \) to \( F(S \setminus \{s_1, \ldots, s_m\}) \) in the strict 2-category \( C \)).

It is now clear that the functor \( F^+ \) is unique if it exists: its values on objects, 1-morphisms, and 2-morphisms of \( \text{Path}^{\leq}_2(Q) \) are determined by conditions (0), (1'), and (2'), respectively.
To prove existence, it will suffice to show that this prescription is well-defined: namely, that the 2-morphism \( \gamma_{S,s_1,\ldots,s_m} \) defined above depends only on the sets \( S \) and \( T = S \setminus \{s_1, \ldots, s_m\} \), and not on the order of the sequence \( (s_1, \ldots, s_m) \) (it then follows easily from the construction that the definition of \( F^+ \) on 2-morphisms is compatible with vertical and horizontal composition). Since the group of all permutations of the set \( \{s_1, \ldots, s_m\} \) is generated by transpositions of adjacent elements, it will suffice to show that we have

\[ \gamma_{S,s_1,\ldots,s_i-1,s_i,s_{i+1},s_{i+2},\ldots,s_m} = \gamma_{S,s_1,\ldots,s_i-1,s_{i+1},s_{i+2},\ldots,s_m} \]
for each \( 1 \leq i < m \). Replacing \( S \) by \( S \setminus \{s_1, \ldots, s_{i-1}\} \), we are reduced to proving that \( \gamma_{S,s,t} = \gamma_{S,t,s} \) whenever \( s < t \) are elements of \( S \setminus \{x_0, x_n\} \). We now distinguish two cases:

- Suppose that the elements \( s \) and \( t \) are non-consecutive elements of \( S \): that is, we have \( s = x_i \) and \( t = x_j \) where \( j > i + 1 \). In this case, we can identify both \( \gamma_{S,s,t} \) and \( \gamma_{S,t,s} \) with the horizontal composition

  \[ \text{id}_{f_{u,n-1}} \circ \cdots \circ F(\mu_{x_{j+1},x_j,x_{j-1}}) \circ \cdots \circ F(\mu_{x_{i+1},x_i,x_{i-1}}) \circ \cdots \circ \text{id}_{f_1,0}. \]

- Suppose that the elements \( s \) and \( t \) are consecutive: that is, we have \( S = \{x_0 < \cdots < r < s < t < u < \cdots < x_n\} \). In this case, to verify the identity \( \gamma_{S,s,t} = \gamma_{S,t,s} \), we can replace \( S \) by the subset \( \{r < s < t < u\} \) and thereby reduce to checking the commutativity of the diagram
in the category $\text{Hom}_C(F(r), F(u))$, which is the coherence condition required by the composition constraints for the lax functor $F$ (axiom $(c)$ of Definition 2.2.4.5).

\[ \square \]

2.4 Simplicial Categories

Let $\text{Top}$ denote the category of topological spaces. By definition, a morphism in the category $\text{Top}$ is a continuous function $f : X \to Y$. In homotopy theory, one is fundamentally concerned not only with continuous functions themselves, but also with homotopies between them: that is, continuous functions $h : [0,1] \times X \to Y$. More generally, for each $n \geq 0$, one can consider the set

$$\text{Hom}_{\text{Top}}(X,Y)_n = \{\text{Continuous functions } \sigma : |\Delta^n| \times X \to Y\};$$

here $|\Delta^n|$ denotes the topological simplex of dimension $n$. The sets $\{\text{Hom}_{\text{Top}}(X,Y)_n\}_{n \geq 0}$ can be assembled into a simplicial set $\text{Hom}_{\text{Top}}(X,Y)_\bullet$, and the construction $(X,Y) \mapsto \text{Hom}_{\text{Top}}(X,Y)_\bullet$ endows $\text{Top}$ with the structure of a simplicial category: that is, a category which is enriched over simplicial sets, in the sense of Definition 2.1.7.1. Much as the singular simplicial set $\text{Sing}_\bullet(X) = \text{Hom}_{\text{Top}}(\ast, X)_\bullet$ can be regarded as a combinatorial encoding of the homotopy type of an individual topological space $X$, the simplicial enrichment of $\text{Top}$ can be regarded as a combinatorial encoding of the homotopy theory of topological spaces.

Our goal in this section is to provide an introduction to the theory of simplicial categories. We begin in §2.4.1 by defining the notion of simplicial category (Definition 2.4.1.1). The collection of (small) simplicial categories can itself be organized into a category $\text{Cat}_\Delta$, in which the morphisms are given by simplicial functors (Definition 2.4.1.11). In §2.4.2 we provide many examples of how simplicial categories arise in nature: in particular, we explain that $\text{Cat}_\Delta$ can be regarded as an enlargement of the usual category $\text{Cat}$ of small categories (Example 2.4.2.3), and also of the category $2\text{Cat}_{\text{Str}}$ of strict 2-categories (Example 2.4.2.7).

Recall that to every category $C$ we can associate a simplicial set $N_\bullet(C)$ called the nerve of $C$ (Construction 1.2.1.1). In §2.4.3 we describe a generalization of this construction (due to Cordier) which associates to each simplicial category $C_\bullet$ a simplicial set $N_{hc}^\bullet(C)$ called the homotopy coherent nerve of $C_\bullet$ (Definition 2.4.3.5). This construction specializes to the ordinary nerve in the case where $C_\bullet$ is an ordinary category (and to the Duskin nerve in the case where $C_\bullet$ arises from a strict 2-category: see Example 2.4.3.11). It is particularly well-behaved in the special case where $C_\bullet$ is locally Kan (meaning that simplicial Hom-sets $\text{Hom}_C(X,Y)_\bullet$ are Kan complexes): in this case, a theorem of Cordier and Porter asserts that the homotopy coherent nerve $N_{hc}^\bullet(C)$ is an $\infty$-category (Theorem 2.4.5.1).

In §2.4.4 we show that the homotopy coherent nerve functor $N_{hc}^\bullet : \text{Cat}_\Delta \to \text{Set}_\Delta$ admits a left adjoint (Corollary 2.4.4.4). This left adjoint carries each simplicial set $S_\bullet$
to a simplicial category $\text{Path}[S]$, which we will refer to as the (simplicial) path category of $S$. The construction $S \mapsto \text{Path}[S]$ is a generalization of the classical path category studied in §1.2.6 when $S$ is the 1-dimensional simplicial set associated to a directed graph $G$, the simplicial category $\text{Path}[S]$ can be identified with the ordinary category $\text{Path}[G]$ of Construction 1.2.6.1 (see Proposition 2.4.4.7). For a general simplicial set $S$, the path category $\text{Path}[S]$ is a complicated object. However, in each fixed simplicial degree $m$ it is relatively simple: the ordinary category $\text{Path}[S]_m$ can be identified with the classical path category of a certain directed graph $G_m$ which can be described concretely in terms of the combinatorics of $S$ (Theorem 2.4.4.10). We will exploit this description in §2.4.5 to carry out the proof of Theorem 2.4.5.1 and again in §2.4.6 to compare the homotopy category of a (locally Kan) simplicial category $C$ to the homotopy category of its associated $\infty$-category $\text{N}_{hc}(C)$ (Proposition 2.4.6.8).

**Warning 2.4.0.1.** The ordinary nerve functor $C \mapsto \text{N}_*(C)$ determines a fully faithful embedding from the category $\text{Cat}$ of small categories to the category $\text{Set}_\Delta$ of simplicial sets (Proposition 1.2.2.1). However, the homotopy coherent nerve $\text{N}_{hc} : \text{Cat}_\Delta \to \text{Set}_\Delta$ is not fully faithful when regarded as a functor of ordinary categories. Phrased differently, the adjoint functors

$$\text{Set}_\Delta \xrightarrow{\text{N}_{hc}} \text{Cat}_\Delta \xleftarrow{\text{Path}[-]}$$

associate to each simplicial category $C$ a counit map $v : \text{Path}[\text{N}_{hc}(C)] \to C$, which is almost never an isomorphism of simplicial categories. However, we will see later that $v$ is a weak equivalence of simplicial categories whenever $C$ is locally Kan (§5). Moreover, the construction $C \mapsto \text{N}_{hc}(C)$ establishes an equivalence from the homotopy theory of (locally Kan) simplicial categories $C$ with the homotopy theory of $\infty$-categories (§5).

### 2.4.1 Simplicial Enrichment

Let $\text{Set}_\Delta$ denote the category of simplicial sets (Definition 1.1.1.12). Then $\text{Set}_\Delta$ admits Cartesian products (Remark 1.1.1.13), and can therefore be endowed with the Cartesian monoidal structure described in Example 2.1.3.2. We will use the term simplicial category to refer to a category which is enriched over $\text{Set}_\Delta$, in the sense of Definition 2.1.7.1. For the reader’s convenience, we spell this definition out in detail (and establish some notation we will use when discussing simplicial categories, which differs somewhat from the general conventions of §2.1.7).

**Definition 2.4.1.1** (Simplicial Categories). A simplicial category $C$ consists of the following data:

1. A collection $\text{Ob}(C)$, whose elements we refer to as objects of $C$. We will often abuse notation by writing $X \in C$ to indicate that $X$ is an element of $\text{Ob}(C)$. 

---

**OOJR**
(2) For every pair of objects \( X, Y \in \text{Ob}(\mathcal{C}) \), a simplicial set \( \text{Hom}_\mathcal{C}(X, Y)_\bullet \).

(3) For every triple of objects \( X, Y, Z \in \text{Ob}(\mathcal{C}) \), a morphism of simplicial sets
\[
c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y, Z)_\bullet \times \text{Hom}_\mathcal{C}(X, Y)_\bullet \to \text{Hom}_\mathcal{C}(X, Z)_\bullet,
\]
which we will refer to as the composition law.

(4) For every object \( X \in \text{Ob}(\mathcal{C}) \), a vertex \( \text{id}_X \in \text{Hom}_\mathcal{C}(X, X)_0 \), which we will refer to as the identity morphism of \( X \).

These data are required to satisfy the following conditions:

(A) For every quadruple of objects \( W, X, Y, Z \in \text{Ob}(\mathcal{C}) \), the diagram of simplicial sets
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(Y, Z)_\bullet \times \text{Hom}_\mathcal{C}(W, X)_\bullet & \xrightarrow{id \times c_{Y,X,W}} & \text{Hom}_\mathcal{C}(W, Y)_\bullet \\
\downarrow{c_{Z,Y,X} \times id} & & \downarrow{c_{Z,X,W}} \\
\text{Hom}_\mathcal{C}(Y, Z)_\bullet \times \text{Hom}_\mathcal{C}(W, Y)_\bullet & & \text{Hom}_\mathcal{C}(X, Z)_\bullet \times \text{Hom}_\mathcal{C}(W, X)_\bullet
\end{array}
\]
commutes (in other words, the composition law of (3) is associative).

(U) For every pair of objects \( X, Y \in \text{Ob}(\mathcal{C}) \), the maps of simplicial sets
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(X, Y)_\bullet \times \{\text{id}_X\} & \xrightarrow{id \times \{\text{id}_Y\}} & \text{Hom}_\mathcal{C}(X, Y)_\bullet \\
\downarrow{c_{Y,X,Y} \times \text{id}} & & \downarrow{c_{Y,Y,X}} \\
\text{Hom}_\mathcal{C}(X, Y)_\bullet \times \text{Hom}_\mathcal{C}(X, Y)_\bullet & & \text{Hom}_\mathcal{C}(X, Y)_\bullet \times \text{Hom}_\mathcal{C}(X, Y)_\bullet
\end{array}
\]
coincide with the projection maps onto the factor \( \text{Hom}_\mathcal{C}(X, Y)_\bullet \).

Opening Warning 2.4.1.2. The terminology of Definition 2.4.1.1 is not standard. Many authors use the term simplicial category to mean a simplicial object of the category \( \text{Cat} \), and the term simplicially enriched category to mean a category enriched over simplicial sets. These notions are closely related: see Remark 2.4.1.12.

Opening Construction 2.4.1.3. Let \( \mathcal{C}_\bullet \) be a simplicial category. For every nonnegative integer \( n \geq 0 \), we define an ordinary category \( \mathcal{C}_n \) as follows:

- The objects of \( \mathcal{C}_n \) are the objects of \( \mathcal{C}_\bullet \).
- Let \( X, Y \in \text{Ob}(\mathcal{C}_n) = \text{Ob}(\mathcal{C}_\bullet) \) be objects of \( \mathcal{C}_n \). A morphism from \( X \) to \( Y \) in the category \( \mathcal{C}_n \) is an \( n \)-simplex of the simplicial set \( \text{Hom}_\mathcal{C}(X, Y)_\bullet \). In other words, we have an equality of sets \( \text{Hom}_{\mathcal{C}_n}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)_n \).
For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}_n$, the composition $g \circ f : X \to Z$ is given by the image of the vertex $(g, f)$ under the map of simplicial sets

$$c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y, Z)_\bullet \times \text{Hom}_\mathcal{C}(X, Y)_\bullet \to \text{Hom}_\mathcal{C}(X, Z)_\bullet.$$ 

For every object $X \in \text{Ob}(\mathcal{C}_n) = \text{Ob}(\mathcal{C}_\bullet)$, the identity morphism from $X$ to itself in the category $\mathcal{C}_n$ is the $n$-simplex of $\text{Hom}_\mathcal{C}(X, X)_\bullet$ which corresponds to the composite map

$$\Delta^n \to \Delta^0 \xrightarrow{id_X} \text{Hom}_\mathcal{C}(X, X)_\bullet.$$ 

Example 2.4.1.4 (The Underlying Category of a Simplicial Category). Let $\mathcal{C}_\bullet$ be a simplicial category. We let $\mathcal{C} = \mathcal{C}_0$ denote the ordinary category obtained by applying Construction 2.4.1.3 in the case $n = 0$. We will refer to $\mathcal{C}$ as the underlying category of the simplicial category $\mathcal{C}_\bullet$. Note that $\mathcal{C}$ can also be obtained from $\mathcal{C}_\bullet$ by applying the general procedure described in Example 2.1.7.5.

We will sometimes abuse terminology by identifying a simplicial category $\mathcal{C}_\bullet$ with its underlying category $\mathcal{C}$. In particular, if $X$ and $Y$ are objects of $\mathcal{C}_\bullet$, we will write $\text{Hom}_\mathcal{C}(X, Y)_\bullet$ to denote the set $\text{Hom}_\mathcal{C}_0(X, Y) = \text{Hom}_\mathcal{C}(X, Y)_0$ of morphisms from $X$ to $Y$ in the category $\mathcal{C}$.

Example 2.4.1.5 (Topological Spaces). Let Top denote the category whose objects are topological spaces and whose morphisms are continuous functions. Then Top can be promoted to a simplicial category $\text{Top}_\bullet$: given a pair of topological spaces $X$ and $Y$, we define the simplicial set $\text{Hom}_\text{Top}(X, Y)_\bullet$ informally by the formula

$$\text{Hom}_\text{Top}(X, Y)_n = \text{Hom}_\text{Top}(|\Delta^n| \times X, Y).$$

In particular, a vertex of $\text{Hom}_\text{Top}(X, Y)_\bullet$ can be identified with a continuous function $f : X \to Y$. Moreover, for any topological space $Y$, we have a canonical isomorphism of simplicial sets $\text{Hom}_\text{Top}(\ast, Y)_\bullet \simeq \text{Sing}_\bullet(Y)$, where $\text{Sing}_\bullet(Y)$ is the singular simplicial set of Construction 1.1.7.1.

Let $\mathcal{C}$ be a category. Roughly speaking, a simplicial enrichment $\mathcal{C}_\bullet$ of $\mathcal{C}$ can be viewed as a datum which allows us to “do homotopy theory” in $\mathcal{C}$. For example, it allows us to define a notion of homotopy between morphisms of $\mathcal{C}$.

Definition 2.4.1.6. Let $\mathcal{C}_\bullet$ be a simplicial category, and let $f, g : X \to Y$ be two morphisms in the underlying category $\mathcal{C} = \mathcal{C}_0$ having the same source and target. A homotopy from $f$ to $g$ is an edge $h \in \text{Hom}_\mathcal{C}(X, Y)_1$ satisfying $d_1(h) = f$ and $d_0(h) = g$.

Example 2.4.1.7. Let $X$ and $Y$ be topological spaces and let $f, g : X \to Y$ be continuous functions, which we regard as morphisms in the simplicial category $\text{Top}_\bullet$ of Example 2.4.1.5.
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Then a homotopy from $f$ to $g$ in the sense of Definition 2.4.1.6 is a homotopy in the usual sense: a continuous function $h : [0, 1] \times X = |\Delta^1| \times X \to Y$ satisfying $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$.

In a general simplicial category $C$, the notion of homotopy (in the sense of Definition 2.4.1.6) need not be well-behaved: for example, the existence of a homotopy from $f$ to $g$ need not imply the existence of a homotopy from $g$ to $f$. To remedy the situation, it is convenient to restrict attention to a special class of simplicial categories:

**Definition 2.4.1.8.** Let $C_\bullet$ be a simplicial category. We will say that $C_\bullet$ is *locally Kan* if, for every pair of objects $X, Y \in C_\bullet$, the simplicial set $\text{Hom}_C(X, Y)_\bullet$ is a Kan complex (Definition 1.1.9.1).

**Remark 2.4.1.9.** Let $C_\bullet$ be a locally Kan simplicial category, and let $f, g : X \to Y$ be a pair of morphisms in the underlying category $C = C_0$ having the same source and target. Invoking Proposition 1.1.9.10 we see that the following conditions are equivalent:

(a) There exists a homotopy from $f$ to $g$, in the sense of Definition 2.4.1.6.

(b) The morphisms $f$ and $g$ belong to the same connected component of the Kan complex $\text{Hom}_C(X, Y)_\bullet$.

In particular, condition (a) defines an equivalence relation on the set $\text{Hom}_C(X, Y)$.

**Exercise 2.4.1.10.** Let $\text{Top}_\bullet$ be the simplicial category of Example 2.4.1.5. Show that $\text{Top}_\bullet$ is locally Kan (hint: generalize the proof of Proposition 1.1.9.8).

Specializing Definition 2.1.7.10 to the setting of simplicial enrichments, we obtain the following:

**Definition 2.4.1.11 (Simplicial Functors).** Let $C_\bullet$ and $D_\bullet$ be simplicial categories. A *simplicial functor* $F : C_\bullet \to D_\bullet$ consists of the following data:

1. For every object $X \in \text{Ob}(C_\bullet)$, and object $F(X) \in \text{Ob}(D_\bullet)$.

2. For every pair of objects $X, Y \in \text{Ob}(C_\bullet)$, a map of simplicial sets $F_{X,Y} : \text{Hom}_C(X, Y)_\bullet \to \text{Hom}_D(F(X), F(Y))_\bullet$.

These data are required to satisfy the following conditions:

- For every object $X \in \text{Ob}(C_\bullet)$, the map of simplicial sets $F_{X,X} : \text{Hom}_C(X, X)_\bullet \to \text{Hom}_D(F(X), F(X))_\bullet$ carries the vertex $\text{id}_X$ to the vertex $\text{id}_{F(X)}$. 
• For every triple of objects \(X, Y, Z \in \text{Ob}(\mathcal{C}_\bullet)\), the diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(Y, Z)_\bullet \times \text{Hom}_\mathcal{C}(X, Y)_\bullet & \to & \text{Hom}_\mathcal{C}(X, Z)_\bullet \\
\downarrow F_{Y, Z} \otimes F_{X, Y} & & \downarrow F_{X, Z} \\
\text{Hom}_\mathcal{D}(F(Y), F(Z))_\bullet \times \text{Hom}_\mathcal{D}(F(X), F(Y))_\bullet & \to & \text{Hom}_\mathcal{D}(F(X), F(Z))_\bullet
\end{array}
\]

is commutative.

We let \(\text{Cat}_\Delta\) denote the category whose objects are (small) simplicial categories and whose morphisms are simplicial functors.

**Remark 2.4.1.12.** Let \(\mathcal{C}_\bullet\) be a (small) simplicial category. Then the construction \([n] \mapsto \mathcal{C}_n\) determines a functor from the simplex category \(\Delta^{\text{op}}\) (Definition 1.1.1.2) to the category \(\text{Cat}\) of (small) categories. Allowing \(\mathcal{C}_\bullet\) to vary, we obtain a functor \(\text{Cat}_\Delta \to \text{Fun}(\Delta^{\text{op}}, \text{Cat})\), which fits into a pullback diagram of categories

\[
\begin{array}{ccc}
\text{Cat}_\Delta & \to & \text{Fun}(\Delta^{\text{op}}, \text{Cat}) \\
\downarrow \text{Ob} & & \downarrow \text{Ob} \\
\text{Set} & \to & \text{Fun}(\Delta^{\text{op}}, \text{Set})
\end{array}
\]

where the lower horizontal map carries each set \(S\) to the constant functor \(\Delta^{\text{op}} \to \text{Set}\) taking the value \(S\).

Phrased more informally: simplicial categories can be identified with simplicial objects \(\mathcal{C}_\bullet\) of \(\text{Cat}\) for which the underlying simplicial set of objects \([n] \mapsto \text{Ob}(\mathcal{C}_n)\) is constant. In particular, the functor \(\text{Cat}_\Delta \to \text{Fun}(\Delta^{\text{op}}, \text{Cat})\) is a fully faithful embedding.

**Proposition 2.4.1.13.** The category \(\text{Cat}_\Delta\) admits small limits and colimits.

**Proof.** The category \(\text{Cat}\) admits small limits and colimits, which are preserved by the forgetful functor \(\text{Ob} : \text{Cat} \to \text{Set}\). It follows that the category \(\text{Fun}(\Delta^{\text{op}}, \text{Cat})\) of simplicial objects in \(\text{Cat}\) also admits small limits and colimits, which are computed pointwise. Remark 2.4.1.12 supplies a fully faithful embedding \(\text{Cat}_\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \text{Cat})\) whose essential image is closed under small limits and colimits, so that \(\text{Cat}_\Delta\) admits small limits and colimits as well.

\[\square\]

### 2.4.2 Examples of Simplicial Categories

We now supply some examples of simplicial categories.
Example 2.4.2.1 (Simplicial Sets). Let $\text{Set}_\Delta$ denote the category of simplicial sets. Then $\text{Set}_\Delta$ can be regarded as (the underlying ordinary category of) a simplicial category, which we will also denote by $\text{Set}_\Delta$: given a pair of simplicial sets $X_\bullet$ and $Y_\bullet$, we define $\text{Hom}_{\text{Set}_\Delta}(X_\bullet, Y_\bullet)$ to be the simplicial set $\text{Fun}(X_\bullet, Y_\bullet)$ parametrizing morphisms from $X_\bullet$ to $Y_\bullet$ (see Construction 1.4.3.1).

Example 2.4.2.2 (Delooping). Let $M_\bullet$ be a simplicial monoid. We let $BM_\bullet$ denote the simplicial category having a single object $X$, with $\text{Hom}_{BM_\bullet}(X, X)\bullet = M_\bullet$ and the composition law is given by the monoid structure on $M_\bullet$. We will refer to $BM_\bullet$ as the delooping of the simplicial monoid $M_\bullet$. Note that the construction $M_\bullet \mapsto BM_\bullet$ induces an equivalence of categories

$$\{\text{Simplicial Monoids}\} \simeq \{\text{Simplicial Categories } \mathcal{C} \text{ with } \text{Ob}(\mathcal{C}) = \{X\}\}.$$ 

We can produce many more examples using the construction of Remark 2.1.7.4. If $\mathcal{A}$ is a monoidal category equipped with a (lax) monoidal functor $F : \mathcal{A} \to \text{Set}_\Delta$, then every $\mathcal{A}$-enriched category can also be regarded as a simplicial category. We now consider four instances of this construction:

- We can take $F : \text{Set} \to \text{Set}_\Delta$ to be the functor which carries each set $S$ to the associated constant simplicial set (Construction 1.1.4.2).
- We can take $F : \text{Cat} \to \text{Set}_\Delta$ to be the functor which carries each category $\mathcal{C}$ to its nerve $N_\bullet(\mathcal{C})$ (Construction 1.2.1.1).
- We can take $F : \text{Set}_\Delta \to \text{Set}_\Delta$ to be the functor which carries each simplicial set $S_\bullet$ to the opposite simplicial set $S_\bullet^{op}$.
- We can take $F : \text{Top} \to \text{Set}_\Delta$ to be the functor which carries each topological space to the singular simplicial set $\text{Sing}_\bullet(\mathcal{C})$ (Construction 1.1.7.1).

Example 2.4.2.3 (Ordinary Categories as Simplicial Categories). Let $\mathcal{C}$ be an ordinary category. We define a simplicial category $\mathcal{C}_\bullet$ as follows:

- The objects of $\mathcal{C}_\bullet$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C}_\bullet) = \text{Ob}(\mathcal{C})$, $\text{Hom}_\mathcal{C}(X, Y)\bullet$ is the constant simplicial set associated to the set $\text{Hom}_\mathcal{C}(X, Y)$ (see Construction 1.1.4.2).
- For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C}_\bullet) = \text{Ob}(\mathcal{C})$, the composition law

$$c_{Z,Y,X} : \text{Hom}_\mathcal{C}(Y, Z)\bullet \times \text{Hom}_\mathcal{C}(X, Y)\bullet \to \text{Hom}_\mathcal{C}(X, Z)\bullet$$

on $\mathcal{C}_\bullet$ is determined by the composition law $\text{Hom}_\mathcal{C}(Y, Z) \times \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(X, Z)$ on $\mathcal{C}$. 
We will refer to $\mathcal{C}_\bullet$ as the \textit{constant simplicial category} associated to $\mathcal{C}$. Under the fully faithful embedding $\text{Cat}_\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \text{Cat})$ of Remark 2.4.1.12, it corresponds to the constant functor $\Delta^{\text{op}} \rightarrow \{\mathcal{C}\} \hookrightarrow \text{Cat}$ (see Construction 1.1.4.2). In particular, the underlying category of $\mathcal{C}_\bullet$ (in the sense of Example 2.4.1.4) is the ordinary category $\mathcal{C}$.

Remark 2.4.2.4. It follows from Corollary 1.1.4.8 and Remark 2.4.1.12 that the construction $\text{Cat} \rightarrow \text{Cat}_\Delta \quad \mathcal{C} \mapsto \mathcal{C}_\bullet$

of Example 2.4.2.3 is fully faithful. Its essential image consists of those simplicial categories $\mathcal{E}_\bullet$ having the property that, for every pair of objects $X, Y \in \text{Ob}(\mathcal{E}_\bullet)$, the simplicial set $\text{Hom}_{\mathcal{E}_\bullet}(X, Y)_\bullet$ is discrete (Definition 1.1.4.9). We will sometimes abuse notation by not distinguishing between the ordinary category $\mathcal{C}$ and the constant simplicial category $\mathcal{C}_\bullet$.

Remark 2.4.2.5. Let $\mathcal{C}$ be an ordinary category and let $\mathcal{D}_\bullet$ be a simplicial category. Applying Proposition 1.1.4.5 (and Remark 2.4.1.12), we deduce that the restriction map

$$\{\text{Simplicial functors } \mathcal{C}_\bullet \rightarrow \mathcal{D}_\bullet\} \simeq \{\text{Functors } \mathcal{C} \rightarrow \mathcal{D}_0\},$$

is bijective. In other words, the fully faithful embedding

$$\text{Cat} \hookrightarrow \text{Cat}_\Delta \quad \mathcal{C} \mapsto \mathcal{C}_\bullet$$

of Remark 2.4.2.4 is left adjoint to the forgetful functor

$$\text{Cat}_\Delta \rightarrow \text{Cat} \quad \mathcal{D}_\bullet \mapsto \mathcal{D}_0.$$ 

of Example 2.4.1.4.

Remark 2.4.2.6. Let $\mathcal{C}$ be an ordinary category. Then the simplicial category $\mathcal{C}_\bullet$ of Example 2.4.2.3 is locally Kan (since constant simplicial sets are Kan complexes; see Example 1.1.9.7).

Example 2.4.2.7 (Strict 2-Categories as Simplicial Categories). Let $\mathcal{C}$ be strict 2-category (Definition 2.2.0.1). Then we can associate to $\mathcal{C}$ a simplicial category $\mathcal{C}_\bullet$ as follows:

- The objects of $\mathcal{C}_\bullet$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C}_\bullet) = \text{Ob}(\mathcal{C})$, the simplicial set $\text{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is the nerve of the category $\text{Hom}_{\mathcal{C}}(X, Y)$.
- For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C}_\bullet) = \text{Ob}(\mathcal{C})$, the composition law

$$\text{Hom}_{\mathcal{C}}(Y, Z)_\bullet \times \text{Hom}_{\mathcal{C}}(X, Y)_\bullet \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)_\bullet$$

of $\mathcal{C}_\bullet$ is given by the nerve of the composition functor $\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$. 


Remark 2.4.2.8. In the situation of Example 2.4.2.7, we will generally abuse notation by identifying the strict 2-category $\mathcal{C}$ with the associated simplicial category $\mathcal{C}_\bullet$. Note that the underlying category of $\mathcal{C}_\bullet$ (in the sense of Example 2.4.1.4) agrees with the underlying category of $\mathcal{C}$ (in the sense of Remark 2.2.0.3). Moreover, since the nerve functor $N_\bullet : \text{Cat} \to \text{Set}_\Delta$ is fully faithful (Proposition 1.2.2.1), the construction of Example 2.4.2.7 supplies a fully faithful embedding

$$\text{2Cat}_{\text{Str}} \hookrightarrow \text{Cat}_\Delta \quad \mathcal{C} \mapsto \mathcal{C}_\bullet,$$

where $2\text{Cat}_{\text{Str}}$ denotes the category of strict 2-categories (see Definition 2.2.5.5).

Remark 2.4.2.9. Let $\mathcal{C}$ be an ordinary category, regarded as a strict 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the simplicial category $\mathcal{C}_\bullet$ associated to $\mathcal{C}$ by Example 2.4.2.7 agrees with the simplicial category associated to $\mathcal{C}$ by Example 2.4.2.3.

Remark 2.4.2.10. Let $\mathcal{C}$ be a strict 2-category. Then the simplicial category $\mathcal{C}_\bullet$ of Example 2.4.2.7 is locally Kan if and only if every 2-morphism in $\mathcal{C}$ is invertible: that is, if and only if $\mathcal{C}$ is a $(2, 1)$-category (in the sense of Definition 2.3.0.1). This follows from Proposition 1.2.4.2.

Example 2.4.2.11 (The Conjugate of a Simplicial Category). Let $\mathcal{C}_\bullet$ be a simplicial category. We define a new simplicial category $\mathcal{C}_\circlearrowleft$ as follows:

- The objects of $\mathcal{C}_\circlearrowleft$ are the objects of $\mathcal{C}_\bullet$.
- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C}_\circlearrowleft) = \text{Ob}(\mathcal{C}_\bullet)$, we have an equality of simplicial sets
  $$\text{Hom}_{\mathcal{C}_\circlearrowleft}(X, Y)_\bullet = \text{Hom}_{\mathcal{C}}(X, Y)^{\text{op}}_\bullet;$$
  here the right hand side denotes the opposite of the simplicial set $\text{Hom}_{\mathcal{C}}(X, Y)_\bullet$ (Construction 1.3.2.2).
- For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C}_\circlearrowleft) = \text{Ob}(\mathcal{C}_\bullet)$, the composition law
  $$\text{Hom}_{\mathcal{C}_\circlearrowleft}(Y, Z)_\bullet \times \text{Hom}_{\mathcal{C}_\circlearrowleft}(X, Y)_\bullet \to \text{Hom}_{\mathcal{C}_\circlearrowleft}(X, Z)_\bullet$$
  on $\mathcal{C}_\circlearrowleft$ is obtained from the composition law on $\mathcal{C}_\bullet$ by passing to opposite simplicial sets.

We will refer to $\mathcal{C}_\circlearrowleft$ as the conjugate of the simplicial category $\mathcal{C}_\bullet$.

Remark 2.4.2.12. Let $\mathcal{C}_\bullet$ be a simplicial category and let $\mathcal{C}_\circlearrowleft$ denote the conjugate simplicial category (Example 2.4.2.11). Then, when regarded as a simplicial object of $\text{Cat}$, the conjugate simplicial category $\mathcal{C}_\circlearrowleft$ is given by the functor

$$\Delta^{\text{op}} \xrightarrow{\text{Op}} \Delta^{\text{op}} \xrightarrow{[n] \mapsto \mathcal{C}_\bullet^n} \text{Cat};$$
here $\text{Op}$ denotes the involution of $\Delta$ described in Notation 1.3.2.1. In particular, the underlying ordinary categories of $\mathcal{C}_{\bullet}$ and $\mathcal{C}_{\bullet}^c$ are the same.

**Remark 2.4.2.13.** Let $\mathcal{C}$ be a strict $2$-category and let $\mathcal{C}_{\bullet}$ denote the associated simplicial category (Example 2.4.2.7). Then the conjugate simplicial category $(\mathcal{C}_{\bullet})^c$ can be identified with the simplicial category $(\mathcal{C}^c)_{\bullet}$ associated to the conjugate $2$-category $\mathcal{C}^c$ of Construction 2.2.3.4. In particular, if $\mathcal{C}$ is an ordinary category, then we have a canonical isomorphism $\mathcal{C}_{\bullet}^c \simeq \mathcal{C}_{\bullet}$.

**Remark 2.4.2.14.** Let $\mathcal{C}_{\bullet}$ be a simplicial category. Then $\mathcal{C}_{\bullet}$ is locally Kan if and only if the conjugate simplicial category $\mathcal{C}_{\bullet}^c$ (Example 2.4.2.11) is locally Kan.

**Example 2.4.2.15** (Topologically Enriched Categories). Let $\text{Top}$ denote the category of topological spaces. The formation of singular simplicial sets (Construction 1.1.7.1) determines a functor

$$\text{Sing}_{\bullet} : \text{Top} \to \text{Set}_{\Delta} \quad X \mapsto \text{Sing}_{\bullet}(X)$$

which preserves finite products (in fact, it preserves all small limits), and can therefore be regarded as a monoidal functor from $\text{Top}$ to $\text{Set}_{\Delta}$ (where we endow both $\text{Top}$ and $\text{Set}_{\Delta}$ with the Cartesian monoidal structure). Applying Remark 2.1.7.4, we see that every topologically enriched category $\mathcal{C}$ can be regarded as a simplicial category $\mathcal{C}_{\bullet}$ having the same objects, with morphism simplicial sets given by

$$\text{Hom}_C(X,Y)_{\bullet} = \text{Sing}_{\bullet}(\text{Hom}_C(X,Y))$$

here $\text{Hom}_C(X,Y)$ denotes the set of morphisms from $X$ to $Y$, endowed with the topology determined by the topological enrichment of $\mathcal{C}$ (see Example 2.1.7.8).

**Remark 2.4.2.16.** Let $\mathcal{C}$ be a topologically enriched category, and let $\mathcal{C}_{\bullet}$ denote the associated simplicial category (Example 2.4.2.15). Then $\mathcal{C}_{\bullet}$ is locally Kan (since the singular simplicial set $\text{Sing}_{\bullet}(X)$ of any topological space $X$ is a Kan complex; see Proposition 1.1.9.8).

**Warning 2.4.2.17.** Let $\text{Top}_{\text{LCH}}$ denote the full subcategory of $\text{Top}$ spanned by the locally compact Hausdorff spaces. Then we can view $\text{Top}_{\text{LCH}}$ as a topologically enriched category, where we endow each of the sets

$$\text{Hom}_{\text{Top}_{\text{LCH}}}(X,Y) = \{\text{Continuous functions } f : X \to Y\}$$

with the compact-open topology, generated by open sets of the form $\{f \in \text{Hom}_{\text{Top}}(X,Y) : f(K) \subseteq U\}$ where $K \subseteq X$ is compact and $U \subseteq Y$ is open. On this subcategory, the simplicial enrichment of Example 2.4.2.15 coincides with the simplicial enrichment of Example 2.4.1.5. Beware that some technical issues arise if we allow spaces which are not locally compact:
Given topological spaces $X$, $Y$, and $Z$, the composition map

$$\text{Hom}_{\text{Top}}(Y,Z) \times \text{Hom}_{\text{Top}}(X,Y) \to \text{Hom}_{\text{Top}}(X,Z) \quad (g,f) \mapsto g \circ f$$

need not be continuous (with respect to the compact-open topologies on $\text{Hom}_{\text{Top}}(X,Y)$, $\text{Hom}_{\text{Top}}(Y,Z)$, and $\text{Hom}_{\text{Top}}(X,Z)$) when $Y \notin \text{Top}_{\text{LCH}}$. Consequently, the construction of compact-open topologies does not determine a topological enrichment of $\text{Top}$ (in the sense of Example 2.1.7.8).

Given topological spaces $X$ and $Y$, a function $|\Delta^n| \to \text{Hom}_{\text{Top}}(X,Y)$ which is continuous (for the compact-open topology on $\text{Hom}_{\text{Top}}(X,Y)$) need not correspond to a continuous function $|\Delta^n| \times X \to Y$ when $X \notin \text{Top}_{\text{LCH}}$.

One can remedy these difficulties by replacing $\text{Top}$ by the subcategory of compactly generated weak Hausdorff spaces introduced in [30].

### 2.4.3 The Homotopy Coherent Nerve

Let $\text{Top}$ denote the category of topological spaces and let $N_\bullet(\text{Top})$ denote its nerve (Construction 1.2.1.1). Then $N_\bullet(\text{Top})$ is a simplicial set whose 2-simplices can be identified with diagrams of topological spaces $\sigma$:

$$
\begin{array}{ccc}
& X_1 & \\
X_0 \downarrow f_{10} & & \downarrow f_{21} \\
& X_0 \downarrow f_{20} & \rightarrow & X_2
\end{array}
$$

which commute in the sense that $f_{20}$ is equal to the composition $f_{21} \circ f_{10}$. In the study of algebraic topology, one often encounters diagrams which commute in the weaker sense that $f_{20}$ is homotopic to the composition $f_{21} \circ f_{10}$. By definition, this means that there exists a continuous function $h : [0,1] \times X_0 \to X_2$ which satisfies the boundary conditions

$$
\begin{align*}
& h|_{\{0\} \times X_0} = f_{20} \\
& h|_{\{1\} \times X_0} = f_{21} \circ f_{10}.
\end{align*}
$$

In this case, we say that the function $h$ is a homotopy from $f_{20}$ to $f_{21} \circ f_{10}$, and that $h$ is a witness to the homotopy commutativity of the diagram $\sigma$. In this section, we will introduce an enlargement $N_{hc}^\bullet(\text{Top})$ of the simplicial set $N_\bullet(\text{Top})$, whose 2-simplices are given by pairs $(\sigma,h)$ where $\sigma$ is a (possibly noncommutative) diagram as above, and $h$ is a witness to the homotopy commutativity of $\sigma$. This is a special case of a general construction (Definition 2.4.3.5) which can be applied to any simplicial category.

**Notation 2.4.3.1** (Simplicial Path Categories). Let $(Q, \leq)$ be a partially ordered set, and let $\text{Path}_{(2)}[Q]$ denote the path 2-category of $Q$ (Construction 2.3.6.1). We let $\text{Path}[Q]_\bullet$. 

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denote the simplicial category obtained from the strict 2-category $\text{Path}_{(2)}[Q]$ by applying the construction of Example 2.4.2.7. More concretely, we can describe the simplicial category $\text{Path}_Q$ as follows:

- The objects of $\text{Path}_Q$ are the elements of the partially ordered set $Q$.
- If $x$ and $y$ are elements of $Q = \text{Ob}(\text{Path}_Q)$, then $\text{Hom}_{\text{Path}_Q}(x,y)$ is the nerve of the partially ordered set of finite linearly ordered subsets $\{x = x_0 < x_1 < \cdots < x_m = y\} \subseteq Q$ with least element $x$ and largest element $y$.
- For each element $x \in Q = \text{Ob}(\text{Path}_Q)$, the identity morphism $\text{id}_x$ is the singleton $\{x\} \in \text{Hom}_{\text{Path}_Q}(x,x)$.
- For $x,y,z \in Q = \text{Ob}(\text{Path}_Q)$, the composition law
  \[ \text{Hom}_{\text{Path}_Q}(y,z) \times \text{Hom}_{\text{Path}_Q}(x,y) \to \text{Hom}_{\text{Path}_Q}(x,z). \]
  is given on vertices by the construction $(S,T) \mapsto S \cup T$.

In the special case where $Q = [n] = \{0 < 1 < \cdots < n\}$, we denote the simplicial category $\text{Path}_Q$ by $\text{Path}_n$.

**Remark 2.4.3.2.** Let $Q$ be a partially ordered set. The simplicial category $\text{Path}_Q$ can be regarded as a “thickened version” of $Q$. For every pair of elements $x,y \in Q$, the simplicial set $\text{Hom}_{\text{Path}_Q}(x,y)$ is empty if $x \not\leq y$, and weakly contractible (see Definition 3.2.6.4) if $x \leq y$ (since it is the nerve of a partially ordered set with a least element $\{x,y\}$). In particular, there is a unique simplicial functor $\pi : \text{Path}_Q \to Q$ which is the identity on objects (where we abuse notation by identifying $Q$ with the associated constant simplicial category of Example 2.4.2.3). The simplicial functor $\pi$ is a prototypical example of a weak equivalence in the setting of simplicial categories (see [?]).

**Remark 2.4.3.3.** A topologically enriched variant of $\text{Path}_Q$ appears in the work of Leitch ([27]); see appendix B of [35] for a related construction.

**Remark 2.4.3.4** (Relationship with Ordinary Path Categories). Let $Q$ be a partially ordered set and let $\text{Gr}(Q)$ denote the associated directed graph, given concretely by

\[
\text{Vert}(\text{Gr}(Q)) = Q \quad \text{Edge}(\text{Gr}(Q)) = \{(x,y) : x < y\}.
\]

Then the path category $\text{Path}[\text{Gr}(Q)]$ of Construction 1.2.6.1 is the underlying category of the simplicial category $\text{Path}_Q$ of Notation 2.4.3.1 (see Remark 2.3.6.2). In other words, we can regard $\text{Path}_Q$ as a simplicially enriched version of $\text{Path}[\text{Gr}(Q)]$. Beware that the simplicial enrichment is nontrivial in general: that is, the simplicial mapping sets $\text{Hom}_{\text{Path}_Q}(x,y)$ are usually not constant.
Definition 2.4.3.5 (The Homotopy Coherent Nerve). Let $C_\bullet$ be a simplicial category. We let $N^{hc}_\bullet(C)$ denote the simplicial set given by the construction

\[(n) \in \Delta^{op} \mapsto \text{Hom}_{\text{Cat}_\Delta}(\text{Path}[n]_\bullet, C_\bullet) = \{\text{Simplicial functors Path}[n]_\bullet \to C_\bullet\}.
\]

We will refer to $N^{hc}_\bullet(C)$ as the homotopy coherent nerve of the simplicial category $C_\bullet$.

Remark 2.4.3.6. The homotopy coherent nerve $N^{hc}_\bullet(C)$ was introduced by Cordier in [5] (motivated by earlier work of Vogt on the theory of homotopy coherence; see [38]).

Remark 2.4.3.7. The homotopy coherent nerve of Definition 2.4.3.5 determines a functor $N^{hc}_\bullet(-)$ from the category $\text{Cat}_\Delta$ of simplicial categories (Definition 2.4.1.11) to the category $\text{Set}_\Delta$ of simplicial sets (Definition 1.1.1.12). This is a special case of the general construction described in Variant 1.1.7.6, associated to the cosimplicial object of $\text{Cat}_\Delta$ given by

\[\Delta \to \text{Cat}_\Delta \quad [n] \mapsto \text{Path}[n]_\bullet.\]

Remark 2.4.3.8 (Comparison with the Nerve). Let $C_\bullet$ be a simplicial category and let $C = C_0$ denote the underlying ordinary category. For every partially ordered set $Q$, composition with the simplicial functor $\text{Path}[Q]_\bullet \to Q$ of Remark 2.4.3.2 induces a monomorphism

\[\{\text{Ordinary functors } Q \to C\} \hookrightarrow \{\text{Simplicial functors Path}[Q]_\bullet \to C_\bullet\}.\]

Restricting this construction to partially ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, we obtain a monomorphism of simplicial sets $N_\bullet(C) \hookrightarrow N^{hc}_\bullet(C)$, where $N_\bullet(C)$ is the nerve of Construction 1.2.1.1 and $N^{hc}_\bullet(C)$ is the homotopy coherent nerve of Definition 2.4.3.5.

Example 2.4.3.9 (Vertices and Edges of the Homotopy Coherent Nerve). In the cases $Q = [0]$ and $Q = [1]$, the map $\pi : \text{Path}[Q]_\bullet \to Q$ is an equivalence of simplicial categories (since a path in $Q$ is uniquely determined by its endpoints). It follows that for every simplicial category $C_\bullet$, the comparison map $N_\bullet(C) \hookrightarrow N^{hc}_\bullet(C)$ of Remark 2.4.3.8 is bijective on vertices and edges. In particular:

- Vertices of the homotopy coherent nerve $N^{hc}_\bullet(C)$ can be identified with objects $X$ of the underlying category $C$.
- Edges of the homotopy coherent nerve $N^{hc}_\bullet(C)$ can be identified with morphisms $f : X \to Y$ of the underlying category $C$.
- The face maps $d_0, d_1 : N^{hc}_\bullet(C) \to N^{hc}_0(C)$ carry a morphism $f : X \to Y$ to its codomain $Y = d_0(f)$ and domain $f = d_1(f)$, respectively.
- The degeneracy map $s_0 : N^{hc}_0(C) \to N^{hc}_1(C)$ carries an object $X \in C$ to the identity morphism $\text{id}_X : X \to X$. 
Example 2.4.3.10 (2-Simplices of the Homotopy Coherent Nerve). Let $Q = \{x_0 < x_1 < x_2\}$ be a linearly ordered set with 2 elements. Then the map $\pi : \text{Path}[Q] \to Q$ is not an equivalence of simplicial categories. In the underlying category $\text{Path}[Q]$, the diagram

\[
\begin{array}{ccc}
{x_0 < x_1} & \xrightarrow{x_1} & {x_1 < x_2} \\
\downarrow & & \downarrow \\
{x_0 < x_2} & \cong & x_2
\end{array}
\]

does not commute: the composition of the diagonal maps is the path $\{x_0 < x_1 < x_2\}$. However, it commutes in a weaker sense: there is an edge joining $\{x_0 < x_2\}$ to $\{x_0 < x_1 < x_2\}$ in the simplicial set $\text{Hom}_{\text{Path}[Q]}(x, z)$. It follows that for any simplicial category $C$, a choice of 2-simplex

$$
\sigma \in N^hc(C) = \text{Hom}_{\text{Cat}_\Delta}(\text{Path}[2], C) \simeq \text{Hom}_{\text{Cat}_\Delta}(\text{Path}[Q], C)
$$
determines a (possibly non-commutative) diagram $\sigma_0$:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_{21}} & X_2 \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{f_{10}} & X_0, \xrightarrow{f_{20}} & X_2,
\end{array}
\]
in $C$, together with a homotopy $h$ from $f_{20}$ to $f_{21} \circ f_{10}$ (in the sense of Definition 2.4.6). In §2.4.6 we will prove the converse: every choice of homotopy from $f_{20}$ to $f_{21} \circ f_{10}$ determines a unique 2-simplex of $N^hc(C)$ (Proposition 2.4.6.9).

Example 2.4.3.11 (Comparison with the Duskin Nerve). Let $C$ be a strict 2-category and let $C^\circ$ denote the associated simplicial category (Example 2.4.2.7). For any partially ordered set $Q$, Remark 2.4.2.8 and Theorem 2.3.6.7 supply bijections

$$
\text{Hom}_{\text{Cat}_\Delta}(\text{Path}[Q], C) \simeq \text{Hom}_{\text{Cat}_{\text{Str}}(\text{Path}[2], C)} \simeq \text{Hom}_{\text{Cat}_{\text{Str}}(\text{Path}(\text{2}), C^c)} \simeq \text{Hom}_{\text{Cat}_{\text{ULax}}(Q, C^c)};
$$

here $C^c$ denotes the conjugate 2-category of $C$ (Construction 2.2.3.4). Restricting these bijections to the special case where $Q$ has the form $[n] = \{0 < 1 < \cdots < n\}$, we obtain an isomorphism of simplicial sets

$$
N^hc(C) \simeq N^D(C^c),
$$

where $N^hc(C)$ is the homotopy coherent nerve of Definition 2.4.3.5 and $N^D(C^c)$ is the Duskin nerve of Construction 2.3.1.1.
2.4. SIMPLICIAL CATEGORIES

Example 2.4.3.12 (The Case of an Ordinary Category). Let \( \mathcal{C} \) be an ordinary category, regarded as a constant simplicial category \( \mathcal{C}^\bullet \) via the construction of Example 2.4.2.3. Combining Examples 2.3.1.3 and Examples 2.4.3.11, we obtain isomorphisms

\[
N^\bullet(\mathcal{C}) \simeq N^D(\mathcal{C}) = N^D(\mathcal{C}^c) \simeq N^{hc}(\mathcal{C}).
\]

Unwinding the definitions, we see that the composite isomorphism \( N^\bullet(\mathcal{C}) \simeq N^{hc}(\mathcal{C}) \) is the comparison map of Remark 2.4.3.8. In other words, when restricted to constant simplicial categories, the homotopy coherent nerve of Definition 2.4.3.5 reduces to the classical nerve of Construction 1.2.1.1.

2.4.4 The Path Category of a Simplicial Set

Let \( G \) be a directed graph, which we identify with a simplicial set \( G^\bullet \) of dimension \( \leq 1 \) (Proposition 1.1.5.9). In §1.2.6, we introduced a category \( \text{Path}[G] \) called the path category of \( G \) (Construction 1.2.6.1). The category \( \text{Path}[G] \) is characterized (up to isomorphism) by a universal property: for any category \( \mathcal{C} \), Proposition 1.2.6.5 supplies a bijection

\[
\{ \text{Functors } F : \text{Path}[G] \to \mathcal{C} \} \simeq \text{Hom}_{\Delta^\text{op}}(G^\bullet, N^{hc}(\mathcal{C})).
\]

In this section, we introduce a generalization of the construction \( G \mapsto \text{Path}[G] \), where we replace directed graphs by arbitrary simplicial sets (not necessarily of dimension \( \leq 1 \)) and categories by simplicial categories.

Definition 2.4.4.1. Let \( S^\bullet \) be a simplicial set and let \( \mathcal{C}^\bullet \) be a simplicial category. We will say that a map of simplicial sets \( u : S^\bullet \to N^{hc}(\mathcal{C}) \) exhibits \( \mathcal{C}^\bullet \) as a path category of \( S^\bullet \) if, for every simplicial category \( D^\bullet \), composition with \( u \) induces a bijection

\[
\{ \text{Simplicial functors } F : \mathcal{C}^\bullet \to D^\bullet \} \to \text{Hom}_{\Delta^\text{op}}(S^\bullet, N^{hc}(D)).
\]

Notation 2.4.4.2 (The Path Category of a Simplicial Set). Let \( S^\bullet \) be a simplicial set. It follows immediately from the definitions that if there exists a map of simplicial sets \( u : S^\bullet \to N^{hc}(\mathcal{C}) \) which exhibits \( \mathcal{C}^\bullet \) as the path category of \( S^\bullet \), then the simplicial category \( \mathcal{C}^\bullet \) (and the morphism \( u \)) are unique up to isomorphism and depend functorially on \( S^\bullet \). We will emphasize this dependence by denoting \( \mathcal{C}^\bullet \) by \( \text{Path}[S]^\bullet \) and referring to it as the path category of the simplicial set \( S^\bullet \).

Proposition 2.4.4.3. Let \( S^\bullet \) be a simplicial set. Then there exists simplicial category \( \mathcal{C}^\bullet \) and a morphism of simplicial sets \( u : S^\bullet \to N^{hc}(\mathcal{C}) \) which exhibits \( \mathcal{C}^\bullet \) as a path category of \( S^\bullet \).
Proof. This is a special case of Proposition 1.1.8.22 since the category \( \text{Cat}_\Delta \) admits small colimits (Proposition 2.4.1.13). Explicitly, the simplicial path category of a simplicial set \( S \) is given by the generalized geometric realization

\[
|S|_{\text{Path}[-]} = \varprojlim_{\Delta^n \to S} \text{Path}[n],
\]

where \( \text{Path}[-] \) denotes the cosimplicial object of \( \text{Cat}_\Delta \) defined in Notation 2.4.3.1. \qed

**Corollary 2.4.4.4.** The homotopy coherent nerve functor \( \mathbb{N}^{hc} : \text{Cat}_\Delta \to \text{Set}_\Delta \) admits a left adjoint

\[
\text{Path}[-] : \text{Set}_\Delta \to \text{Cat}_\Delta,
\]

which associates to each simplicial set \( S \) the path category \( \text{Path}[S] \) of Notation 2.4.4.2.

**Warning 2.4.4.5.** We have now introduced several different notions of path category:

(a) To every directed graph \( G \), Construction 1.2.6.1 associates an ordinary category \( \text{Path}[G] \).

(b) To every partially ordered set \( Q \), Notation 2.4.3.1 associates a simplicial category \( \text{Path}[Q] \).

(c) To every simplicial set \( S \), Proposition 2.4.4.3 associates a simplicial category \( \text{Path}[S] \).

We will show below that these constructions are closely related:

(1) If \( G \) is a directed graph and \( S \) denotes the associated simplicial set of dimension \( \leq 1 \) (Proposition 1.1.5.9), then the simplicial category \( \text{Path}[S] \) of (c) is constant, associated to the ordinary category \( \text{Path}[G] \) of (a) (Proposition 2.4.4.7).

(2) If \( Q \) is a partially ordered set, then the simplicial category \( \text{Path}[Q] \) of (b) can be identified with the simplicial category \( \text{Path}[\mathbb{N}(Q)] \) of (c), where \( \mathbb{N}(Q) \) denotes the nerve of \( Q \) (Proposition 2.4.4.15).

(3) For any simplicial set \( S \), the simplicial category \( \text{Path}[S] \) of (c) has an underlying ordinary category \( \text{Path}[S]_0 \), which can be described as the category \( \text{Path}[G] \) associated by (a) to the underlying directed graph \( G = \text{Gr}(S) \) of \( S \) (Proposition 2.4.4.13).

Assertions (1) and (2) imply that the path category constructions of §1.2.6 and §2.4.3 can be regarded as special cases of the construction \( S \mapsto \text{Path}[S] \). Assertion (3) is a partial converse, which guarantees that the simplicial path category \( \text{Path}[S] \) can be regarded as a simplicially enriched version of the classical path category studied in §1.2.6.

In the special case where \( Q \) is a linearly ordered set of the form \( [n] = \{ 0 < 1 < \cdots < n \} \), assertion (2) of Warning 2.4.4.5 is immediate from the definitions:
Example 2.4.4.6 (The Path Category of a Simplex). Let $n \geq 0$ be a nonnegative integer and let $\text{Path}[n]_\bullet$ denote the simplicial category of Notation 2.4.3.1. For any simplicial category $\mathcal{C}_\bullet$, we have canonical bijections

$$\text{Hom}_{\text{Cat}_{\Delta}}(\text{Path}[n]_\bullet, \mathcal{C}_\bullet) \simeq N^\text{hc}_n(\mathcal{C}) \simeq \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, N^\text{hc}_\bullet(\mathcal{C})).$$

It follows that $\text{Path}[n]_\bullet$ is a path category for the standard simplex $\Delta^n$, in the sense of Definition 2.4.4.1.

Proposition 2.4.4.7. Let $G$ be a directed graph, let $\text{Path}[G]$ denote the path category of Construction 1.2.6.1 and let $\text{Path}[G]_\bullet$ denote the associated constant simplicial category (Example 2.4.2.3). Then the comparison map $u : G_\bullet \to N_\bullet(\text{Path}[G]) \simeq N^\text{hc}_\bullet(\text{Path}[G])$ exhibits $\text{Path}[G]_\bullet$ as a path category of the simplicial set $G_\bullet$.

Proof. Unwinding the definitions, we must show that for every simplicial category $\mathcal{D}_\bullet$, the composite map

$$\text{Hom}_{\text{Cat}_{\Delta}}(\text{Path}[G]_\bullet, \mathcal{D}_\bullet) \to \text{Hom}_{\text{Cat}}(\text{Path}[G], \mathcal{D}) \to \text{Hom}_{\text{Set}_{\Delta}}(G_\bullet, N_\bullet(\mathcal{D})) \to \text{Hom}_{\text{Set}_{\Delta}}(G_\bullet, N^\text{hc}_\bullet(\mathcal{D}))$$

is a bijection. Here the first map is bijective because the simplicial category $\text{Path}[G]_\bullet$ is constant (Remark 2.4.2.5), the second by virtue of Proposition 1.2.6.5 and the third because $G_\bullet$ has dimension $\leq 1$ and the comparison map $N_\bullet(\mathcal{D}) \to N^\text{hc}_\bullet(\mathcal{D})$ is an isomorphism on simplices of dimension $\leq 1$ (Example 2.4.3.9).

Warning 2.4.4.8. It follows from Proposition 2.4.4.7 that if $S_\bullet$ is a simplicial set of dimension $\leq 1$, then the simplicial category $\text{Path}[S]_\bullet$ is constant. Beware that this is never true for simplicial sets of dimension $> 1$ (see Theorem 2.4.4.10 below).

The proof of Proposition 2.4.4.3 given above is somewhat unsatisfying: it constructs the path category of a simplicial set $S_\bullet$ abstractly, as the colimit of a certain diagram in $\text{Cat}_{\Delta}$. In general, colimits in $\text{Cat}_{\Delta}$ (like colimits in $\text{Cat}$) can be difficult to describe. However, the (simplicial) path category $\text{Path}[S]_\bullet$ actually has a relatively simple structure. For each nonnegative integer $m$, the category $\text{Path}[S]_m$ is free in the sense of Definition 1.2.6.7 that is, it can be realized as the (ordinary) path category of a directed graph. To formulate a more precise statement, we need a bit of (temporary) notation.

Notation 2.4.4.9. Let $S_\bullet$ be a simplicial set. For each nonnegative integer $m$, we let $E(S, m)$ denote the collection of pairs $(\sigma, \bar{T})$, where $\sigma : \Delta^n \to S_\bullet$ is a nondegenerate simplex of $S_\bullet$ of dimension $n > 0$ and $\bar{T} = (I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m-1} \subseteq I_m)$ is a chain of subsets of $[n]$.
satisfying $I_0 = \{0, n\}$ and $I_m = [n]$. Here we will view $\overrightarrow{T}$ as a $m$-simplex of the simplicial set $\text{Hom}_{\text{Path}[n]}(0, n)\bullet$.

Let $\mathcal{C}_\bullet$ be a simplicial category and let $u : S_\bullet \rightarrow N^{hc}(\mathcal{C})$ be a morphism of simplicial sets. For each element $(\sigma, \overrightarrow{T}) \in E(S, m)$, the composite map

$$\Delta^n \xrightarrow{\sigma} S_\bullet \xrightarrow{u} N^{hc}(\mathcal{C})$$

can be identified with a simplicial functor $u(\sigma) : \text{Path}[n] \rightarrow \mathcal{C}_m$, which we will denote by $u(\sigma, \overrightarrow{T})$.

**Theorem 2.4.4.10.** Let $S_\bullet$ be a simplicial set and let $u : S_\bullet \rightarrow N^{hc}(\text{Path}[S])$ be a map of simplicial sets which exhibits $\text{Path}[S]_\bullet$ as a path category of $S_\bullet$. Then:

1. The map $u$ induces a bijection from the set of vertices of $S_\bullet$ to the set of objects of $\text{Path}[S]_\bullet$.

2. For each nonnegative integer $m \geq 0$, the category $\text{Path}[S]_m$ is free (in the sense of Definition 1.2.6.7).

3. For each nonnegative integer $m \geq 0$, the construction $(\sigma, \overrightarrow{T}) \mapsto u(\sigma, \overrightarrow{T})$ of Notation 2.4.4.9 induces a bijection from $E(S, m)$ to the set of indecomposable morphisms of the category $\text{Path}[S]_m$.

**Remark 2.4.4.11.** Let $S_\bullet$ be a simplicial set. Then the path category $\text{Path}[S]_\bullet$ is characterized (up to isomorphism) by properties (1), (2), and (3) of Theorem 2.4.4.10. More precisely, suppose that $\mathcal{C}_\bullet$ is a simplicial category and that we are given a comparison map $u' : S_\bullet \rightarrow N^{hc}(\mathcal{C})$ satisfying the following three conditions:

1. The map $u'$ induces a bijection from the set of vertices of $S_\bullet$ to the set of objects of $\mathcal{C}_\bullet$.

2. For each nonnegative integer $m \geq 0$, the category $\mathcal{C}_m$ is free.

3. For each nonnegative integer $m \geq 0$, the construction $(\sigma, \overrightarrow{T}) \mapsto u'(\sigma, \overrightarrow{T})$ induces a bijection from $E(S, m)$ to the set of indecomposable morphisms of the category $\mathcal{C}_m$.

Then $u'$ exhibits $\mathcal{C}_\bullet$ as a path category of $S_\bullet$, in the sense of Definition 2.4.4.1. To prove this, we invoke the definition of $\text{Path}[S]_\bullet$ to deduce that there is a unique simplicial functor $F : \text{Path}[S]_\bullet \rightarrow \mathcal{C}_\bullet$ for which the composite map

$$S_\bullet \xrightarrow{u} N^{hc}(\text{Path}[S]) \xrightarrow{N^{hc}(F)} N^{hc}(\mathcal{C})$$

is equal to $u'$. Combining Theorem 2.4.4.10 with assumptions (1'), (2'), and (3'), we deduce that for each $m \geq 0$, the induced functor $\text{Path}[S]_m \rightarrow \mathcal{C}_m$ is a map between free categories which is bijective on objects and indecomposable morphisms, and is therefore an isomorphism of categories.
Remark 2.4.4.12. Let \( u : S_{\bullet} \rightarrow S'_{\bullet} \) be a monomorphism of simplicial sets. Then, for each \( m \geq 0 \), \( u \) induces a monomorphism of sets \( E(S, m) \leftarrow E(S', m) \) (see Notation 2.4.4.9). It follows from Theorem 2.4.4.10 that if \( x \) and \( y \) are vertices of \( S_{\bullet} \), then the induced map of simplicial sets \( \text{Hom}_{\text{Path}[S]}(x, y) \rightarrow \text{Hom}_{\text{Path}[S']}(u(x), u(y)) \) is a monomorphism.

Before giving the proof of Theorem 2.4.4.10, let us use it to deduce assertions (2) and (3) of Warning 2.4.4.5.

Proposition 2.4.4.13. Let \( S_{\bullet} \) be a simplicial set and let \( G \) be its underlying directed graph (Example 1.1.5.4), so that \( G_{\bullet} \) can be identified with the 1-skeleton of \( S_{\bullet} \). Let \( u : S_{\bullet} \rightarrow N_{\text{hc}}(\text{Path}[S]) \) denote the unit map. Then:

- The restriction \( u|_{G_{\bullet}} \) factors uniquely as a composition
  \[ G_{\bullet} \xrightarrow{u_0} N_{\bullet}(\text{Path}[S]_0) \rightarrow N_{\text{hc}}(\text{Path}[S]). \]

- The map \( u_0 \) induces an isomorphism of categories \( \text{Path}[G] \simeq \text{Path}[S]_0 \).

Proof. The first assertion follows immediately from Example 2.4.3.9 since \( G_{\bullet} \) is a simplicial set of dimension \( \leq 1 \). To prove the second assertion, we note that Theorem 2.4.4.10 guarantees that \( \text{Path}[S]_0 \) is a free category, whose objects can be identified with the vertices of \( S_{\bullet} \) and whose indecomposable morphisms can be identified with elements of the set \( E(S, 0) \) of Notation 2.4.4.9. By definition, \( E(S, m) \) consists of pairs \((\sigma, \overrightarrow{T})\), where \( \sigma \) is a nondegenerate \( n \)-simplex of \( S_{\bullet} \) for \( n > 0 \) and \( \overrightarrow{T} = (I_0 \subseteq \cdots \subseteq I_m) \) is a chain of subsets of \([n]\) satisfying \( I_0 = \{0, n\} \) and \( I_m = [n] \). In the case \( m = 0 \), the equality \( I_0 = I_m \) forces \( n = 1 \), so that \( E(S, 0) \) can be identified (via the map \( u_0 \)) with the collection of nondegenerate 1-simplices of \( S_{\bullet} \); that is, with the collection of edges of the graph \( G \). The freeness of \( \text{Path}[S]_0 \) now guarantees that the induced map \( \text{Path}[G] \simeq \text{Path}[S]_0 \) is an isomorphism of categories (see Proposition 1.2.6.11).

Exercise 2.4.4.14. Use Theorem 2.4.4.10 to give a different proof of Proposition 2.4.4.7 (show that if \( S_{\bullet} \) is a simplicial set of dimension \( \leq 1 \), then the sets \( E(S, m) \) appearing in Notation 2.4.4.9 do not depend on \( m \)).

Let \( Q \) be a partially ordered set. Note that every \( n \)-simplex \( \sigma \in N_{\bullet}(Q) \) can be identified with a map of partially ordered sets \([n] \rightarrow Q\), and therefore induces a simplicial functor \( \text{Path}[n]_{\bullet} \rightarrow \text{Path}[Q]_{\bullet} \) which we can view as an \( n \)-simplex of the homotopy coherent nerve \( N_{\text{hc}}(\text{Path}[Q]) \). This construction determines a map of simplicial sets \( u : N_{\bullet}(Q) \rightarrow N_{\text{hc}}(\text{Path}[Q]) \).

Proposition 2.4.4.15. Let \( Q \) be a partially ordered set. Then the comparison map \( u : N_{\bullet}(Q) \rightarrow N_{\text{hc}}(\text{Path}[Q]) \) described above exhibits \( \text{Path}[Q]_{\bullet} \) as a path category for the simplicial set \( N_{\bullet}(Q) \) (in the sense of Definition 2.4.4.1).
Proposition 2.4.4.15 follows immediately from Remark 2.4.4.11 together with the following:

**Lemma 2.4.4.16.** Let $Q$ be a partially ordered set. Then the comparison map $u : N_\bullet(Q) \to N_\bullet^h(\text{Path}[Q])$ satisfies conditions (1'), (2'), and (3') of Remark 2.4.4.11.

**Proof.** Assertion (1') is immediate (the map $u$ is bijective on vertices by construction). For each $m \geq 0$, the category $\text{Path}[Q]_m$ can be described concretely as follows:

- The objects of $\text{Path}[Q]_m$ are the elements of $Q$.
- If $x$ and $y$ are elements of $Q$, then a morphism from $x$ to $y$ in $\text{Path}[Q]_m$ is a chain $\mathcal{J} = (J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m)$ of finite linearly ordered subsets of $Q$, where each $S_i$ has least element $x$ and greatest element $y$.

Note that a morphism $\mathcal{J}$ from $x$ to $y$ is indecomposable (in the sense of Definition 1.2.6.8) if and only if $x < y$ and $J_0 = \{x, y\}$. Moreover, an arbitrary morphism $\mathcal{J}$ from $x$ to $y$ with $J_0 = \{x = x_0 < x_1 < \cdots < x_k = y\}$ decomposes uniquely as a composition of indecomposable morphisms

$$x_0 \xrightarrow{J(1)} x_1 \xrightarrow{J(2)} x_2 \to \cdots \xrightarrow{J(k)} x_k$$

where $J(a)_b = \{z \in J_b : x_{a-1} \leq z \leq x_a\}$. Applying Proposition 1.2.6.11, we deduce that the category $\text{Path}[Q]_m$ is free, which proves (2'). To prove (3'), we observe that every indecomposable morphism $\mathcal{J}$ can be written uniquely in the form $u(\sigma, \overrightarrow{I})$, where $(\sigma, \overrightarrow{I})$ is an element of the set $E(S, m)$ of Notation 2.4.4.9. Writing $J_m = \{x = x_0 < \cdots < x_n = y\}$, we see that $\sigma$ must be the nondegenerate $n$-simplex of $N_\bullet(Q)$ given by the map

$$[n] \to Q \quad i \mapsto x_i,$$

and $\overrightarrow{I}$ must be the chain $(\sigma^{-1}(J_0) \subseteq \sigma^{-1}(J_1) \subseteq \cdots \subseteq \sigma^{-1}(J_m))$ of subsets of $[n]$. □

**Proof of Theorem 2.4.4.10.** Let $m$ be a nonnegative integer, which we regard as fixed throughout the proof. Let $G(S)$ denote the directed graph given by

$$\text{Vert}(G(S)) = S_0 \quad \text{Edge}(G(S)) = E(S, m),$$

where we regard each element

$$(\sigma : \Delta^n \to S_\bullet, \overrightarrow{I} \in \text{Hom}_{\text{Path}[n]}(0, n)_m) \in \text{Edge}(G(S))$$
as having source $\sigma(0) \in \text{Vert}(G(S))$ and target $\sigma(n) \in \text{Vert}(G(S))$. Let $u_{S_\bullet} : S_\bullet \to N^\text{hc}(\text{Path}[S])$ denote the unit map. Then $u_{S_\bullet}$ induces a map of simplicial sets $G(S)_\bullet \to N_\bullet(\text{Path}[S]_m)$, which we can identify with a functor of ordinary categories

$$F_{S_\bullet} : \text{Path}[G] \to \text{Path}[S]_m.$$ 

Let us say that the simplicial set $S_\bullet$ is \textit{good} if $F_{S_\bullet}$ is an isomorphism of categories. Theorem 2.4.4.10 is equivalent to the assertion that every simplicial set is good (for every choice of nonnegative integer $m$). We will prove this by verifying that the collection of good simplicial sets satisfies the hypotheses of Lemma 1.1.8.15:

- Suppose we are given a pushout diagram of simplicial sets $\sigma :$

\[
\begin{array}{ccc}
S_\bullet & \longrightarrow & T_\bullet \\
\downarrow & & \downarrow \\
S'_\bullet & \longrightarrow & T'_\bullet
\end{array}
\]

where the horizontal maps are monomorphisms. Suppose that $S_\bullet$, $T_\bullet$, and $S'_\bullet$ are good; we wish to show that $T'_\bullet$ is good. Note that the horizontal maps induce monomorphisms of directed graphs

$$G(S) \hookrightarrow G(T) \quad G(S') \hookrightarrow G(T').$$

Define subgraphs $G_0(S) \subseteq G(S)$ and $G_0(T) \subseteq G(T)$ by the formulae

$$\text{Vert}(G_0(S)) = \text{Vert}(G(S)) = S_0 \quad \text{Vert}(G_0(T)) = \text{Vert}(G(T)) = T_0$$

$$\text{Edge}(G_0(S)) = \emptyset \quad \text{Edge}(G_0(T)) = \text{Edge}(G(T)) \setminus \text{Edge}(G(S)).$$

We then have a commutative diagram of categories

\[
\begin{array}{ccc}
\text{Path}[G_0(S)] & \longrightarrow & \text{Path}[G_0(T)] \\
\downarrow & & \downarrow \\
\text{Path}[G(S')] & \longrightarrow & \text{Path}[G(T')] \\
\uparrow_{F_{S'}} & & \uparrow_{F_{T'}} \\
\text{Path}[S'_m] & \longrightarrow & \text{Path}[T'_m].
\end{array}
\]

We wish to show that the functor $F_{T'}$ is an isomorphism of categories, and the map $F_{S'}$ is an isomorphism by assumption. It will therefore suffice to show that the lower square in this diagram is a pushout. Note that the upper square is a pushout (since it is obtained from a pushout diagram in the category of directed graphs by passing to path categories). We are therefore reduced to showing that the outer rectangle is a
pushout. We can rewrite this as the outer rectangle in another commutative diagram of categories

\[
\begin{array}{c}
\text{Path}[G_0(S)] \twoheadrightarrow \text{Path}[G_0(T)] \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Path}[G(S)] \twoheadrightarrow \text{Path}[G(T)] \\
\downarrow F_S \quad \downarrow F_T \\
\text{Path}[S]_m \twoheadrightarrow \text{Path}[T]_m \\
\downarrow \quad \downarrow \\
\text{Path}[S']_m \twoheadrightarrow \text{Path}[T']_m.
\end{array}
\]

We now conclude by observing that the upper square in this diagram is a pushout (because it is obtained from a pushout diagram of directed graphs by passing to path categories), the middle square is a pushout (since \( F_S \) and \( F_T \) are isomorphisms), and the lower square is a pushout (since the construction \( X \mapsto \text{Path}[X]_m \) preserves colimits).

- Suppose we are given a sequence of monomorphisms of simplicial sets

\[ S(0)_\bullet \hookrightarrow S(1)_\bullet \hookrightarrow S(2)_\bullet \hookrightarrow \cdots \]

with colimit \( S_\bullet \). Then the functor \( F_{S_\bullet} : \text{Path}[G(S_\bullet)] \to \text{Path}[S]_m \) can be written as a filtered colimit of functors \( F_{S(i)_\bullet} : \text{Path}[G(S(i)_\bullet)] \to \text{Path}[S(i)]_m \). Consequently, if each \( S(i)_\bullet \) is good, then \( S_\bullet \) is good.

- Let \( S_\bullet \) be a simplicial set which can be written as a coproduct \( \coprod_{i \in I} \Delta^n \); we must show that \( S_\bullet \) is good. Without loss of generality, we may assume that \( I \) is a singleton, so that \( S_\bullet = \Delta^n \). In this case, Example 2.4.4.6 supplies an equivalence of simplicial categories \( \text{Path}[S]_\bullet \simeq \text{Path}[n]_\bullet \). The desired result now follows from Lemma 2.4.4.16.

Remark 2.4.4.17. Let \( S_\bullet \) be a simplicial set. For each \( m \geq 0 \), Theorem 2.4.4.10 guarantees that \( \text{Path}[S]_m \) can be realized as the path category of a directed graph \( G_m \) (Construction 1.2.6.1), which can be described explicitly as follows:

- The vertices of \( G_m \) are the vertices of the simplicial set \( S_\bullet \).
- The edges of \( G_m \) are the elements of the set \( E(S, m) \) of Notation 2.4.4.9.

It follows that we can regard the construction \( [m] \mapsto \text{Path}[G_m] \) as a simplicial object of \( \text{Cat} \). The face and degeneracy operators on this simplicial object can be described as follows:
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- For $0 \leq i \leq m$, the degeneracy operator $s_i : \text{Path}[G_m] \to \text{Path}[G_{m+1}]$ is induced by a map of directed graphs from $G_m$ to $G_{m+1}$, which is the identity on vertices and given on edges by the construction

  $$(\sigma, I_0 \subseteq \cdots \subseteq I_m) \mapsto (\sigma, I_0 \subseteq \cdots \subseteq I_{i-1} \subseteq I_i \subseteq I_{i+1} \subseteq \cdots \subseteq I_m).$$

- For $0 < i < m$, the face map $d_i : \text{Path}[G_m] \to \text{Path}[G_{m-1}]$ is induced by a map of directed graphs from $G_m$ to $G_{m-1}$, which is the identity on vertices and given on edges by the construction

  $$(\sigma, I_0 \subseteq \cdots \subseteq I_m) \mapsto (\sigma, I_0 \subseteq \cdots \subseteq I_{i-1} \subseteq I_{i+1} \subseteq \cdots \subseteq I_m).$$

- Each of the face maps $d_m : \text{Path}[G_m] \to \text{Path}[G_{m-1}]$ is induced by a map directed graphs $f : G_m \to G_{m-1}$ which is the identity on vertices. Let $(\sigma, \vec{T})$ be an edge of $G_m$, given by a nondegenerate simplex $\sigma : \Delta^n \to S_\bullet$ and a chain of subsets $\vec{T} = (I_0 \subseteq \cdots \subseteq I_m)$ of $[n]$. Then the subset $I_{m-1} \subseteq I_m = [n]$ is the image of a unique monotone injection $\alpha : [n'] \hookrightarrow [n]$, and the composite map $\Delta^{n'} \xrightarrow{\alpha} \Delta^n \xrightarrow{\sigma} S_\bullet$ factors uniquely as a composition $\Delta^{n'} \to \Delta^{n''} \xrightarrow{\tau} S_\bullet$, where the first map is surjective on vertices and $\tau$ is a nondegenerate $n''$-simplex of $S_\bullet$. For $0 \leq i < m$, let $J_i \subseteq [n'']$ denote the image of the composite map $I_i \hookrightarrow I_{m-1} \xrightarrow{\alpha^{-1}} [n'] \hookrightarrow [n'']$, and set $\vec{J} = (J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{m-1})$. In the case $n'' = 0$, the morphism $f$ carries $(\sigma, \vec{T})$ to the vertex $\tau \in \text{Vert}(G_{m-1}) = S_0$. In the case $n'' > 0$ the morphism $f$ carries $(\sigma, \vec{T})$ to the edge $(\tau, \vec{J}) \in \text{Edge}(G_{m-1})$.

- The face maps $d_0 : \text{Path}[G_m] \to \text{Path}[G_{m-1}]$ are generally not induced by maps of directed graphs $G_m \to G_{m-1}$: that is, they do not carry indecomposable morphisms of $\text{Path}[G_m]$ to indecomposable morphisms of $\text{Path}[G_{m-1}]$. More precisely, if $(\sigma, \vec{T})$ is an edge of $G_n$ with $I_1 = \{0 = i_0 < i_1 < \cdots < i_k = m\}$, then $d_0$ carries $(\sigma, \vec{T})$ to a path of length $k$ in the category $\text{Path}[G_{m-1}]$.

2.4.5 From Simplicial Categories to $\infty$-Categories

Our goal in this section is to prove the following result (see [6]):

\textbf{Theorem 2.4.5.1} (Cordier-Porter). Let $\mathcal{C}_\bullet$ be a simplicial category. If $\mathcal{C}_\bullet$ is locally Kan, then the homotopy coherent nerve $N^{hc}_\bullet(\mathcal{C})$ is an $\infty$-category.

The proof of Theorem \textbf{2.4.5.1} will require some preliminaries. We begin by analyzing the relationship of the simplicial path category $\text{Path}[\Delta^n]_\bullet \simeq \text{Path}[n]_\bullet$ with the subcategory $\text{Path}[\Lambda^n_i]_\bullet$, where $\Lambda^n_i \subseteq \Delta^n$ is an inner horn.
CHAPTER 2. EXAMPLES OF ∞-CATEGORIES

Notation 2.4.5.2 (Cubes as Simplicial Sets). Let $I$ be a set. We let $\square^I$ denote the simplicial set given by the product $\prod_{i \in I} \Delta^1$. We will refer to $\square^I$ as the $I$-cube. Equivalently, we can describe $\square^I$ as the nerve of the power set $P(I) = \{I_0 \subseteq I\}$, where we regard $P(I)$ as partially ordered with respect to inclusion.

In the special case where $I$ is the set $\{1, 2, \ldots, n\}$ for some nonnegative integer $n$, we will denote the simplicial set $\square^I$ by $\square^n$ and refer to it as the standard $n$-cube.

Remark 2.4.5.3. Let $I$ be a finite set and let $\square^I$ be the $I$-cube of Notation 2.4.5.2. Then the geometric realization $|\square^I|$ can be identified with the topological space $\prod_{i \in I}[0, 1]$. In particular, the geometric realization $|\square^n|$ is homeomorphic to the standard cube $\{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_i \leq 1\}$.

This is a tautology in the case $n = 1$, and follows in general from the compatibility of geometric realizations with products of finite simplicial sets (see Corollary 3.5.2.2).

Remark 2.4.5.4. Let $n \geq 0$ be a nonnegative integer. For $0 \leq i < j \leq n$, we have a canonical isomorphism of simplicial sets

$$
\text{Hom}_{\text{Path}[n]}(i, j)_* = N_*(\{\text{Subsets } S \subseteq [n] \text{ with } \min(S) = i \text{ and } \max(S) = j\})
\cong N_*(P(\{i + 1, i + 2, \ldots, j - 2, j - 1\}))
\cong \square^{i+1,i+2,\ldots,j-2,j-1}
\cong \square^{j-i-1},
$$

where the second map is given by the construction $S \mapsto S \setminus \{i, j\}$. In particular, we have a canonical isomorphism of simplicial sets $\text{Hom}_{\text{Path}[n]}(0, n)_* \cong \square^n$.

Under these isomorphisms, the composition law on $\text{Path}[n]_*$ is given for $i < j < k$ by the construction

$$
\text{Hom}_{\text{Path}[n]}(j, k)_* \times \text{Hom}_{\text{Path}[n]}(i, j)_* \cong \square^{j+1,\ldots,k-1} \times \square^{i+1,\ldots,j-1}
\cong \square^{j+1,\ldots,k-1} \times \{1\} \times \square^{i+1,\ldots,j-1}
\cong \square^{j+1,\ldots,k-1} \times \Delta^1 \times \square^{0+1,\ldots,j-1}
\cong \square^{i+1,\ldots,k-1}
\cong \text{Hom}_{\text{Path}[n]}(i, k)_*.
$$

Notation 2.4.5.5 (Subsets of the $I$-Cube). Let $I$ be a finite set and let $\square^I$ denote the $I$-cube of Notation 2.4.5.2. For each element $i \in I$, we can identify $\square^I$ with the product $\Delta^1 \times \square^{I \setminus \{i\}}$. Using this identification, we obtain simplicial subsets

$$
\{0\} \times \square^{I \setminus \{i\}} \subseteq \square^I \supseteq \{1\} \times \square^{I \setminus \{i\}}.
$$
which we will refer to as *faces* of $\square^I$. The (disjoint) union of these two faces is another simplicial subset of $\square^I$, which we can identify with the product $\partial \Delta^1 \times \square^{I\setminus \{i\}}$.

We let $\partial \square^I$ denote the simplicial subset of $\square^I$ given by the union

$$\bigcup_{i \in I}(\partial \Delta^1 \times \square^{I\setminus \{i\}})$$

of all its faces. We will refer to $\partial \square^I$ as the *boundary of the* $I$-*cube* $\square^I$.

For $i \in I$, we let $\cap_i^I \subseteq \square^I$ denote the simplicial subset of $\square^I$ given by the union of the face ($\{0\} \times \square^{I\setminus \{i\}}$) with $\bigcup_{j \in I\setminus \{i\}}(\partial \Delta^1 \times \square^{I\setminus \{j\}})$. Similarly, we let $\cup_i^I$ denote the simplicial subset of $\square^I$ given by the union of the face ($\{1\} \times \square^{I\setminus \{i\}}$) with $\bigcup_{j \in I\setminus \{i\}}(\partial \Delta^1 \times \square^{I\setminus \{j\}})$. We will refer to the simplicial subsets $\cap_i^I$, $\cup_i^I \subseteq \square^I$ as *hollow* $I$-*cubes*.

In the special case where $I = \{1, \ldots, n\}$ for some nonnegative integer $n$, we will denote the simplicial sets $\partial \square^n$, $\cap_i^n$, and $\cup_i^n$ by $\partial \square^n$, $\cap_i^n$, and $\cup_i^n$, respectively.

**Remark 2.4.5.6.** Roughly speaking, one can think of the simplicial set $\partial \square^n$ as obtained from the $n$-cube $\square^n$ by removing its interior, while the subsets $\cap_i^n$, $\cup_i^n$ are obtained from $\square^n$ by removing the interior together with a single face.

**Example 2.4.5.7.** The standard 2-cube $\square^2 \simeq \Delta^1 \times \Delta^1$ is depicted in the diagram

```
• — — — — — — •
|            |
• — — — — — — •
|            |
• — — — — — — •
```

It is a simplicial set of dimension 2, having exactly two nondegenerate 2-simplices (represented by the triangular regions in the preceding diagram) and five nondegenerate edges. The boundary $\partial \square^2$ is a 1-dimensional simplicial subset of $\square^2$, obtained by removing the nondegenerate 2-simplices along with the "internal" edge to obtain the directed graph depicted in the diagram

```
• — — — — — — •
|            |
• — — — — — — •
```

Each of the hollow 2-cubes $\cap_1^2$, $\cap_2^2$, $\cup_1^2$, $\cup_2^2$ can be obtained from $\partial \square^2$ by further deletion of
Proposition 2.4.5.8. Let $0 < i < n$ be positive integers and let $F : \text{Path}[\Lambda^n_i] \to \text{Path}[\Delta^n]$ be the simplicial functor induced by the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$. Then:

(a) The functor $F$ is bijective on objects; in particular, we can identify the objects of Path[\Lambda^n_i] with elements of the set $[n] = \{0 < 1 < \cdots < n\}$.

(b) For $(j,k) \neq (0,n)$, the functor $F$ induces an isomorphism of simplicial sets

$$\text{Hom}_{\text{Path}[\Lambda^n_i]}(j,k) \simeq \text{Hom}_{\text{Path}[\Delta^n]}(j,k).$$

(c) The functor $F$ induces a monomorphism of simplicial sets

$$\text{Hom}_{\text{Path}[\Lambda^n_i]}(0,n) \hookrightarrow \text{Hom}_{\text{Path}[\Delta^n]}(0,n),$$

whose image can be identified with the hollow cube

$$\sqcup_i^{n-1} \subseteq [n-1] \simeq \text{Hom}_{\text{Path}[\Delta^n]}(0,n).$$

introduced in Notation 2.4.5.5.

Proof. Assertion (a) is immediate from Theorem 2.4.4.10. To prove (b) and (c), fix an integer $m \geq 0$. Using Lemma 2.4.4.16, we see that Path[\Delta^n]_m can be identified with the path category Path[G] of a directed graph $G$ which can be described concretely as follows:

- The vertices of $G$ are the elements of the set $[n] = \{0 < 1 < \cdots < n\}$. 

a single edge, represented in the diagrams

\[ \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_1^n \]

\[ \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_2^n \]

\[ \bullet \quad \longrightarrow \quad \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_1^n \quad \longrightarrow \quad \bullet \]

\[ \bullet \quad \longrightarrow \quad \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_2^n \quad \longrightarrow \quad \bullet \]

\[ \bullet \quad \longrightarrow \quad \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_1^n \quad \longrightarrow \quad \bullet \]

\[ \bullet \quad \longrightarrow \quad \bullet \quad \longrightarrow \quad \bullet \]
\[ \sqcup_2^n \quad \longrightarrow \quad \bullet \]
For $0 \leq j < k \leq n$, an edge of $G$ with source $j$ and target $k$ is a chain of subsets

$$\{j < k\} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m \subseteq \{j, j+1, \ldots, k-1, k\}$$

Using Theorem 2.4.4.10, we see that $\text{Path}[\Lambda^n_m]$ can be identified with the path category of the directed subgraph $G' \subseteq G$ having the same vertices, where an edge $\vec{I} = (I_0 \subseteq \cdots \subseteq I_m)$ of $G$ belongs to $G'$ if and only if the subset $I_m$ corresponds to a face of $\Delta^n$ which belongs to the subset $\Lambda^n_i$: that is, if and only if $[n] \setminus \{i\} \notin I_m$. In particular, we see that for $(j, k) \neq (0, n)$, every edge of $G$ with source $j$ and target $k$ is contained in $G'$. It follows that the simplicial functor $F$ induces a bijection

$$\text{Hom}_{\text{Path}[\Lambda^n_i]}(j, k)_m \rightarrow \text{Hom}_{\text{Path}[\Delta^n]}(j, k)_m$$

for $(j, k) \neq (0, n)$, which proves (b). Moreover, the map

$$\text{Hom}_{\text{Path}[\Lambda^n_i]}(0, n)_m \rightarrow \text{Hom}_{\text{Path}[\Delta^n]}(0, n)_m$$

is a monomorphism, whose image consists of those chains

$$\vec{I} = (I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m)$$

where either $I_0 \neq \{0, n\}$ or $([n] \setminus \{i\}) \notin I_m$. Under the identification of $\text{Hom}_{\text{Path}[\Delta^n]}(0, n)_\bullet$ with the cube $\square^{i-1} \simeq N_\bullet(P(\{1, \ldots, n-1\}))$ supplied by Remark 2.4.5.4, this corresponds to collection of $m$-simplices of $\square^{i-1}$ given by chains of subsets

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m \subseteq \{1, \ldots, n-1\}$$

where either $J_0 \neq \emptyset$ or $\{1, 2, \ldots, i-1, i+1, \ldots, n-1\} \notin J_m$, which is exactly the set of $m$-simplices which belong to the hollow cube $\sqcup^{i-1}$.

To apply Proposition 2.4.5.8, we record the following elementary observation about simplicial categories:

**Proposition 2.4.5.9.** Let $\mathcal{E}_\bullet$ be a simplicial category containing a pair of objects $x, y \in \text{Ob}(\mathcal{E}_\bullet)$. Assume that, for each object each object $z \in \text{Ob}(\mathcal{E}_\bullet)$, we have

$$\text{Hom}_{\mathcal{E}_\bullet}(z, x)_\bullet = \begin{cases} \{\text{id}_x\} & \text{if } z = x \\ \emptyset & \text{otherwise}. \end{cases} \quad \text{Hom}_{\mathcal{E}_\bullet}(y, z)_\bullet = \begin{cases} \{\text{id}_y\} & \text{if } z = y \\ \emptyset & \text{otherwise}. \end{cases}$$

Let $\mathcal{D}_\bullet \subseteq \mathcal{E}_\bullet$ denote a simplicial subcategory having the same objects, which satisfies

$$\text{Hom}_{\mathcal{D}_\bullet}(a, b)_\bullet = \text{Hom}_{\mathcal{E}_\bullet}(a, b)_\bullet.$$
unless \((a, b) = (x, y)\). Let \(F : \mathcal{D} \to \mathcal{C}\) be a functor of simplicial categories carrying \(x\) to an object \(X = F(x)\) and \(y\) to an object \(Y \in F(y)\), so that \(F\) induces a map of simplicial sets \(F_{x,y} : \text{Hom}_\mathcal{D}(x, y) \to \text{Hom}_\mathcal{C}(X, Y)\). Then the construction \(\overline{F} \mapsto \overline{F}_{x,y}\) induces a bijection

\[
\{\text{Simplicial functors } F : \mathcal{E} \to \mathcal{C} \text{ extending } F\} \\ \sim \\
\{\text{Maps } \lambda : \text{Hom}_\mathcal{E}(x, y) \to \text{Hom}_\mathcal{C}(X, Y) \text{ extending } F_{x,y}\}\.
\]

**Proof.** Fix a map of simplicial sets \(\lambda : \text{Hom}_\mathcal{E}(x, y) \to \text{Hom}_\mathcal{C}(X, Y)\) which extends \(F_{x,y}\). We wish to show that there is a unique simplicial functor \(F : \mathcal{E} \to \mathcal{C}\) such that \(F = \overline{F}\) and \(\overline{F}_{x,y} = \lambda\). The uniqueness is clear: the simplicial functor \(F\) must coincide with \(F\) on objects and satisfy \(F_{x', y'} = \overline{F}_{x', y'}\) for \((x', y') \neq (x, y)\). To prove existence, one must show that this prescription defines a simplicial functor: that is, that for every triple of objects \(a, b, c \in \text{Ob}(\mathcal{E})\), the resulting diagram of simplicial sets

\[
\begin{array}{c}
\text{Hom}_\mathcal{E}(b, c) \times \text{Hom}_\mathcal{E}(a, b) \to \text{Hom}_\mathcal{E}(a, c) \\
\downarrow F_{a,b} \otimes F_{b,c} \\
\text{Hom}_\mathcal{C}(F(b), F(c)) \times \text{Hom}_\mathcal{C}(F(a), F(b)) \to \text{Hom}_\mathcal{C}(F(a), F(c))
\end{array}
\]

is commutative. We consider several cases:

- Suppose that \((a, b) = (x, y)\). If \(c \neq y\), then the simplicial set \(\text{Hom}_\mathcal{E}(b, c)\) is empty and the commutativity of the diagram is automatic. If \(c = y\), then both compositions can be identified with the map

  \[\{\text{id}_y\} \times \text{Hom}_\mathcal{E}(x, y) \simeq \text{Hom}_\mathcal{E}(x, y) \xrightarrow{\lambda} \text{Hom}_\mathcal{C}(X, Y)\,\cdot\,
  \]

- Suppose that \((b, c) = (x, y)\). If \(a \neq x\), then the simplicial set \(\text{Hom}_\mathcal{E}(a, b)\) is empty and the commutativity of the diagram is automatic. If \(a = x\), then both compositions can be identified with the map

  \[\text{Hom}_\mathcal{E}(x, y) \times \{\text{id}_x\} \simeq \text{Hom}_\mathcal{E}(x, y) \xrightarrow{\lambda} \text{Hom}_\mathcal{C}(X, Y)\,\cdot\,
  \]

- If neither \((a, b) = (x, y)\) nor \((b, c) = (x, y)\), then the desired result follows from the commutativity of the diagram

  \[
  \begin{array}{c}
  \text{Hom}_\mathcal{D}(b, c) \times \text{Hom}_\mathcal{D}(a, b) \to \text{Hom}_\mathcal{D}(a, c) \\
  \downarrow F_{a,b} \otimes F_{b,c} \\
  \text{Hom}_\mathcal{C}(F(b), F(c)) \times \text{Hom}_\mathcal{C}(F(a), F(b)) \to \text{Hom}_\mathcal{C}(F(a), F(c))
  \end{array}
  \]

(since \(F\) is assumed to be a simplicial functor).
2.4. SIMPLICIAL CATEGORIES

It follows from Proposition 2.4.5.8 that for \( 0 < i < n \), the hypotheses of Proposition 2.4.5.9 are satisfied by the inclusion \( D_i = \text{Path}[\Lambda^n_i] \rightarrow \text{Path}[\Delta^n] = \mathcal{E} \) and the objects \( x = 0 \) and \( y = n \). We therefore obtain the following:

**Corollary 2.4.5.10.** Let \( \mathcal{C} \) be a simplicial category, let \( 0 < i < n \), and let \( \sigma_0 : \Lambda^n_i \rightarrow N^{hc}(\mathcal{C}) \) be a map of simplicial sets, which we can identify with a simplicial functor \( F : \text{Path}[\Lambda^n_i] \rightarrow \mathcal{C} \) inducing a map of simplicial sets

\[
\lambda_0 : \sqcup_i^{n-1} \simeq \text{Hom}_{\text{Path}[\Lambda^n_i]}(0, n) \rightarrow \text{Hom}_\mathcal{C}(F(0), F(n)).
\]

Then we have a canonical bijection

\[
\{ \text{Maps } \sigma : \Delta^n \rightarrow N^{hc}(\mathcal{C}) \text{ with } \sigma_0 = \sigma|_{\Lambda^n_i} \} \quad \xrightarrow{\sim} \quad \{ \text{Maps } \lambda : \square^{n-1} \rightarrow \text{Hom}_\mathcal{C}(F(0), F(n)) \text{ with } \lambda_0 = \lambda|_{\sqcup_i^{n-1}} \}.
\]

To deduce Theorem 2.4.5.1 from Corollary 2.4.5.10 we will need the following standard characterization of Kan complexes (for a proof, see Proposition 4.4.2.1):

**Theorem 2.4.5.11 (Homotopy Extension Lifting Property).** Let \( X \) be a simplicial set. The following conditions are equivalent:

1. The simplicial set \( X \) is a Kan complex.
2. The inclusion of simplicial sets \( \{0\} \rightarrow \Delta^1 \) induces a trivial Kan fibration \( \text{Fun}(\Delta^1, X) ightarrow \text{Fun}(\{0\}, X) \simeq X \).
3. The inclusion of simplicial sets \( \{1\} \rightarrow \Delta^1 \) induces a trivial Kan fibration \( \text{Fun}(\Delta^1, X) ightarrow \text{Fun}(\{1\}, X) \simeq X \).

**Corollary 2.4.5.12.** Let \( X \) be a Kan complex and let \( I \) be a finite set containing a distinguished element \( i \). Then:

(a) Every map of simplicial sets \( f : \sqcup I \rightarrow X \) can be extended to a map \( \overline{f} : \square^I \rightarrow X \).

(b) Every map of simplicial sets \( g : \prod I \rightarrow X \) can be extended to a map \( \overline{g} : \square^I \rightarrow X \).

**Proof.** Unwinding the definitions, we see that \( \sqcup I \) can be identified with the pushout

\[
\left( \{1\} \times \square^I \setminus \{i\} \right) \sqcup_{\{1\} \times \partial \square^I \setminus \{i\}} (\Delta^1 \times \partial \square^I \setminus \{i\}).
\]
Consequently, a map of simplicial sets $f : \Lambda^I_i \to X_\bullet$ can be identified with a commutative diagram of solid arrows

$$
\begin{array}{ccc}
\partial \Lambda^I_i & \longrightarrow & \text{Fun}(\Delta^1, X_\bullet) \\
\downarrow & & \downarrow \\
\Lambda^I_i & \longrightarrow & \text{Fun}(\{1\}, X_\bullet),
\end{array}
$$

and an extension $\mathcal{F} : \square^I \to X_\bullet$ of $f$ can be identified with a solution to the associated lifting problem. If $X_\bullet$ is a Kan complex, then the right vertical arrow is a trivial Kan fibration (Theorem 2.4.5.11), so the desired extension exists by virtue of Proposition 1.4.5.3. This proves (a); the proof of (b) is similar.

Proof of Theorem 2.4.6.1. Let $C_\bullet$ be a locally Kan simplicial category; we wish to show that the homotopy coherent nerve $N^{hc}_\bullet(C)$ is an $\infty$-category. Fix positive integers $0 < i < n$; we wish to show that every map of simplicial sets $\sigma_0 : \Lambda^n_i \to N^{hc}_\bullet(C)$ can be extended to an $n$-simplex $\sigma : \Delta^n \to N^{hc}_\bullet(C)$. Let us identify $\sigma_0$ with a simplicial functor $F : \text{Path}[\Lambda^n_i] \to C_\bullet$ inducing a map of simplicial sets $\lambda_0 : \Lambda^{n-1}_i \to \text{Hom}_{C}(F(0), F(n))_\bullet$. By virtue of Corollary 2.4.5.10, it will suffice to show that $\lambda_0$ can be extended to a map of simplicial sets $\lambda : \square^{n-1} \to \text{Hom}_{C}(F(0), F(n))_\bullet$. The existence of this extension follows from Corollary 2.4.5.12, by virtue of our assumption that $\text{Hom}_{C}(F(0), F(n))_\bullet$ is a Kan complex.

2.4.6 The Homotopy Category of a Simplicial Category

For every simplicial set $S_\bullet$, we let $\pi_0(S_\bullet)$ denote the set of connected components of $S_\bullet$ (Definition 1.1.6.8). Recall that the functor $\pi_0 : \text{Set}_\Delta \to \text{Set}$ preserves finite products (Corollary 1.1.6.26). Applying Remark 2.1.7.4, we obtain the following:

Construction 2.4.6.1 (The Homotopy Category of a Simplicial Category). Let $C_\bullet$ be a simplicial category. We define an ordinary category $hC$ as follows:

- The objects of $hC$ are the objects of $C$.
- For every pair of objects $X, Y \in \text{Ob}(hC) = \text{Ob}(C)$, we have
  $$\text{Hom}_{hC}(X, Y) = \pi_0(\text{Hom}_{C}(X, Y)_\bullet).$$
- For every triple of objects $X, Y, Z \in \text{Ob}(hC) = \text{Ob}(C)$, the composition map
  $$\circ : \text{Hom}_{hC}(Y, Z) \times \text{Hom}_{hC}(X, Y) \to \text{Hom}_{hC}(X, Z)$$
is given by the composition

\[ \text{Hom}_{hC}(Y, Z) \times \text{Hom}_{hC}(X, Y) = \pi_0(\text{Hom}_C(Y, Z)_\bullet) \times \pi_0(\text{Hom}_C(X, Y)_\bullet) \]

\[ \cong \pi_0(\text{Hom}_C(Y, Z)_\bullet) \times \pi_0(\text{Hom}_C(X, Y)_\bullet) \]

\[ \to \pi_0(\text{Hom}_C(X, Z)_\bullet) \]

\[ = \text{Hom}_{hC}(X, Z). \]

We will refer to \( hC \) as the *homotopy category of* \( C \).

**Remark 2.4.6.2 (The Component Functor).** Let \( C_\bullet \) be a simplicial category and let \( hC \) be its homotopy category (Construction 2.4.6.1). For every pair of objects \( X, Y \in \text{Ob}(C_\bullet) = \text{Ob}(hC) \), Construction 1.1.6.18 supplies a map of simplicial sets

\[ u_{X,Y} : \text{Hom}_C(X, Y)_\bullet \to \text{Hom}_{hC}(X, Y)_\bullet. \]

Here \( \text{Hom}_{hC}(X, Y)_\bullet \) denotes the constant simplicial set associated to the set \( \text{Hom}_{hC}(X, Y) \), and \( u_{X,Y} \) carries each \( n \)-simplex of \( \text{Hom}_C(X, Y)_\bullet \) to the connected component which contains it. Allowing \( X \) and \( Y \) to vary, we obtain a simplicial functor \( u : C_\bullet \to hC_\bullet \) which is the identity on objects; we will refer to \( u \) as the *component functor*.

**Remark 2.4.6.3.** Let \( C_\bullet \) be a simplicial category with underlying category \( C = C_0 \). Then the simplicial functor \( u : C_\bullet \to hC_\bullet \) induces a functor of ordinary categories \( u_0 : C \to hC \), which can be described as follows:

- On objects, the functor \( u_0 \) is the identity map from \( \text{Ob}(C) = \text{Ob}(hC) \) to itself.
- For every pair of objects \( X, Y \in \text{Ob}(C) = \text{Ob}(hC) \), the induced map \( \text{Hom}_C(X, Y) \to \text{Hom}_{hC}(X, Y) \) is a surjection, which we will denote by \( f \mapsto [f] \).
- Given a pair of morphisms \( f, g : X \to Y \) in \( C \) having the same source and target, we have \( [f] = [g] \) if and only if \( f \) and \( g \) belong to the same connected component of the simplicial set \( \text{Hom}_C(X, Y)_\bullet \).

**Remark 2.4.6.4.** Let \( C_\bullet \) be a simplicial category with underlying category \( C = C_0 \), and let \( f, g : X \to Y \) be a pair of morphisms of \( C \) having the same source and target. Using Remark 1.1.6.23, we see that the following conditions are equivalent:

(a) The morphisms \( f \) and \( g \) represent the same morphism in the homotopy category \( hC \); that is, we have \( [f] = [g] \).

(b) There exists a sequence of morphisms \( f = f_0, f_1, f_2, \ldots, f_n = g \in \text{Hom}_C(X, Y) \) such that, for \( 1 \leq i \leq n \), either there exists a homotopy from \( f_{i-1} \) to \( f_i \) or a homotopy from \( f_i \) to \( f_{i-1} \) (in the sense of Definition 2.4.1.6).
If \( C \) is locally Kan, then we can replace (b) by the following simpler condition:

(c) There exists a homotopy from \( f \) to \( g \) (in the sense of Definition 2.4.1.6).

See Remark 2.4.1.9.

**Example 2.4.6.5** (The Homotopy Category of Top). Let Top denote the category of topological spaces and continuous functions, endowed with the simplicial enrichment \( \text{Top}_\bullet \) described in Example 2.4.1.5. Then the homotopy category \( \text{hTop} \) is the homotopy category of all topological spaces: the objects of \( \text{hTop} \) are topological spaces, and the morphisms of \( \text{hTop} \) are homotopy classes of continuous maps.

The homotopy category of a simplicial category can be characterized by a universal mapping property:

**Proposition 2.4.6.6.** Let \( C \) be a simplicial category and let \( u : C \rightarrow \text{h}C \) be the simplicial functor described in Remark 2.4.6.2. Then, for any category \( D \), composition with \( u \) induces a bijection

\[
\{ \text{Ordinary Functors } f : \text{h}C \rightarrow D \} \rightarrow \{ \text{Simplicial Functors } F : C \rightarrow D \}.
\]

**Proof.** Use Proposition 1.1.6.19. \( \square \)

**Corollary 2.4.6.7.** The fully faithful embedding

\[
\text{Cat} \hookrightarrow \text{Cat}_\Delta \quad D \mapsto D_\bullet
\]

of Example 2.4.2.3 admits a left adjoint, given on objects by the formation of homotopy categories \( C \rightarrow \text{h}C \).

We have now introduced two different notions of homotopy category:

- The homotopy category \( \text{h}C \) of a simplicial category \( C \), given by Construction 2.4.1.1.
- The homotopy category \( \text{h}S \) of a simplicial set \( S \), defined in Definition 1.2.5.1 (and described more explicitly in §1.3.5 when \( S \) is an \( \infty \)-category).

These constructions are related. Let \( C \) be a simplicial category. Applying the homotopy coherent nerve to the component functor \( u \) of Remark 2.4.6.2 we obtain a map of simplicial sets

\[
N^{\text{hc}}_\bullet(C) \xrightarrow{N^{\text{hc}}_\bullet(u)} N^{\text{hc}}_\bullet(\text{h}C) \simeq N_\bullet(\text{h}C),
\]

which we can identify with a functor of ordinary categories \( U : \text{h}N^{\text{hc}}_\bullet(C) \rightarrow \text{h}C \).

**Proposition 2.4.6.8.** Let \( C \) be a locally Kan simplicial category. Then the construction above induces an isomorphism of categories \( U : \text{h}N^{\text{hc}}_\bullet(C) \simeq \text{h}C. \)
2.4. SIMPLICIAL CATEGORIES

To prove Proposition 2.4.6.8, we need to analyze the 2-simplices of the homotopy coherent nerve $N^{hc}(C)$. Recall that the vertices and edges of $N^{hc}(C)$ can be identified with objects and morphisms in the underlying category $C = C_0$ (Example 2.4.3.9). In particular, a map of simplicial sets $\sigma_0 : \partial \Delta^2 \to N^{hc}(C)$ can be identified with a (possibly noncommutative) diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{f_{10}} & X_1 \\
& \nearrow_{f_{21}} & \searrow^{f_{20}} \\
& & X_2
\end{array}
$$

in the category $C$. We will need the following:

**Proposition 2.4.6.9.** Let $C_\bullet$ be a simplicial category and let $\sigma_0 : \partial \Delta^2 \to N^{hc}(C)$ be a map of simplicial sets, which we identify with a diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{f_{10}} & X_1 \\
& \nearrow_{f_{21}} & \searrow^{f_{20}} \\
& & X_2
\end{array}
$$

as above. Then the construction of Example 2.4.3.10 induces a bijection

$$
\{\text{Maps } \sigma : \Delta^2 \to N^{hc}(C) \text{ with } \sigma|_{\partial \Delta^2} = \sigma_0\} \sim \{\text{Homotopies from } f_{20} \text{ to } f_{21} \circ f_{10}\}.
$$

It is not difficult to deduce Proposition 2.4.6.9 directly from the definition of the homotopy coherent nerve. We will instead deduce it from a more general result (Corollary 2.4.6.12), which supplies an analogous description of the $n$-simplices of $N^{hc}(C)$ for all $n > 0$. First, let us note some consequences of Proposition 2.4.6.9.

**Example 2.4.6.10.** Let $C_\bullet$ be a locally Kan simplicial category, so that the homotopy coherent nerve $N^{hc}(C)$ is an $\infty$-category (Theorem 2.4.5.1). Suppose we are given a pair of morphisms $f, g : X \to Y$ in the underlying category $C = C_0$ having the same source and target. Let $\sigma_0 : \partial \Delta^2 \to N^{hc}(C)$ be the map corresponding to the (possibly noncommutative) diagram

$$
\begin{array}{ccc}
& & Y \\
& f & \searrow^{\text{id}_Y} \\
X & \xrightarrow{g} & Y
\end{array}
$$

Applying Proposition 2.4.6.9 we obtain a bijection from the set of homotopies from $g$ to $f$ in the $\infty$-category $N^{hc}(C)$ (in the sense of Definition 1.3.3.1) to the set of homotopies from
$f$ to $g$ in the simplicial category $\mathcal{C}_\bullet$ (in the sense of Definition 2.4.1.6). In particular, we see that $f$ and $g$ are homotopic in $N_{\text{hc}}^\bullet(\mathcal{C})$ if and only if they are homotopic in $\mathcal{C}_\bullet$.

**Proof of Proposition 2.4.6.8.** Let $\mathcal{C}_\bullet$ be a locally Kan simplicial category; we wish to show that the comparison map $U : hN_{\text{hc}}^\bullet(\mathcal{C}) \overset{\sim}{\rightarrow} h\mathcal{C}$ is an isomorphism of categories. By construction, $U$ is bijective on objects. It will therefore suffice to show that for every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, the induced map

$$U_{X,Y} : \text{Hom}_{hN_{\text{hc}}^\bullet(\mathcal{C})}(X,Y) \rightarrow \text{Hom}_{h\mathcal{C}}(X,Y)$$

is a bijection. This is precisely the content of Example 2.4.6.10.

We will deduce Proposition 2.4.6.9 from the following variant of Proposition 2.4.5.8:

**Proposition 2.4.6.11.** Let $n$ be a positive integer and let $F : \text{Path}[^\partial \Delta^n]_\bullet \rightarrow \text{Path}[\Delta^n]_\bullet$ be the simplicial functor induced by the boundary inclusion $\partial \Delta^n \hookrightarrow \Delta^n$. Then:

(a) The functor $F$ is bijective on objects; in particular, we can identify objects of $\text{Path}[^\partial \Delta^n]_\bullet$ with elements of the set $[n] = \{0 < 1 < \cdots < n\}$.

(b) For $(j,k) \neq (0,n)$, the functor $F$ induces an isomorphism of simplicial sets

$$\text{Hom}_{\text{Path}[^\partial \Delta^n]_\bullet}(j,k)_\bullet \simeq \text{Hom}_{\text{Path}[\Delta^n]_\bullet}(j,k)_\bullet.$$

(c) The functor $F$ induces a monomorphism of simplicial sets $\text{Hom}_{\text{Path}[^\partial \Delta^n]_\bullet}(0,n)_\bullet \hookrightarrow \text{Hom}_{\text{Path}[\Delta^n]_\bullet}(0,n)_\bullet$, whose image can be identified with the boundary $\partial \Delta^{n-1} \subseteq \Delta^{n-1} \simeq$ introduced in Notation 2.4.5.5.

**Proof.** Assertion (a) is immediate from Theorem 2.4.4.10. To prove (b) and (c), fix an integer $m \geq 0$ and let us identify $\text{Path}[\Delta^n]_m$ with the path category $\text{Path}[G]$ of the directed graph $G$ appearing in the proof of Proposition 2.4.5.8. Using Theorem 2.4.4.10, we see that $\text{Path}[\partial \Delta^n]_m$ can be identified with the path category of the directed subgraph $G' \subseteq G$ having the same vertices, where an edge $\overrightarrow{I} = (I_0 \subseteq \cdots \subseteq I_m)$ of $G$ belongs to $G'$ if and only if $I_m \neq [n]$. In particular, we see that for $(j,k) \neq (0,n)$, every edge of $G$ with source $j$ and target $k$ is contained in $G'$. It follows that the simplicial functor $F$ induces a bijection

$$\text{Hom}_{\text{Path}[\partial \Delta^n]_\bullet}(j,k)_m \rightarrow \text{Hom}_{\text{Path}[\Delta^n]_\bullet}(j,k)_m$$

for $(j,k) \neq (0,n)$, which proves (b). Moreover, the map

$$\text{Hom}_{\text{Path}[\partial \Delta^n]_\bullet}(0,n)_m \rightarrow \text{Hom}_{\text{Path}[\Delta^n]_\bullet}(0,n)_m$$

is a monomorphism, whose image consists of those chains

$$\overrightarrow{I} = (I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m)$$
2.4. SIMPLICIAL CATEGORIES

where either $I_0 \neq \{0, n\}$ or $I_m \neq [n]$. Under the identification of $\text{Hom}_{\mathcal{C}}(\partial \Delta^n, \Delta^n)$ with the cube $\square^{n-1} \simeq N\bullet(P(\{1, \ldots, n-1\}))$, this is exactly the set of $m$-simplices which belong to the boundary $\partial \square^{n-1} \subseteq \square^n$.

Combining Propositions 2.4.6.11 and 2.4.5.9, we obtain the following:

**Corollary 2.4.6.12.** Let $\mathcal{C}$ be a simplicial category, let $n > 0$, and let $\sigma_0 : \partial \Delta^n \to N\bullet^h(\mathcal{C})$ be a map of simplicial sets, which we identify with a simplicial functor $F : \mathcal{C}$ inducing a map of simplicial sets

$$\lambda_0 : \partial \square^{n-1} \simeq \text{Hom}_{\mathcal{C}}(\partial \Delta^n) \to \text{Hom}_{\mathcal{C}}(F(0), F(n))\bullet.$$

Then we have a canonical bijection

$$\{\text{Maps } \sigma : \Delta^n \to N\bullet^h(\mathcal{C}) \text{ with } \sigma_0 = \sigma|_{\partial \Delta^n} \} \to \{\text{Maps } \lambda : \partial \square^{n-1} \to \text{Hom}_{\mathcal{C}}(F(0), F(n))\bullet \text{ with } \lambda_0 = \lambda|_{\partial \square^{n-1}} \}.$$

**Example 2.4.6.13 (1-Simplices of the Homotopy Coherent Nerve).** Let $\mathcal{C}$ be a simplicial category. By definition, giving a map of simplicial sets $\sigma_0 : \partial \Delta^1 \to N\bullet^h(\mathcal{C})$ is equivalent to giving a pair of objects $X_0 = \sigma_0(0)$ and $X_1 = \sigma_0(1)$ of the underlying category $\mathcal{C} = \mathcal{C}_0$. Applying Corollary 2.4.6.12, we deduce that extending $\sigma_0$ to a 1-simplex of $N\bullet^h(\mathcal{C})$ is equivalent to supplying a morphism $f : X_0 \to X_1$ in the category $\mathcal{C}$ (see Example 2.4.3.9).

**Proof of Proposition 2.4.6.9.** Apply Corollary 2.4.6.12 in the case $n = 2$. 

**Example 2.4.6.14 (3-Simplices of the Homotopy Coherent Nerve).** Let $\mathcal{C}$ be a simplicial category. Using Proposition 2.4.6.9, we see that a map of simplicial sets $\sigma_0 : \partial \Delta^3 \to N\bullet^h(\mathcal{C})$ can be identified with the following data:

- A collection of four objects $\{X_i \in \mathcal{C}\}_{0 \leq i \leq 3}$.
- A collection of six morphisms $\{f_{ji} \in \text{Hom}_\mathcal{C}(X_i, X_j)\}_{0 \leq i < j \leq 3}$.
- A collection of four 1-simplices $\{h_{kji} \in \text{Hom}_\mathcal{C}(X_i, X_k)\}_{0 \leq i < j < k \leq 3}$, where each $h_{kji}$ is a homotopy from $f_{ki}$ to $f_{kj} \circ f_{ji}$.

From this data, we can assemble a map of simplicial sets $\lambda_0 : \partial \square^2 \to \text{Hom}_\mathcal{C}(X_0, X_3)\bullet$, which is represented by the diagram

$$\begin{array}{ccc}
  f_{30} & \xrightarrow{h_{310}} & f_{31} \circ f_{10} \\
  h_{320} & \downarrow & \downarrow h_{321} \circ f_{10} \\
  f_{32} \circ f_{20} & \xrightarrow{f_{32} \circ h_{210}} & f_{32} \circ f_{21} \circ f_{10};
\end{array}$$
here we slightly abuse notation by identifying the morphisms \( f_{10} \) and \( f_{32} \) with the corresponding degenerate edges of the simplicial sets \( \Hom_{C}(X_0, X_1) \) and \( \Hom_{C}(X_2, X_3) \), respectively. Corollary \textbf{2.4.6.12} then asserts that extending \( \sigma_0 \) to a 3-simplex of the homotopy coherent nerve \( \N^{hc}_C(C) \) is equivalent to extending \( \lambda_0 \) to a map of simplicial sets \( \lambda : \square^2 \to \Hom_{C}(X_0, X_3) \).

Stated more informally, the map \( \sigma_0 \) supplies two potentially different paths from \( f_{30} \) to the composition \( f_{32} \circ f_{21} \circ f_{10} \) in the simplicial set \( \Hom_{C}(X_0, X_3) \). To extend \( \sigma_0 \) to a 3-simplex of \( \N^{hc}_C(C) \), one must supply additional data which “witnesses” that these paths are homotopic.

\section{Example: Braid Monoids}

In general, the path category \( \Path[S] \), associated to a simplicial set \( S \), is a fairly complicated object. In this section, we describe one situation in which it admits a particularly concrete description, which arises in the theory of Coxeter groups. Let us begin by reviewing some terminology.

\begin{definition}
A \textit{Coxeter system} is a pair \((W, S)\), where \( W \) is a group and \( S \subseteq W \) is

\begin{itemize}
\item Each element of \( S \) has order 2.
\item For each \( s,t \in S \), let \( m_{s,t} \in \mathbb{Z}_{>0} \cup \{\infty\} \) denote the order of the product \( st \) in the group \( W \). Then the inclusion \( S \hookrightarrow W \) exhibits \( W \) as the quotient of the free group generated by \( S \) by the relations \( (st)^{m_{s,t}} = 1 \) (indexed by those pairs \( s,t \) with \( m_{s,t} < \infty \)).
\end{itemize}

\end{definition}

\begin{remark}
We will use the term \textit{Coxeter group} to refer to a group \( W \) together with a choice of subset \( S \subseteq W \) for which the pair \((W, S)\) is a Coxeter system. Beware that the subset \( S \) is not determined by the structure of \( W \) as an abstract group: for example, if \((W, S)\) is a Coxeter system, then so is \((W, wSw^{-1})\) for each \( w \in W \). In other words, a Coxeter group is not merely a group, but a group equipped with some additional structure (namely, the structure of a Coxeter system \((W, S)\)).
\end{remark}

\begin{notation}[Lengths]
Let \((W, S)\) be a Coxeter system. Then the group \( W \) is generated by \( S \): that is, every element of \( W \) can be written as a product of elements of \( S \). For each \( w \in W \), we let \( \ell(w) \) denote the smallest nonnegative integer \( n \) for which \( w \) factors as a product \( s_1s_2 \cdots s_n \), where each \( s_i \) belongs to \( S \). We will refer to \( \ell(w) \) as the \textit{length} of \( w \).
\end{notation}

\begin{remark}
Let \((W, S)\) be a Coxeter system. Then the length function \( \ell : W \to \mathbb{Z}_{\geq 0} \) has the following properties:

\begin{itemize}
\item An element \( w \in W \) satisfies \( \ell(w) = 0 \) if and only if \( w = 1 \) is the identity element of \( W \).
\end{itemize}
\end{remark}
• An element \( w \in W \) satisfies \( \ell(w) = 1 \) if and only if \( w \) belongs to \( S \).

• For every pair of elements \( w, w' \in W \), we have \( \ell(ww') \leq \ell(w) + \ell(w') \). Moreover, we also have \( \ell(ww') \equiv \ell(w) + \ell(w') \) \( \text{(mod 2)} \).

**Construction 2.4.7.5** (The Braid Group). Let \((W, S)\) be a Coxeter system. We let \( \text{Br}(W) \) denote the quotient of the free group generated by \( S \) by the relations \((st)^{m_{s,t}} = 1\), where \( s \) and \( t \) range over distinct elements of \( S \) satisfying \( m_{s,t} < \infty \); here \( m_{s,t} \) denotes the order of the product \( st \) in the group \( W \). We will refer to \( \text{Br}(W) \) as the braid group of the Coxeter system \((W, S)\). By construction, the braid group \( \text{Br}(W) \) is equipped with a surjective group homomorphism \( \text{Br}(W) \to W \), which exhibits \( W \) as the quotient of \( \text{Br}(W) \) by the relations \( s^2 = 1 \) for \( s \in S \).

Let \( \text{Br}^+(W) \) denote the submonoid of \( \text{Br}(W) \) generated by the elements of \( S \). We will refer to \( \text{Br}^+(W) \) as the braid monoid of the Coxeter system \((W, S)\).

In [7], Deligne gave a convenient simplicial presentation for the braid monoid \( \text{Br}^+(W) \) in the case where the Coxeter group \( W \) is finite. To formulate it, we need a bit more terminology.

**Notation 2.4.7.6.** Let \( W \) be a Coxeter group with identity element 1. We let \( M_0(W) \) denote the free monoid generated by the set \( W \setminus \{1\} \). We will identify the elements of \( M(W) \) with finite sequences \( \vec{w} = (w_1, w_2, \ldots, w_n) \), where each \( w_i \) is an element of \( W \setminus \{1\} \).

Let \( \vec{v} = (v_1, v_2, \ldots, v_m) \) and \( \vec{w} = (w_1, \ldots, w_n) \) be elements of \( M_0(W) \). We will say that \( \vec{w} \) is a refinement of \( \vec{v} \) if there exists a strictly increasing function \( \varphi : [m] \to [n] \) satisfying \( \varphi(0) = 0, \varphi(n) = n \), and

\[
v_i = w_{\varphi(i-1)+1}w_{\varphi(i-1)+2} \cdots w_{\varphi(i)}
\]

for \( 1 \leq i \leq m \). We write \( \vec{v} \preceq \vec{w} \) to indicate that \( \vec{w} \) is a refinement of \( \vec{v} \). Then \( \preceq \) determines a partial ordering on the set \( M_0(W) \). We denote the nerve of this partially ordered set by \( M_\bullet(W) \). Note that the multiplication on \( M_0(W) \) (given by concatenation) endows \( M_\bullet(W) \) with the structure of a simplicial monoid.

**Exercise 2.4.7.7.** Let \( W \) be a Coxeter group, and let \( \vec{u} = (u_1, u_2, \ldots, u_m) \) and \( \vec{w} = (w_1, \ldots, w_n) \) be elements of \( M(W) \). Show that, if \( \vec{w} \) is a refinement of \( \vec{u} \), then there is a unique sequence of integers \( 0 = j_0 < j_1 < \cdots < j_m = n \) satisfying the condition specified in Notation 2.4.7.6.

**Remark 2.4.7.8.** Let \((W, S)\) be a Coxeter system. Then an element \( \vec{w} = (w_1, w_2, \ldots, w_m) \) of \( M_0(W) \) is maximal (with respect to the refinement ordering \( \preceq \)) if and only if each \( w_i \)
Then:

Theorem 2.4.7.9 (Deligne). Let \((W, S)\) be a Coxeter system for which the underlying Coxeter group \(W\) is finite, and let \(\text{Br}^+(W)\) denote the braid monoid of Construction 2.4.7.5. Then:

(a) There is an isomorphism of monoids \(f : \pi_0(M_\bullet(W)) \to \text{Br}^+(W)\) which is uniquely determined by the following property: if \(\vec{s} = (s_1, s_2, \ldots, s_n) \in M_0(W)\) is a sequence of elements of \(S\), then \(f\) carries the connected component of \(\vec{s}\) to the product \(s_1 s_2 \cdots s_n \in \text{Br}^+(W)\).

(b) Each connected component of \(M_\bullet(W)\) is weakly contractible (Definition 3.2.6.4).

In other words, the isomorphism \(f\) determines a weak homotopy equivalence of simplicial monoids \(M_\bullet(W) \to \text{Br}^+(W)\).

Proof. This is a special case of Théorème 2.4 of [7].

We now reformulate the definition of the simplicial monoid \(M_\bullet(W)\) using the theory of simplicial path categories.

Notation 2.4.7.10. Let \((W, S)\) be a Coxeter system and let \(B_\bullet W\) denote the classifying simplicial set of the group \(W\) (Example 1.2.4.3). For each nonnegative integer \(n\), let us identify \(B_n W\) with the collection of all \(n\)-tuples \((w_n, w_{n-1}, \ldots, w_1)\) of elements of \(W\). Let \(B^n_n W\) denote the subset of \(B_n W\) consisting of those sequences \((w_n, w_{n-1}, \ldots, w_1)\) satisfying the identity

\[\ell(w_1 w_2 \cdots w_n) = \ell(w_1) + \ell(w_2) + \cdots + \ell(w_n).\]

It is easy to see that the collection of subsets \(B^n_n W \subseteq B_n W\) are stable under the face and degeneracy operators of \(B_\bullet W\), and therefore determine a simplicial subset \(B^n_\bullet W \subseteq B_\bullet W\).

Construction 2.4.7.11. Let \((W, S)\) be a Coxeter system, let \(M_\bullet(W)\) be the simplicial monoid of Notation 2.4.7.6 and let \(B M_\bullet(W)\) denote the simplicial category obtained by delooping \(M_\bullet(W)\) (Example 2.4.2.2), having a single object \(X\) with \(\text{Hom}_{BM(W)}(X, X)_\bullet = M_\bullet(W)\).

Let \(\sigma = (w_n, \ldots, w_1)\) be a nondegenerate \(n\)-simplex of the simplicial set \(B^n_\bullet(W)\) (Notation 2.4.7.10). Then \(\sigma\) determines a simplicial functor \(u(\sigma) : \text{Path}[n]_\bullet \to BM_\bullet(W)\), which carries each object of \(\text{Path}[n]_\bullet\) to the unique object \(X\) of \(BM_\bullet(W)\), and each morphism \(I = \{i_0 < \ldots < i_k\} \in \text{Hom}_{\text{Path}[n]}(i_0, i_k)\) to the sequence

\[(v_1, v_2, \ldots, v_k) \in M_0(W) \quad v_j = w_{i_{j-1}+1} w_{i_{j-1}+2} \cdots w_{i_j} \quad (v_1, v_2, \ldots, v_k) \in M_0(W).\]
Regarding \( u(\sigma) \) as an \( n \)-simplex of the homotopy coherent nerve \( N^{hc}_\bullet(BM(W)) \), the construction \( \sigma \mapsto u(\sigma) \) extends to a map of simplicial sets \( u : B^\bullet_\circ(W) \to N^{hc}_\circ(BM(W)) \).

**Proposition 2.4.7.12.** Let \((W,S)\) be a Coxeter system. Then the map of simplicial sets \( u : B^\bullet_\circ(W) \to N^{hc}_\circ(BM(W)) \) of Construction 2.4.7.11 exhibits \( BM_\bullet(W) \) as a path category of the simplicial set \( B^\bullet_\circ(W) \), in the sense of Definition 2.4.4.1.

**Proof.** Fix an integer \( m \geq 0 \). Then \( BM_m(W) \) is the delooping of the monoid \( M_m(W) \) whose elements are tuples \( \vec{w}_0 \preceq \vec{w}_1 \preceq \vec{w}_2 \preceq \cdots \preceq \vec{w}_m \), where each \( \vec{w}_i \in M_0(W) \) is a sequence \( (w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}) \) of elements of \( W \setminus \{1\} \). Moreover, the monoid structure on \( M_m(W) \) is given by concatenation. From this description, it is easy to see that the monoid \( M_m(W) \) is freely generated by its indecomposable elements, which are precisely those sequences for which \( \vec{w}_0 \) has length 1. In this case, the relation \( \vec{w}_0 \preceq \vec{w}_m \) guarantees that \( \vec{w}_m \) is a nondegenerate \( n_m \)-simplex of the simplicial set \( B^\bullet_\circ(W) \). It follows that the map \( u \) induces a bijection from the set \( E(B^\circ(W), m) \) of Notation 2.4.4.9 to the set of indecomposable elements of the monoid \( M_m(W) \). The desired result now follows from the criterion of Remark 2.4.4.11.

**Corollary 2.4.7.13.** Let \( W \) be a finite Coxeter group, and let \( B^\bullet_\circ(W) \subseteq B_\bullet(W) \) be the simplicial subset of Notation 2.4.7.10. Then the simplicial path category \( \text{Path}[B^\circ(W)]_\bullet \) has a single object \( X \), whose endomorphism monoid \( \text{Hom}_{\text{Path}[B^\circ(W)]}(X,X)_\bullet \) is weakly homotopy equivalent to the braid monoid \( Br^+(W) \) of Construction 2.4.7.5.

**Proof.** Combine Proposition 2.4.7.12 with Theorem 2.4.7.9.

### 2.5 Differential Graded Categories

Homological algebra provides a plentiful supply of examples of \( \infty \)-categories. Let us begin by reviewing some terminology.

**Definition 2.5.0.1.** Let \( \mathcal{A} \) be an additive category (Definition [?]). A **chain complex with values in** \( \mathcal{A} \) is a pair \((C_\bullet, \partial)\), where \( C_\bullet = \{C_n\}_{n \in \mathbb{Z}} \) is a collection of objects of \( \mathcal{A} \) and \( \partial = \{\partial_n\}_{n \in \mathbb{Z}} \) is a collection of morphisms \( \partial_n : C_n \to C_{n-1} \) in \( \mathcal{A} \) with the property that each composition \( \partial_n \circ \partial_{n+1} \) is the zero morphism from \( C_{n+1} \) to \( C_{n-1} \).

**Notation 2.5.0.2.** Let \( \mathcal{A} \) be an additive category. Then a chain complex \((C_\bullet, \partial)\) with values in \( \mathcal{A} \) can be graphically represented by a diagram

\[ \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots \]
in which each successive composition is equal to zero. We will generally abuse terminology by identifying \((C_\bullet, \partial)\) with the underlying collection \(C_\bullet = \{C_n\}_{n \in \mathbb{Z}}\), which we will refer to as a graded object of \(\mathcal{A}\). We view \(\partial = \{\partial_n\}_{n \in \mathbb{Z}}\) as an endomorphism of \(C_\bullet\) which is homogeneous of degree \(-1\), which we refer to as the differential or the boundary operator of the chain complex \(C_\bullet\). We will generally abuse notation by omitting the subscript from the expression \(\partial_n\); that is, we denote each of the boundary operators \(C_n \to C_{n-1}\) by the same symbol \(\partial\) (or \(\partial_C\), when we need to emphasize that its association with the particular chain complex \(C_\bullet\)).

Chain complexes with values in an additive category \(\mathcal{A}\) can themselves be organized into a category.

**Definition 2.5.0.3.** Let \((C_\bullet, \partial_C)\) and \((D_\bullet, \partial_D)\) be chain complexes with values in an additive category \(\mathcal{A}\). A chain map from \((C_\bullet, \partial_C)\) and \((D_\bullet, \partial_D)\) is a collection \(f = \{f_n\}_{n \in \mathbb{Z}}\), where each \(f_n\) is a morphism from \(C_n\) to \(D_n\) in the category \(\mathcal{A}\), for which each of the diagrams

\[
\begin{array}{ccc}
C_n & \xrightarrow{\partial_C} & C_{n-1} \\
\downarrow f_n & & \downarrow f_{n-1} \\
D_n & \xrightarrow{\partial_D} & D_{n-1}
\end{array}
\]

is commutative.

If \(\mathcal{A}\) is an additive category, we let \(\text{Ch}(\mathcal{A})\) denote the category whose objects are chain complexes with values in \(\mathcal{A}\) and whose morphisms are chain maps.

**Notation 2.5.0.4.** Let \(k\) be a commutative ring. We will write \(\text{Ch}(k)\) for the category \(\text{Ch}(\mathcal{A})\), where \(\mathcal{A}\) is the category of \(k\)-modules and \(k\)-module homomorphisms. In particular, we will write \(\text{Ch}(\mathbb{Z})\) for the category of chain complexes of abelian groups.

**Definition 2.5.0.5** (Chain Homotopy). Let \(\mathcal{A}\) be an additive category and let \((C_\bullet, \partial_C)\) and \((D_\bullet, \partial_D)\) be chain complexes with values in \(\mathcal{A}\). Let \(f = \{f_n\}_{n \in \mathbb{Z}}\) and \(f' = \{f'_n\}_{n \in \mathbb{Z}}\) be chain maps from \(C_\bullet\) to \(D_\bullet\). A chain homotopy from \(f\) to \(f'\) is a collection of maps \(h = \{h_n : C_n \to D_{n+1}\}\) which satisfy the identity

\[
f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C
\]

for every integer \(n\).

We say that \(f\) and \(f'\) are chain homotopic if there exists a chain homotopy from \(f\) to \(f'\). We will say that \(f\) is a chain homotopy equivalence if there exists a chain map \(g : D_\bullet \to C_\bullet\) such that \(g \circ f\) and \(f \circ g\) are chain homotopic to the identity morphisms \(\text{id}_{C_\bullet}\) and \(\text{id}_{D_\bullet}\), respectively.
Remark 2.5.0.6. Let $C_\ast$ and $D_\ast$ be chain complexes with values in an additive category $\mathcal{A}$. Then chain homotopy determines an equivalence relation on the set of chain maps $f : C_\ast \to D_\ast$. More precisely:

- Every chain map $f : C_\ast \to D_\ast$ is chain homotopic to itself, via the chain homotopy given by the collection of zero maps $\{0 : C_n \to D_{n+1}\}$.
- Let $f, f' : C_\ast \to D_\ast$ be chain maps. If $f$ is chain homotopic to $f'$, then $f'$ is chain homotopic to $f$. More precisely, if $h$ is a chain homotopy from $f$ to $f'$, then $-h$ is a chain homotopy from $f'$ to $f$.
- Let $f, f', f'' : C_\ast \to D_\ast$ be chain maps. If $f$ is chain homotopic to $f'$ and $f'$ is chain homotopic to $f''$, then $f$ is chain homotopic to $f''$. More precisely, if $h$ is a chain homotopy from $f$ to $f'$ and $h'$ is a chain homotopy from $f'$ to $f''$, then $h + h'$ is a chain homotopy from $f$ to $f''$.

Remark 2.5.0.7. Let $C_\ast$ and $D_\ast$ be chain complexes with values in an additive category $\mathcal{A}$, and let $f, f' : C_\ast \to D_\ast$ be chain maps which are chain homotopic. Then:

- For every chain map $g : D_\ast \to E_\ast$, the composite maps $g \circ f$ and $g \circ f'$ are chain homotopic. More precisely, if $h = \{h_n\}_{n \in \mathbb{Z}}$ is a chain homotopy from $f$ to $f'$, then the collection of composite maps $\{g_{n+1} \circ h_n\}$ is a chain homotopy from $g \circ f$ to $g \circ f'$.
- For every chain map $e : B_\ast \to C_\ast$, the composite maps $f \circ e$ and $f' \circ e$ are chain homotopic. More precisely, if $h = \{h_n\}_{n \in \mathbb{Z}}$ is a chain homotopy from $f$ to $f'$, then the collection of composite maps $\{h_n \circ e_n\}$ is a chain homotopy from $f \circ e$ to $f' \circ e$.

Construction 2.5.0.8 (The Homotopy Category of Chain Complexes). Let $\mathcal{A}$ be an additive category. We define a category $\text{hCh}(\mathcal{A})$ as follows:

- The objects of $\text{hCh}(\mathcal{A})$ are chain complexes with values in $\mathcal{A}$.
- If $C_\ast$ and $D_\ast$ are chain complexes with values in $\mathcal{A}$, then $\text{Hom}_{\text{hCh}(\mathcal{A})}(C_\ast, D_\ast)$ is the quotient of $\text{Hom}_{\mathcal{A}}(C_\ast, D_\ast)$ by the relation of chain homotopy equivalence. If $f : C_\ast \to D_\ast$ is a chain map, we denote its equivalence class by $[f] \in \text{Hom}_{\mathcal{A}}(C_\ast, D_\ast)$.
- If $C_\ast, D_\ast,$ and $E_\ast$ are chain complexes with values in $\mathcal{A}$, then the composition law

$$\circ : \text{Hom}_{\text{hCh}(\mathcal{A})}(D_\ast, E_\ast) \times \text{Hom}_{\text{hCh}(\mathcal{A})}(C_\ast, D_\ast) \to \text{Hom}_{\text{hCh}(\mathcal{A})}(C_\ast, E_\ast)$$

is uniquely determined by the requirement that $[g] \circ [f] = [g \circ f]$ for every pair of chain maps $f : C_\ast \to D_\ast$ and $g : D_\ast \to E_\ast$ (the well-definedness of this operation follows from Remark 2.5.0.7).
We will refer to $h\text{Ch}(A)$ as the homotopy category of $\text{Ch}(A)$.

The definition of the homotopy category $h\text{Ch}(A)$ of chain complexes is analogous to the definition of the homotopy category $h\text{Top}$ of topological spaces: the latter is obtained by working with continuous functions up to homotopy, and the former by working with chain maps up to chain homotopy. As with its topological counterpart, passage from $\text{Ch}(A)$ to $h\text{Ch}(A)$ is a destructive procedure. By enforcing the equality $[f] = [f']$ whenever there exists a chain homotopy $h$ from $f$ to $f'$, we sacrifice the ability to extract information which depends on a particular choice of chain homotopy. The situation can be remedied by contemplating a more elaborate structure.

**Construction 2.5.0.9** (Mapping Complexes). Let $(C\_*, \partial_C)$ and $(D\_*, \partial_D)$ be chain complexes with values in an additive category $A$. For each integer $d$, we let $[C\_*,D\_*]_d$ denote the abelian group $\prod_{n \in \mathbb{Z}} \text{Hom}_A(C_n, D_{n+d})$ consisting of maps from $C_\_*$ to $D_\_*$ which are homogeneous of degree $d$. These abelian groups can be organized into a chain complex

$$\cdots \xrightarrow{\partial} [C\_*,D\_*]_2 \xrightarrow{\partial} [C\_*,D\_*]_1 \xrightarrow{\partial} [C\_*,D\_*]_0 \xrightarrow{\partial} [C\_*,D\_*]_{-1} \xrightarrow{\partial} [C\_*,D\_*]_{-2} \xrightarrow{\partial} \cdots,$$

whose boundary operator $\partial : [C\_*,D\_*]_d \to [C\_*,D\_*]_{d-1}$ is given by the formula $\partial \{f_n : C_n \to D_{n+d}\}_{n \in \mathbb{Z}} = \{\partial_D \circ f_n - (-1)^d f_{n-1} \circ \partial_C\}_{n \in \mathbb{Z}}$. We will refer to $[C\_*,D\_*]_\_*$ as the mapping complex associated to the chain complexes $C\_*$ and $D\_*$.

Note that from the mapping complexes $[C\_*,D\_*]_\_*$, we can extract both the set of chain maps $\text{Hom}_\text{Ch}(A)(C\_*,D\_*)$ and the set of homotopy equivalence classes $\text{Hom}_{h\text{Ch}(A)}(C\_*,D\_*)$:

- Chain maps from $C\_*$ to $D\_*$ can be identified with 0-cycles of the chain complex $[C\_*,D\_*]_\_*$: that is, with elements $f = \{f_n\}_{n \in \mathbb{Z}} \in [C\_*,D\_*]_0$ satisfying $\partial(f) = 0$.

- Given a pair of chain maps $f,f' : C\_* \to D\_*$, a chain homotopy from $f$ to $f'$ is an element $h = \{h_n\}_{n \in \mathbb{Z}} \in [C\_*,D\_*]_1$ satisfying $\partial(h) = f' - f$. In particular, $f$ and $f'$ are chain homotopic if and only if they are homologous when viewed as 0-cycles of the complex $[C\_*,D\_*]_\_*$, so $\text{Hom}_{h\text{Ch}(A)}(C\_*,D\_*)$.

Moreover, the mapping complexes of Construction 2.5.0.9 are equipped with maps

$$\circ : [D\_*,E\_*]_m \times [C\_*,D\_*]_n \to [C\_*,E\_*]_{m+n},$$

which refine the composition laws on the categories $\text{Ch}(A)$ and $h\text{Ch}(A)$. In §2.5.2 we axiomatize this structure by introducing the notion of a differential graded category (Definition 2.5.2.1). By definition, a differential graded category is a category which is enriched over the category $\text{Ch}(\mathbb{Z})$ of graded abelian groups (endowed with the monoidal structure given by the tensor product of chain complexes, which we review in §2.5.1). The category of
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chain complexes $\text{Ch}(\mathcal{A})$ is a prototypical example of a differential graded category (Example 2.5.2.5), with the enrichment supplied by the mapping complexes of Construction 2.5.0.9.

Let $\mathcal{C}$ be a differential graded category. To every pair of objects $X,Y \in \mathcal{C}$, the enrichment of $\mathcal{C}$ supplies a chain complex $\text{Hom}_\mathcal{C}(X,Y)_*$, whose 0-cycles are the morphisms from $X$ to $Y$ in $\mathcal{C}$. Heuristically, one can think of this data as endowing $\mathcal{C}$ with the structure of a higher category, whose $n$-morphisms (for $n \geq 2$) are given by the elements of $\text{Hom}_\mathcal{C}(X,Y)_{n-1}$ (for varying $X$ and $Y$). In §2.5.3 we make this heuristic precise by constructing a simplicial set $N^\text{dg}_\bullet(\mathcal{C})$ called the differential graded nerve of $\mathcal{C}$ (Definition 2.5.3.7), and proving that it is an $\infty$-category in the sense of Definition 1.3.0.1 (Theorem 2.5.3.10). In §2.5.4 we show that the homotopy category of $N^\text{dg}_\bullet(\text{Ch}(\mathcal{A}))$ can be identified with the homotopy category of chain complexes $\text{hCh}(\mathcal{A})$ of Construction 2.5.0.8.

The remainder of this section is devoted to studying the relationship between the differential graded nerve $N^\text{dg}_\bullet(\mathcal{C})$ and the homotopy coherent nerve of §2.4. This will require a somewhat lengthy detour through the theory of simplicial abelian groups. In §2.5.5 we will associate to each simplicial set $S_\bullet$ its normalized chain complex $N_*(S;\mathbb{Z})$, given in each degree $n$ by the free abelian group on the set of nondegenerate $n$-simplices of $S_\bullet$ (Construction 2.5.5.9). The construction $S_\bullet \mapsto N_*(S;\mathbb{Z})$ determines a functor from the category of simplicial sets to the category $\text{Ch}(\mathbb{Z})$ of chain complexes of abelian groups. In §2.5.6 we show that this functor has a right adjoint $K : \text{Ch}(\mathbb{Z}) \to \text{Set}_\Delta$, which we will refer to as the Eilenberg-MacLane functor (Construction 2.5.6.3). To each chain complex of abelian groups $M_*$, this functor associates a simplicial abelian group $K(M_*)$, which we will refer to as the (generalized) Eilenberg-MacLane space of $M_*$. Moreover, by celebrated Dold-Kan correspondence (Theorem 2.5.6.1), the Eilenberg-MacLane functor restricts to an equivalence

$$\text{Ch}(\mathbb{Z})_{\geq 0} \xrightarrow{\sim} \{\text{Simplicial Abelian Groups}\},$$

where $\text{Ch}(\mathbb{Z})_{\geq 0} \subset \text{Ch}(\mathbb{Z})$ denotes the full subcategory spanned by those chain complexes which are concentrated in nonnegative degrees (Definition 2.5.1.1).

Let $S_\bullet$ and $T_\bullet$ be simplicial sets. In §2.5.8 we review the classical Alexander-Whitney construction, which supplies a chain map

$$\text{AW} : N_*(S \times T;\mathbb{Z}) \to N_*(S;\mathbb{Z}) \boxtimes N_*(T;\mathbb{Z});$$

here the right hand side denotes the tensor product of the normalized chain complexes $N_*(S;\mathbb{Z})$ and $N_*(T;\mathbb{Z})$. Allowing $S_\bullet$ and $T_\bullet$ to vary, these maps determine a lax monoidal structure on the Eilenberg-MacLane functor $K : \text{Ch}(\mathbb{Z}) \to \text{Set}_\Delta$. Using this structure, we will associate to each differential graded category $\mathcal{C}$ a simplicial category $\mathcal{C}_\Delta^\bullet$ having the same objects, with simplicial mapping sets given by $\text{Hom}_{\mathcal{C}_\Delta^\bullet}(X,Y) = K(\text{Hom}_\mathcal{C}(X,Y)_*)$ (Construction 2.5.9.2). In §2.5.9 we construct a comparison map $\mathfrak{Z}$ from the homotopy
coherent nerve $N^\text{hc}(C^\Delta)$ to the differential graded nerve $N^\text{dg}(C)$ (Proposition 2.5.9.10), and show that it is a trivial Kan fibration (Theorem 2.5.9.17). The proof of this result (and the construction of the map $Z$) rely heavily on the shuffle product $\nabla: N^*_{+}(S; \mathbb{Z}) \times N^*_{+}(T; \mathbb{Z}) \to N^*_{+}(S \times T; \mathbb{Z})$ introduced by Eilenberg and MacLane, which we review in §2.5.7.

Warning 2.5.0.10. The differential graded nerve construction $C \mapsto N^\text{dg}_{+}(C)$ can be used to produce many interesting examples of $\infty$-categories. However, not every $\infty$-category can be obtained in this way (even up to equivalence). Put differently, $\infty$-categories of the form $N^\text{dg}_{+}(C)$ have some special features, which are not shared by general $\infty$-categories. For example, if $C$ is a pretriangulated differential graded category (Definition [?]), then the differential graded nerve $N^\text{dg}_{+}(C)$ is a stable $\infty$-category (see Proposition [?]).

2.5.1 Generalities on Chain Complexes

In this section, we provide a brief review of some of the homological algebra which will be needed throughout §2.5.

Definition 2.5.1.1. Let $\mathcal{A}$ be an additive category, let $C_{*}$ be a chain complex with values in $\mathcal{A}$, and let $n$ be an integer. We will say that $C_{*}$ is concentrated in degrees $\geq n$ if objects $C_{m} \in \mathcal{A}$ are zero for $m < n$. Similarly, we say that $C_{*}$ is concentrated in degrees $\leq n$ if the objects $C_{m}$ are zero for $m > n$. We let $\text{Ch}(\mathcal{A})_{\geq n}$ denote the full subcategory of $\text{Ch}(\mathcal{A})$ spanned by those chain complexes which are concentrated in degree $n$, and $\text{Ch}(\mathcal{A})_{\leq n}$ the full subcategory spanned by those chain complexes which are concentrated in degrees $\leq n$.

Example 2.5.1.2. Let $\mathcal{A}$ be an additive category, let $C \in \mathcal{A}$ be an object, and let $n$ be an integer. We will write $C[n]$ for the chain complex given by

$$C[n]_{*} = \begin{cases} C & \text{if } * = n \\ 0 & \text{otherwise,} \end{cases}$$

where each differential is the zero morphism. Note that a chain complex $M_{*}$ is isomorphic to $C[n]$ (for some object $C \in \mathcal{A}$) if and only if it both concentrated in degrees $\geq n$ and concentrated in degrees $\leq n$.

Notation 2.5.1.3 (Cycles and Boundaries). Let $\mathcal{A}$ be an abelian category (Definition [?]) and let $C_{*}$ be a chain complex with values in $\mathcal{A}$. For each integer $n$, we let $Z_{n}(C)$ denote the kernel of the boundary operator $\partial : C_{n} \to C_{n-1}$, and $B_{n}(C)$ the image of the boundary operator $\partial : C_{n+1} \to C_{n}$. We regard $Z_{n}(C)$ and $B_{n}(C)$ as subobjects of $C_{n}$. Note that we have $B_{n}(C) \subseteq Z_{n}(C)$ (this is a reformulation of the identity $\partial^{2} = 0$).

In the special case where $\mathcal{A} = \text{Ab}$ is the category of abelian groups, we will refer to the elements of $C_{n}$ as $n$-chains of $C_{*}$, to the elements of $Z_{n}(C)$ as $n$-cycles of $C_{*}$, and to the elements of $B_{n}(C)$ as $n$-boundaries of $C_{*}$.
Definition 2.5.1.4 (Homology). Let $\mathcal{A}$ be an abelian category and let $C_*$ be a chain complex with values in $\mathcal{A}$. For every integer $n$, we let $H_n(C)$ denote the quotient $Z_n(C)/B_n(C)$. We will refer to $H_n(C)$ as the $n$th homology of the chain complex $C_*$. We say that the chain complex $C_*$ is acyclic if the homology objects $H_n(C)$ vanish for every integer $n$.

If $\mathcal{A} = \text{Ab}$ is the category of abelian groups and if $x \in Z_n(C)$ is an $n$-cycle of $C_*$, we let $[x]$ denote its image in the homology group $H_n(C)$: we refer to $[x]$ as the homology class of $x$. We say that a pair of $n$-cycles $x, x' \in Z_n(C)$ are homologous if $[x] = [x']$: that is, if there exists an $(n+1)$-chain $y$ satisfying $x' = x + \partial(y)$.

Definition 2.5.1.5 (Quasi-Isomorphisms). Let $\mathcal{A}$ be an abelian category, let $C_*$ and $D_*$ be chain complexes with values in $\mathcal{A}$, and let $f : C_* \to D_*$ be a chain map. We say that $f$ is a quasi-isomorphism if, for every integer $n$, the induced map of homology objects $H_n(C) \to H_n(D)$ is an isomorphism.

Remark 2.5.1.6. Let $C_*$ be a chain complex with values in an abelian category $\mathcal{A}$. In practice, the homology objects $H_n(C)$ are often primary objects of interest, while the chain complex $C_*$ itself plays an ancillary role. The terminology of Definition 2.5.1.5 emphasizes this perspective: a chain map $f : C_* \to D_*$ which induces an isomorphism on homology should allow us to view the chain complexes $C_*$ and $D_*$ as “the same” for many purposes (this idea is the starting point for Verdier’s theory of derived categories, which we will discuss in §[?]).

Remark 2.5.1.7 (Two-out-of-Three). Let $\mathcal{A}$ be an abelian category and suppose we are given a commutative diagram of chain complexes

$$
\begin{array}{cccccc}
0 & \longrightarrow & C'_* & \longrightarrow & C_* & \longrightarrow & C''_* & \longrightarrow & 0 \\
& & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & \\
0 & \longrightarrow & D'_* & \longrightarrow & D_* & \longrightarrow & D''_* & \longrightarrow & 0
\end{array}
$$

in which the rows are exact. If any two of the chain maps $f, f', f''$ are quasi-isomorphisms, then so is the third. This follows by comparing the long exact homology sequences associated to the upper and lower rows (see Construction [?]).

Proposition 2.5.1.8. Let $C_*$ and $D_*$ be chain complexes with values in an abelian category $\mathcal{A}$, and let $f, f' : C_* \to D_*$ be a pair of chain maps. If $f$ and $f'$ are chain homotopic, then they induce the same map from $H_n(C)$ to $H_n(D)$ for every integer $n$.

Proof. Let $h = \{h_m\}_{m \in \mathbb{Z}}$ be a chain homotopy from $f$ to $f'$, so that $f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C$. It follows that, when restricted to the subobject $Z_n(C) \subseteq C_n$, the difference $f'_n - f_n = \partial_D \circ h_n$. factors through the subobject $B_n(D) \subseteq Z_n(D)$, so the induced maps $H_n(f), H_n(f') : H_n(C) \to H_n(D)$ are the same. \qed
Corollary 2.5.1.9. Let $f : C_* \to D_*$ be a chain map between chain complexes with values in an abelian category $\mathcal{A}$. If $f$ is a chain homotopy equivalence, then it is a quasi-isomorphism.

For later use, we record the following elementary fact:

Proposition 2.5.1.10. Let $P_*$ be a chain complex taking values in an abelian category $\mathcal{A}$. Assume that $P_*$ is acyclic, concentrated in degrees $\geq 0$, and that each $P_n$ is a projective object of $\mathcal{A}$. Then $P_*$ is a projective object of the category $\text{Ch}(\mathcal{A})$. In other words, every epimorphism of chain complexes $f : M_* \to P_*$ admits a section.

Proof. Our assumption that $P_*$ is acyclic guarantees that for every integer $n \geq 0$, we have a short exact sequence

$$0 \to Z_n(P) \to P_n \xrightarrow{\partial} Z_{n-1}(P) \to 0.$$ 

It follows by induction on $n$ that each of these exact sequences splits and that each $Z_n(P)$ is also a projective object of $\mathcal{A}$. We can therefore choose a direct sum decomposition $P_n \simeq Z_n(P) \oplus Q_n$, where the differential on $P_*$ restricts to isomorphisms $\partial : Q_n \simeq Z_{n-1}(P)$. Since each $Q_n$ is projective and $f$ is an epimorphism in each degree, we can choose maps $u_n : Q_n \to M_n$ for which the composition $f_n \circ u_n$ equal to the identity on $Q_n$. The maps $u_n$ then extend uniquely to a map of chain complexes $s = \{s_n\}_{n \in \mathbb{Z}}$, characterized by the requirement that each composition

$$Q_{n+1} \oplus Q_n \xrightarrow{\partial \oplus \text{id}} Z_n(P) \oplus Q_n = P_n \xrightarrow{s_n} M_n$$

is the sum of the maps $u_{n+1}$ and $u_n$. \qed

We now specialize our attention to the category $\text{Ch}(\mathbb{Z})$ of chain complexes of abelian groups, which we will endow with a monoidal structure.

Notation 2.5.1.11. Let $C_*$ and $D_*$ be graded abelian groups. We define a new graded abelian group $(C \boxtimes D)_* = C_* \boxtimes D_*$ by the formula

$$(C \boxtimes D)_n = \bigoplus_{n = n' + n''} C_{n'} \otimes D_{n''}.$$ 

Here the direct sum is taken over the set $\{(n', n'') \in \mathbb{Z} \times \mathbb{Z} : n = n' + n''\}$ of all decompositions of $n$ as a sum of two integers $n'$ and $n''$, and $C_{n'} \boxtimes D_{n''}$ denotes the tensor product of $C_{n'}$ with $D_{n''}$ (formed in the category of abelian groups). For every pair of elements $x \in C_m$ and $y \in D_n$, we let $x \boxtimes y$ denote the image of the pair $(x, y)$ under the canonical map

$$C_m \times D_n \to C_m \otimes D_n \leftarrow (C \boxtimes D)_{m+n}.$$
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Proposition 2.5.1.12. Let \((C_\ast, \partial)\) and \((D_\ast, \partial)\) be chain complexes. Then there is a unique homomorphism of graded abelian groups

\[ \partial : (C \boxtimes D)_\ast \to (C \boxtimes D)_{\ast-1} \]

satisfying the identity

\[ \partial(x \boxtimes y) = (\partial(x) \boxtimes y) + (-1)^m(x \boxtimes \partial(y)) \]

for \(x \in C_m\) and \(y \in D_n\). Moreover, this homomorphism satisfies \(\partial^2 = 0\), so we can regard the pair \(((C \boxtimes D)_\ast, \partial)\) as a chain complex.

Proof. For every pair of integers \(m, n \in \mathbb{Z}\), the construction

\[(x, y) \mapsto (\partial x \boxtimes y) + (-1)^m(x \boxtimes \partial y)\]

determines a bilinear map \(C_m \times D_n \to (C \boxtimes D)_{m+n-1}\). Invoking the universal property of tensor products and direct sums, we deduce that there is a unique map \(\partial : (C \boxtimes D)_\ast \to (C \boxtimes D)_{\ast-1}\) with the desired properties. The identity \(\partial^2 = 0\) follows from the calculation

\[
\partial^2(x \boxtimes y) = \partial((\partial x \boxtimes y) + (-1)^m(x \boxtimes \partial y)) \\
= (\partial^2 x \boxtimes y) + (-1)^m\partial(x \boxtimes \partial y) + (-1)^m(\partial x \boxtimes \partial y) + (-1)^{2m}(x \boxtimes \partial^2 y) \\
= 0.
\]

\[\square\]

Notation 2.5.1.13. In the situation of Proposition 2.5.1.12, we will refer to \(((C \boxtimes D)_\ast, \partial)\) as the tensor product of the chain complexes \((C_\ast, \partial)\) and \((D_\ast, \partial)\).

Warning 2.5.1.14 (The Koszul Sign Rule). Let \((C_\ast, \partial)\) and \((D_\ast, \partial)\) be chain complexes. There is a unique isomorphism of graded abelian groups \(\tau : C_\ast \boxtimes D_\ast \to D_\ast \boxtimes C_\ast\) satisfying \(\tau(x \boxtimes y) = y \boxtimes x\) for all \(x \in C_m, y \in C_n\). Beware that \(\tau\) is usually not a chain map: we have

\[\partial \tau(x \boxtimes y) = \partial(y \boxtimes x) = (\partial y \boxtimes x) + (-1)^n(y \boxtimes \partial x)\]

\[\tau(\partial(x \boxtimes y)) = \tau((\partial x \boxtimes y) + (-1)^m(x \boxtimes \partial y)) = (-1)^m(\partial y \boxtimes x) + (\partial x \boxtimes y).\]

This can be remedied by modifying the isomorphism \(\tau\): there is another isomorphism of graded abelian groups

\[\sigma : C_\ast \boxtimes D_\ast \simeq D_\ast \boxtimes C_\ast; \quad \sigma(x \boxtimes y) = (-1)^{mn}(y \boxtimes x).\]
The isomorphism of $\sigma$ is a chain map (hence an isomorphism of chain complexes) by virtue of the calculation

$$
\partial \sigma(x \boxtimes y) = \partial((-1)^{mn} y \boxtimes x) \\
= (-1)^{mn} (\partial y \boxtimes x) + (-1)^{m+n} (y \boxtimes \partial x) \\
= (-1)^n \sigma(x \boxtimes \partial y) + \sigma(\partial x \boxtimes y) \\
= \sigma(\partial(x \boxtimes y)).
$$

**Exercise 2.5.1.15 (Universal Property of the Tensor Product).** Let $(C_\ast, \partial), (D_\ast, \partial)$, and $(E_\ast, \partial)$ be chain complexes. We will say that a collection of bilinear maps

$$
\{ f_{m,n} : C_m \times D_n \to E_{m+n} \}_{m,n \in \mathbb{Z}}
$$

satisfies the Leibniz rule if, for every pair of elements $x \in C_m$ and $y \in D_n$, the identity

$$
\partial f_{m,n}(x, y) = f_{m-1,n}(\partial x, y) + (-1)^m f_{m,n-1}(x, \partial y)
$$

holds in the abelian group $E_{m+n-1}$. Show that there is a canonical bijection from the collection of chain maps $f : C_\ast \boxtimes D_\ast \to E_\ast \boxtimes E_\ast$ to the collection of systems of bilinear maps $\{ f_{m,n} : C_m \times D_n \to E_{m+n} \}_{m,n \in \mathbb{Z}}$ satisfying the Leibniz rule, given by the construction $f_{m,n}(x, y) = f(x \boxtimes y)$.

**Remark 2.5.1.16 (Associativity Isomorphisms).** Let $(C_\ast, \partial), (D_\ast, \partial)$, and $(E_\ast, \partial)$ be chain complexes of abelian groups. Then there is a unique isomorphism of graded abelian groups

$$
\alpha : C_\ast \boxtimes (D_\ast \boxtimes E_\ast) \to (C_\ast \boxtimes D_\ast) \boxtimes E_\ast
$$

satisfying the identity $\alpha(x \boxtimes (y \boxtimes z)) = (x \boxtimes y) \boxtimes z$. Moreover, $\alpha$ is an isomorphism of chain complexes: this follows from observation that $\alpha(\partial(x \boxtimes (y \boxtimes z)))$ and $\partial \alpha(x \boxtimes (y \boxtimes z))$ are both given by the sum

$$
(\partial x \boxtimes y) \boxtimes z + (-1)^m (x \boxtimes \partial y) \boxtimes z + (-1)^{m+n} ((x \boxtimes y) \boxtimes \partial z)
$$

for $x \in C_m$, $y \in D_n$, $z \in E_p$.

**Construction 2.5.1.17 (The Monoidal Structure on Chain Complexes).** Let $\text{Ch}(\mathbb{Z})$ denote the category of chain complexes of abelian groups (Definition 2.5.0.3). We define a monoidal structure on $\text{Ch}(\mathbb{Z})$ as follows:

- The tensor product functor $\boxtimes : \text{Ch}(\mathbb{Z}) \times \text{Ch}(\mathbb{Z}) \to \text{Ch}(\mathbb{Z})$ carries each pair of chain complexes $(C_\ast, \partial)$ and $(D_\ast, \partial)$ to the tensor product chain complex $(C_\ast \boxtimes D_\ast, \partial)$ of Proposition 2.5.1.12 and carries a pair of chain maps $f : C_\ast \to C'_\ast$, $g : D_\ast \to D'_\ast$ to the tensor product map

$$
(f \boxtimes g) : C_\ast \boxtimes D_\ast \to C'_\ast \boxtimes D'_\ast
$$

$$(f \boxtimes g)(x \boxtimes y) = f(x) \boxtimes g(y).$$
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- For every triple of chain complexes \( C = (C_*, \partial) \), \( D = (D_*, \partial) \), and \( E = (E_*, \partial) \), the associativity constraint
  \[
  \alpha_{C,D,E} : C_* \otimes (D_* \otimes E_*) \cong (C_* \otimes D_*) \otimes E_*
  \]
is the isomorphism of Remark 2.5.1.16.

- The unit object of \( \text{Ch}(\mathbb{Z}) \) is the chain complex \( \mathbb{Z}[0] \) of Example 2.5.1.2, and the unit constraint \( \upsilon : \mathbb{Z}[0] \otimes \mathbb{Z}[0] \cong \mathbb{Z}[0] \) is the isomorphism classified by the bilinear map
  \[
  \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \quad (m,n) \mapsto mn.
  \]

**Remark 2.5.1.18.** Let \((C_*, \partial)\) and \((D_*, \partial)\) be chain complexes. The tensor product chain complex \((C_* \otimes D_*, \partial)\) of Proposition 2.5.1.12 is characterized up to (unique) isomorphism by the universal property of Exercise 2.5.1.15. However, the construction of this tensor product complex (and, by extension, the monoidal structure on \( \text{Ch}(\mathbb{Z}) \)) depends on auxiliary choices. These choices are ultimately irrelevant in the sense that they do not change the isomorphism class of the monoidal category \( \text{Ch}(\mathbb{Z}) \) or, equivalently, of the classifying simplicial set \( B_* \text{Ch}(\mathbb{Z}) \) of Example 2.3.1.18. This simplicial set can be described concretely (without auxiliary choices): its \( n \)-simplices can be identified with systems of chain complexes \( \{C(j,i)_*\}_{0 \leq i < j \leq n} \) together with bilinear maps
  \[
  C(k,j)_q \times C(j,i)_p \rightarrow C(k,i)_{q+p} \quad (y,z) \mapsto yz
  \]
for \( 0 \leq i < j < k \leq n \) which satisfy the Leibniz rule \( \partial(yz) = (\partial y)z + (-1)^q y(\partial z) \) together with the associative law \( x(yz) = (xy)z \) for \( x \in C(\ell,k)_r \), \( y \in C(k,j)_q \), \( z \in C(j,i)_p \) with \( 0 \leq i < j < k < \ell \leq n \).

### 2.5.2 Differential Graded Categories

Let \( \text{Ch}(\mathbb{Z}) \) denote the category of chain complexes of abelian groups, equipped with the monoidal structure described in Construction 2.5.1.17. A **differential graded category** is a category enriched over \( \text{Ch}(\mathbb{Z}) \) (in the sense of Definition 2.1.7.1). For the convenience of the reader, we spell out this definition in detail.

**Definition 2.5.2.1** (Differential Graded Categories). A **differential graded category** \( \mathcal{C} \) consists of the following data:

1. A collection \( \text{Ob}(\mathcal{C}) \), whose elements we refer to as **objects** of \( \mathcal{C} \). We will often abuse notation by writing \( X \in \mathcal{C} \) to indicate that \( X \) is an element of \( \text{Ob}(\mathcal{C}) \).

2. For every pair of objects \( X, Y \in \text{Ob}(\mathcal{C}) \), a chain complex \( (\text{Hom}_\mathcal{C}(X,Y)_*, \partial) \). For each integer \( n \), we refer to the elements of \( \text{Hom}_\mathcal{C}(X,Y)_n \) as **morphisms of degree** \( n \) **from** \( X \) **to** \( Y \).
(3) For every triple of objects $X, Y, Z \in \text{Ob}(C)$ and every pair of integers $m, n \in \mathbb{Z}$, a function
\[ c_{Z,Y,X} : \text{Hom}_{C}(Y, Z)_n \times \text{Hom}_{C}(X, Y)_m \to \text{Hom}_{C}(X, Z)_{m+n}, \]
which we will refer to as the composition law. Given a pair of morphisms $f \in \text{Hom}_{C}(X, Y)_m$ and $g \in \text{Hom}_{C}(Y, Z)_n$, we will often denote the image $c_{Z,Y,X}(g, f) \in \text{Hom}_{C}(X, Z)_{m+n}$ by $g \circ f$ or $gf$.

(4) For every object $X \in \text{Ob}(C)$, a morphism $\text{id}_X \in \text{Hom}_{C}(X, X)_0$, which we will refer to as the identity morphism.

These data are required to satisfy the following conditions:

- The composition law on $C$ is associative in the following sense: for every triple of elements $f \in \text{Hom}_{C}(W, X)_{\ell}$, $g \in \text{Hom}_{C}(X, Y)_m$, and $h \in \text{Hom}_{C}(Y, Z)_n$, we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$ (in the abelian group $\text{Hom}_{C}(W, Z)_{\ell+m+n}$).

- The composition law on $C$ is unital on both sides: for every element $f \in \text{Hom}_{C}(X, Y)_n$, we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$.

- For every triple of objects $X, Y, Z \in \text{Ob}(C)$, the composition maps $\text{Hom}_{C}(Y, Z)_n \times \text{Hom}_{C}(X, Y)_m \to \text{Hom}_{C}(X, Z)_{m+n}$ are bilinear and satisfy the Leibniz rule of Exercise 2.5.1.15. In other words, we have
\[ g \circ (f + f') = (g \circ f) + (g \circ f') \quad (g + g') \circ f = (g \circ f) + (g' \circ f) \]
\[ \partial(g \circ f) = (\partial g) \circ f + (-1)^n g \circ (\partial f). \]

\textbf{Remark 2.5.2.2.} Let $C$ be a differential graded category. For each object $X \in \text{Ob}(C)$, the identity morphism $\text{id}_X$ is a 0-cycle of the chain complex $\text{Hom}_{C}(X, X)_*$: that is, it satisfies $\partial(\text{id}_X) = 0$. This follows from the calculation
\[ \partial(\text{id}_X) = \partial(\text{id}_X \circ \text{id}_X) = \partial(\text{id}_X) \circ \text{id}_X + \text{id}_X \circ \partial(\text{id}_X) = \partial(\text{id}_X) + \partial(\text{id}_X). \]

\textbf{Remark 2.5.2.3.} Let $C$ be a differential graded category containing a pair of morphisms $f \in \text{Hom}_{C}(X, Y)_m$ and $g \in \text{Hom}_{C}(Y, Z)_n$. It follows from the Leibniz rule
\[ \partial(g \circ f) = (\partial g) \circ f + (-1)^n g \circ (\partial f) \]
that if $f$ and $g$ are cycles (that is, if they satisfy $\partial f = 0$ and $\partial g = 0$), then $g \circ f$ is also a cycle. In particular, we have a bilinear composition map
\[ Z_n(\text{Hom}_{C}(Y, Z)) \times Z_m(\text{Hom}_{C}(X, Y)) \to Z_{m+n}(\text{Hom}_{C}(X, Z)). \]
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Construction 2.5.2.4 (The Underlying Category of a Differential Graded Category). To every differential graded category $\mathcal{C}$, we can associate an ordinary category $\mathcal{C}^\circ$ as follows:

- The objects of $\mathcal{C}^\circ$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C})^\circ$, a morphism from $X$ to $Y$ in $\mathcal{C}^\circ$ is a 0-cycle of chain complex $\text{Hom}_\mathcal{C}(X,Y)^\circ$.
- For each object $X \in \text{Ob}(\mathcal{C}^\circ) = \text{Ob}(\mathcal{C})$, the identity morphism from $X$ to itself in $\mathcal{C}^\circ$ is the identity morphism $\text{id}_X \in \text{Hom}_\mathcal{C}(X,X)_0$ (which is a cycle by virtue of Remark 2.5.2.2).
- Composition of morphisms in $\mathcal{C}^\circ$ is given by the composition law on $\mathcal{C}$ (which preserves 0-cycles by virtue of Remark 2.5.2.3).

We will refer to $\mathcal{C}^\circ$ as the underlying category of the differential graded category $\mathcal{C}$ (note that $\mathcal{C}^\circ$ can also be obtained by applying the general procedure described in Example 2.1.7.5).

Example 2.5.2.5 (Chain Complexes). Let $\mathcal{A}$ be an additive category. We define a differential graded category $\text{Ch}(\mathcal{A})$ as follows:

- The objects of $\text{Ch}(\mathcal{A})$ are chain complexes with values in $\mathcal{A}$ (Definition 2.5.0.1).
- If $C^\bullet$ and $D^\bullet$ are chain complexes with values in $\mathcal{A}$, then $\text{Hom}_{\text{Ch}(\mathcal{A})}(C^\bullet,D^\bullet)^\circ$ is the chain complex of abelian groups $[C,D]^\bullet$ defined in Construction 2.5.0.9.
- If $C^\bullet, D^\bullet, D_\circ$, and $E^\bullet$ are chain complexes with values in $\mathcal{A}$, then the composition law $\circ : [D,E]^e \times [C,D]^d \rightarrow [C,E]_{d+e}$ is given by the formula $\{g_n\}_{n\in\mathbb{Z}} \circ \{f_n\}_{n\in\mathbb{Z}} = \{g_{n+d} \circ f_n\}_{n\in\mathbb{Z}}$.

Note that if $C^\bullet$ and $D^\bullet$ are chain complexes with values in $\mathcal{A}$, then a collection of maps $f = \{f_n : C_n \rightarrow D_n\}_{n\in\mathbb{Z}}$ is a 0-cycle of the chain complex $[C,D]^\bullet$ if and only if it is a chain map from $C^\bullet$ to $D^\bullet$. Consequently, applying Construction 2.5.2.4 to the differential graded category $\text{Ch}(\mathcal{A})$ yields the ordinary category of chain complexes and chain maps. In other words, this construction supplies a $\text{Ch}(\mathbb{Z})$-enrichment of the category $\text{Ch}(\mathcal{A})$ introduced in Definition 2.5.0.3.

Example 2.5.2.6 (Differential Graded Algebras). A differential graded algebra is a (not necessarily commutative) graded ring $A^\bullet = \{A_n\}_{n\in\mathbb{Z}}$ equipped with a differential $\partial : A^\bullet \rightarrow A^\bullet_{-1}$ satisfying $\partial^2 = 0$ and the Leibniz rule $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^m x \cdot (\partial y)$ for $x \in A_m$ and $y = A_n$. If $\mathcal{C}$ is a differential graded category containing an object $X$, then the composition law on $\mathcal{C}$ endows the chain complex $\text{End}_{\mathcal{C}}(X)^\bullet = \text{Hom}_{\mathcal{C}}(X,X)^\circ$ with the structure of a
differential graded algebra. Conversely, for every differential graded algebra \((A_\ast, \partial)\), there is a unique differential graded category \(\mathcal{C}\) with \(\text{Ob}(\mathcal{C}) = \{X\}\). In other words, the construction \(\mathcal{C} \mapsto \text{End}_C(X)_\ast\) induces a bijective correspondence
\[
\{\text{Differential graded categories } \mathcal{C}\text{ with } \text{Ob}(\mathcal{C}) = \{X\}\} \sim \{\text{Differential graded algebras}\}.
\]

**Example 2.5.2.7.** Let \(B_\ast \text{Ch}(\mathbb{Z})\) denote the classifying simplicial set of the monoidal category of chain complexes. For each nonnegative integer \(n \geq 0\), we can use the analysis of Remark 2.5.1.18 to identify \(n\)-simplices of \(B_\ast \text{Ch}(\mathbb{Z})\) with differential graded categories \(\mathcal{C}\) satisfying \(\text{Ob}(\mathcal{C}) = \{0, 1, \ldots, n\}\) and
\[
\text{Hom}_\mathcal{C}(i, j)_\ast = \begin{cases} 
\mathbb{Z}[0] & \text{if } i = j \\
0 & \text{if } i > j.
\end{cases}
\]

**Definition 2.5.2.8 (Differential Graded Functors).** Let \(\mathcal{C}\) and \(\mathcal{D}\) be differential graded categories. A differential graded functor \(F\) from \(\mathcal{C}\) to \(\mathcal{D}\) consists of the following data:

- For each object \(X \in \text{Ob}(\mathcal{C})\), an object \(F(X) \in \text{Ob}(\mathcal{D})\).
- For each pair of objects \(X, Y \in \text{Ob}(\mathcal{C})\), a chain map \(F_{X,Y} : \text{Hom}_\mathcal{C}(X, Y)_\ast \to \text{Hom}_\mathcal{D}(F(X), F(Y))_\ast\).

These data are required to satisfy the following conditions:

- For every object \(X \in \text{Ob}(\mathcal{C})\), the chain map
  \[
  F_{X,X} : \text{Hom}_\mathcal{C}(X, X)_\ast \to \text{Hom}_\mathcal{D}(F(X), F(X))_\ast
  \]
  carries the identity morphism \(\text{id}_X\) to the identity morphism \(\text{id}_{F(X)}\).
- For every triple of objects \(X, Y, Z \in \text{Ob}(\mathcal{C})\) and pair of morphisms \(f \in \text{Hom}_\mathcal{C}(X, Y)_m\), \(g \in \text{Hom}_\mathcal{C}(Y, Z)_n\), we have \(F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)\).

We let \(\text{Cat}^{dg}\) denote the category whose objects are (small) differential graded categories and whose morphisms are differential graded functors.

**Remark 2.5.2.9.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be differential graded categories. Then differential graded functors from \(\mathcal{C}\) to \(\mathcal{D}\) (in the sense of Definition 2.5.2.8) can be identified with \(\text{Ch}(\mathbb{Z})\)-enriched functors from \(\mathcal{C}\) to \(\mathcal{D}\) (in the sense of Definition 2.1.7.10).
2.5.3 The Differential Graded Nerve

We now explain how to associate to each differential graded category \( \mathcal{C} \) an \( \infty \)-category \( \text{N}^{\text{dg}}_{\bullet}(\mathcal{C}) \), which we will refer to as the differential graded nerve of \( \mathcal{C} \). We begin by describing the simplices of \( \text{N}^{\text{dg}}_{\bullet}(\mathcal{C}) \).

**Construction 2.5.3.1.** Let \( \mathcal{C} \) be a differential graded category. For \( n \geq 0 \), we let \( \text{N}^{\text{dg}}_{n}(\mathcal{C}) \) denote the collection of all ordered pairs \( \{ \{ X_i \}_{0 \leq i \leq n}, \{ f_I \} \} \), where:

- Each \( X_i \) is an object of the differential graded category \( \mathcal{C} \).
- For every subset \( I = \{ i_k > i_{k-1} > \cdots > i_0 \} \subseteq [n] \) having at least two elements, \( f_I \) is an element of the abelian group \( \text{Hom}_\mathcal{C}(X_{i_0}, X_{i_k})_{k-1} \) which satisfies the identity

\[
\partial f_I = \sum_{0 < a < k} (-1)^a f_{I \setminus \{ i_a \}} + (-1)^{k(a+1)} f_{\{ i_k > i_{k-1} > \cdots > i_a \}} \circ f_{\{ i_a > \cdots > i_1 > i_0 \}}
\]

**Example 2.5.3.2** (Vertices of the Differential Graded Nerve). Let \( \mathcal{C} \) be a differential graded category. Then \( \text{N}^{\text{dg}}_{0}(\mathcal{C}) \) can be identified with the collection \( \text{Ob}(\mathcal{C}) \) of objects of \( \mathcal{C} \).

**Example 2.5.3.3** (Edges of the Differential Graded Nerve). Let \( \mathcal{C} \) be a differential graded category. Then \( \text{N}^{\text{dg}}_{1}(\mathcal{C}) \) can be identified with the collection of all triples \( (X_0, X_1, f) \) where \( X_0 \) and \( X_1 \) are objects of \( \mathcal{C} \) and \( f \) is a 0-cycle in the chain complex \( \text{Hom}_\mathcal{C}(X_0, X_1)_{\bullet} \). In other words, \( \text{N}^{\text{dg}}_{1}(\mathcal{C}) \) is the collection of all morphisms in the the underlying category \( \mathcal{C}^\circ \) of Construction 2.5.2.4.

**Example 2.5.3.4** (2-Simplices of the Differential Graded Nerve). Let \( \mathcal{C} \) be a differential graded category. Then an element of \( \text{N}^{\text{dg}}_{2}(\mathcal{C}) \) is given by the following data:

- A triple of objects \( X_0, X_1, X_2 \in \text{Ob}(\mathcal{C}) \).
- A triple of 0-cycles

\[
f_{10} \in \text{Hom}_\mathcal{C}(X_0, X_1)_0 \quad f_{20} \in \text{Hom}_\mathcal{C}(X_0, X_2)_0 \quad f_{21} \in \text{Hom}_\mathcal{C}(X_1, X_2)_0.
\]

- A 1-chain \( f_{210} \in \text{Hom}_\mathcal{C}(X_0, X_2)_1 \) satisfying the identity

\[
\partial(f_{210}) = (f_{21} \circ f_{10}) - f_{20}.
\]

Here the 1-chain \( f_{210} \) can be regarded as a witness to the assertion that the 0-cycles \( f_{20} \) and \( f_{21} \circ f_{10} \) are homologous: that is, they represent the same element of the homology group...
\( H_0(\text{Hom}_C(X_0, X_2)) \). We can present this data graphically by the diagram

\[
\begin{array}{c}
X_1 \\
\downarrow f_{21} \\
\downarrow f_{20} \\
X_0 \\
\uparrow f_{10} \\
\end{array}
\]

\[
\begin{array}{c}
X_2 \\
\end{array}
\]

We now explain how to organize the collection \( \{N^\text{dg}_n(C)\}_{n \geq 0} \) into a simplicial set.

**Proposition 2.5.3.5.** Let \( C \) be a differential graded category. Let \( m, n \geq 0 \) be nonnegative integers and let \( \alpha : [n] \to [m] \) be a nondecreasing function. Then the construction

\[
(\{X_i\}_{0 \leq i \leq m}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0 \leq j \leq n}, \{g_J\}),
\]

\[
g_J = \begin{cases} 
  f_{\alpha(j)} & \text{if } \alpha|_J \text{ is injective} \\
  \text{id}_{X_i} & \text{if } J = \{j_1 > j_0\} \text{ with } \alpha(j_1) = i = \alpha(j_0) \\
  0 & \text{otherwise.}
\end{cases}
\]

determines a map of sets \( \alpha^* : N^\text{dg}_m(C) \to N^\text{dg}_n(C) \).

**Proof.** Let \( (\{X_i\}_{0 \leq i \leq m}, \{f_I\}) \) be an element of \( N^\text{dg}_m(C) \). For each subset \( J \subseteq [n] \) with at least two elements, define \( g_J \) as in the statement of Proposition 2.5.3.5. We wish to show that \( (\{X_{\alpha(j)}\}_{0 \leq j \leq n}, \{g_J\}) \) is an element of \( N^\text{dg}_n(C) \). For this, we must show that for each subset \( J = \{j_k > j_{k-1} > \cdots > j_1 > j_0\} \subseteq [n] \) having at least two elements, we have an equality

\[
\partial g_J = \sum_{1 < a < k} (-1)^a g_J \setminus \{j_a\} + (-1)^{k(a+1)} g_{\{j_k > j_{k-1} > \cdots > j_1 > j_0\}} \circ g_{\{j_a > \cdots > j_1 > j_0\}}. \tag{2.2}
\]

We distinguish three cases:

- **Suppose that the restriction \( \alpha|_J \) is injective.** In this case, we can rewrite (2.2) as an equality

\[
\partial f_{\alpha(J)} = \sum_{0 < a < k} (-1)^a f_{\alpha(J) \setminus \{j_a\}} + (-1)^{k(a+1)} f_{\{\alpha(j_k) > \cdots > j_{a} > \alpha(j_0)\}} \circ f_{\{\alpha(j_1) > \cdots > \alpha(j_0)\}},
\]

which follows from our assumption that \( (\{X_i\}_{0 \leq i \leq m}, \{f_I\}) \) is an element of \( N^\text{dg}_m(C) \).

- **Suppose that \( J = \{j_1 > j_0\} \) is a two-element set satisfying \( \alpha(j_1) = i = \alpha(j_0) \) for some \( 0 \leq i \leq m \).** In this case, we can rewrite (2.2) as an equality \( \partial (\text{id}_{X_i}) = 0 \), which follows from Remark 2.5.2.2.
Suppose that \( J = \{ j_k > j_{k-1} > \cdots > j_1 > j_0 \} \) has at least three elements and that \( \alpha|_J \) is not injective, so that \( g_J = 0 \). We now distinguish three (possibly overlapping) cases:

- The map \( \alpha \) is not injective because \( \alpha(j_i) = i = \alpha(j_0) \) for some \( 0 \leq i \leq m \). In this case, the expressions \( g_{\mathcal{J}\setminus\{j_a\}} \) and \( g_{\{j_a,\ldots,j_i,j_0\}} \) vanish for \( 1 < a < k \). We can therefore rewrite (2.2) as an equality

\[
g_{\mathcal{J}\setminus\{j_1\}} = g_{\{j_1,\ldots,j_i,j_0\}} \circ g_{\{j_i,j_0\}},
\]

which follows from the identities \( g_{\mathcal{J}\setminus\{j_1\}} = g_{\{j_1,\ldots,j_i,j_0\}} \) and \( g_{\{j_i,j_0\}} = \text{id}_{X_i} \).

- The map \( \alpha \) is not injective because \( \alpha(j_k) = i = \alpha(j_{k-1}) \) for some \( 0 \leq i \leq m \). In this case, the expressions \( g_{\mathcal{J}\setminus\{j_a\}} \) and \( g_{\{j_{k-1},\ldots,j_0\}} \) vanish for \( 0 < a < k - 1 \). We can therefore rewrite (2.2) as an equality

\[
g_{\mathcal{J}\setminus\{j_{k-1}\}} = g_{\{j_{k-1},\ldots,j_0\}} \circ g_{\{j_0\}},
\]

which follows from the identities \( g_{\mathcal{J}\setminus\{j_{k-1}\}} = g_{\{j_{k-1},\ldots,j_0\}} \) and \( g_{\{j_0\}} = \text{id}_{X_i} \).

- The map \( \alpha \) is not injective because we have \( \alpha(j_b) = \alpha(j_{b+1}) \) for some \( 0 < b < k - 1 \). In this case, the chains \( g_{\mathcal{J}\setminus\{j_a\}} \) vanish for \( a \notin \{ b, b+1 \} \), and the compositions \( g_{\{j_{b-1},\ldots,j_0\}} \circ g_{\{j_0\}} \) vanish for all \( 0 < a < k \). We can therefore rewrite (2.2) as an equality \( g_{\mathcal{J}\setminus\{j_b\}} = g_{\mathcal{J}\setminus\{j_{b+1}\}} \), which is clear.

\( \square \)

**Exercise 2.5.3.6.** Let \( \mathcal{C} \) be a differential graded category. Suppose we are given a pair of nondecreasing functions \( \alpha : [k] \to [m] \) and \( \beta : [m] \to [n] \). Show that the function \((\beta \circ \alpha)^*\) of Proposition 2.5.3.5 coincides with the composition \( \alpha^* \circ \beta^* \).

**Definition 2.5.3.7.** Let \( \mathcal{C} \) be a differential graded category. We let \( N^\text{dg}_*(\mathcal{C}) \) denote the simplicial set whose value on an object \([n] \in \Delta^\text{op}\) is the set \( N^\text{dg}_n(\mathcal{C}) \) of Construction 2.5.3.1 and whose value on a nondecreasing function \( \alpha : [n] \to [m] \) is the function \( \alpha^* : N^\text{dg}_m(\mathcal{C}) \to N^\text{dg}_n(\mathcal{C}) \) of Proposition 2.5.3.5. We will refer to \( N^\text{dg}_*(\mathcal{C}) \) as the differential graded nerve of \( \mathcal{C} \).

**Remark 2.5.3.8** (Comparison with the Nerve). Let \( \mathcal{C} \) be a differential graded category and let \( \mathcal{C}^\circ \) denote its underlying ordinary category (Construction 2.5.2.4). Suppose that \( \sigma \) is an \( n \)-simplex of the nerve \( N_*(\mathcal{C}^\circ) \), consisting of objects \( \{ X_i \}_{0 \leq i \leq n} \) and 0-cycles \( \{ f_{ji} \in \text{Hom}_{\mathcal{C}}(X_i, X_j) \}_0 \) satisfying \( f_{ii} = \text{id}_{X_i} \) and \( f_{kj} = f_{kj} \circ f_{ji} \) for \( 0 \leq i \leq j \leq k \leq n \). We can then construct an \( n \)-simplex \( U(\sigma) \) of the differential graded nerve \( N^\text{dg}_*(\mathcal{C}) \), given by

\[
U(\sigma) = \{ \{ X_i \}_{0 \leq i \leq n}, \left\{ f_{I} \right\} \} ; \quad f_{I} = \begin{cases} f_{ji} & \text{if } I = \{ j > i \} \\ 0 & \text{otherwise.} \end{cases}
\]
The construction $\sigma \mapsto U(\sigma)$ determines a map of simplicial sets $U : N_{\bullet}(C^\circ) \to N_{\bullet}^{dg}(C)$. This map is a monomorphism, whose image is the simplicial subset of $N_{\bullet}^{dg}(C)$ spanned by those $n$-simplices $\{\{X_i\}_{0 \leq i \leq n}, \{f_I\}\}$ with the property that $f_I = 0$ for $|I| > 2$.

**Remark 2.5.3.9.** Let $C$ be a differential graded category and let $K_{\bullet}$ be a simplicial set. To give a map of simplicial sets $f : K_{\bullet} \to N_{\bullet}^{dg}(C)$, one must supply the following data:

- For each vertex $x$ of $K_{\bullet}$, an object $f(x)$ of the differential graded category $C$.
- For each $k > 0$ and each $k$-simplex $\sigma : \Delta^k \to K_{\bullet}$ with initial vertex $x = \sigma(0)$ and final vertex $y = \sigma(k)$, a $(k - 1)$-chain $f(\sigma) \in \text{Hom}_C(f(x), f(y))_{k-1}$.

Moreover, this data must satisfy the following conditions:

- If $e$ is a degenerate edge of $K_{\bullet}$ connecting a vertex $x$ to itself, then $f(e)$ is the identity morphism $id_{f(x)} \in \text{Hom}_C(f(x), f(x))_0$.
- If $\sigma$ is a degenerate simplex of $K_{\bullet}$ having dimension $\geq 2$, then $f(\sigma) = 0$.
- Let $k > 0$ and let $\sigma : \Delta^k \to K_{\bullet}$ be an $n$-simplex of $K_{\bullet}$. For $0 < a < k$, let $\sigma_{\leq a} : \Delta^a \to K_{\bullet}$ denote the composition of $\sigma$ with the inclusion map $\Delta^a \hookrightarrow \Delta^k$ (which is the identity on vertices), and let $\sigma_{\geq a} : \Delta^{k-a} \hookrightarrow K_{\bullet}$ denote the composition of $\sigma$ with the map $\Delta^{k-a} \hookrightarrow \Delta^k$ given on vertices by $i \mapsto i + a$. Then we have

$$\partial f(\sigma) = \sum_{0 < a < k} (-1)^a f(d(a) + (-1)^{k(a+1)} f(\sigma_{\geq a}) \circ f(\sigma_{\leq a}).$$

**Theorem 2.5.3.10.** Let $C$ be a differential graded category. Then the simplicial set $N_{\bullet}^{dg}(C)$ is an $\infty$-category.

**Proof.** Suppose we are given $0 < j < n$ and a map of simplicial sets $\sigma_0 : \Lambda^n_j \to N_{\bullet}^{dg}(C)$. Using Remark 2.5.3.9, we see that $\sigma_0$ can be identified with the data of a pair $\{(X_i)_{0 \leq i \leq n}, \{f_I\}\}$, where $\{X_i\}_{0 \leq i \leq n}$ is a collection of objects of $C$ and $f_I \in \text{Map}_C(X_{i_0}, X_{i_k})_{k-1}$ is defined for every subset $I = \{i_k > i_{k-1} > \cdots > i_0\} \subseteq [n]$ for which $k > 0$ and $[n] \neq I \neq [n] \setminus \{j\}$, satisfying the identity

$$\partial f_I = \sum_{0 < a < k} (-1)^a f_I(\{i_a\}) + (-1)^{k(a+1)} f_{\{i_k > \cdots > i_a\}} \circ f_{\{i_a > \cdots > i_0\}}. \quad (2.3)$$

We wish to show that $\sigma_0$ can be extended to an $n$-simplex of $N_{\bullet}^{dg}(C)$. To give such an extension, we must supply chains $f_{\{i_a\}} \in \text{Hom}_C(X_0, X_n)_{n-1}$ and $f_{\{i_a\} \setminus \{j\}} \in \text{Hom}_C(X_0, X_n)_{n-2}$ which satisfy (2.3) in the cases $I = [n]$ and $I = [n] \setminus \{j\}$. We claim that there is a unique
such extension which also satisfies $f_{[n]} = 0$. Applying (2.3) in the case $I = [n]$, we deduce that $f_{[n]\setminus\{j\}}$ is necessarily given by

$$(-1)^{j+1}f_{[n]\setminus\{j\}} = \sum_{0 < b < n} (-1)^{n(b+1)}(f_{(n)\ldots>b}) \circ f_{(b)\ldots>0}) + \sum_{0 < b < n, b \neq j} (-1)^b f_{[n]\setminus\{b\}}.$$

To complete the proof, it will suffice to verify that this prescription also satisfies (2.3) in the case $I = [n] \setminus \{j\}$. In what follows, for $0 \leq a < b \leq n$, let us write $[ba]$ for the set \{b > b - 1 > \cdots > a\}. We now compute

$$(-1)^{j+1}\partial f_{[n]\setminus\{j\}} = \sum_{0 < b < n} (-1)^{n(b+1)}(\partial f_{[n]}) f_{[b]} + \sum_{0 < b < n, b \neq j} (-1)^b \partial f_{[n]\setminus\{b\}}$$

$$= \sum_{0 < b < c < n} (-1)^{nb+n+c-b}(f_{[n]}) f_{[c]} f_{[b]} + \sum_{0 < b < c < n} (-1)^{(n-b)c} f_{[nc]} f_{[c]} f_{[b]}$$

$$- \sum_{0 < a < b < n} (-1)^{nb+b-a}(f_{[n]}) f_{[b]} f_{[a]} - \sum_{0 < a < b < n} (-1)^{(n-a)b} f_{[n]} f_{[ba]} f_{[a]}$$

$$+ \sum_{0 < a < b < n, b \neq j} (-1)^{b+a} f_{[n]\setminus\{a,b\}} - \sum_{0 < a < b < n, b \neq j} (-1)^{n+a+b-a} f_{[na]} f_{[b]} f_{[a]}$$

$$- \sum_{0 < b < c < n, b \neq j} (-1)^{b+c} f_{[n]\setminus\{b,c\}} + \sum_{0 < b < c < n, b \neq j} (-1)^{n+c+b} f_{[nc]} f_{[c]} f_{[b]}$$

Here the second and fourth terms cancel, the sixth term cancels with first except for those summands with $c = j$, the eighth term cancels with the third except for those summands with $a = j$, and the fifth term cancels the seventh except for those terms with $a = j$ and $c = j$, respectively. After multiplying by $(-1)^{j+1}$, we can rewrite this identity as

$$\partial f_{[n]\setminus\{j\}} = \sum_{0 < b < j} (-1)^{(n-1)(b+1)}(f_{[n]}) f_{[b]} + \sum_{j < b < n} (-1)^{(n-1)b} f_{[nb]} f_{[b] \setminus\{j\}}$$

$$+ \sum_{j < b < n} (-1)^{b-1} f_{[n] \setminus\{b,j\}} + \sum_{0 < b < j} (-1)^b f_{[n] \setminus\{b,j\}},$$

which recovers equation (2.3) in the case $I = [n] \setminus \{j\}$.

**Remark 2.5.3.11.** The theory of differential graded categories can be regarded as a special case of the more general theory of $A_\infty$-categories (see [16]). Definition 2.5.3.10 and Theorem 2.5.3.10 have been extended to the setting of $A_\infty$-categories by Faonte; we refer the reader to [15] for details.
2.5.4 The Homotopy Category of a Differential Graded Category

Let $\mathcal{C}$ be a differential graded category, and let $N^\bullet_{dg}(\mathcal{C})$ denote its differential graded nerve (Definition 2.5.3.7). Then $N^\bullet_{dg}(\mathcal{C})$ is an $\infty$-category (Theorem 2.5.3.10). Moreover:

- The objects of the $\infty$-category $N^\bullet_{dg}(\mathcal{C})$ are the objects of $\mathcal{C}$ (Example 2.5.3.2).
- If $X$ and $Y$ are objects of $\mathcal{C}$, then a morphism from $X$ to $Y$ in the $\infty$-category $N^\bullet_{dg}(\mathcal{C})$ can be identified with a 0-cycle in the chain complex $\text{Hom}_\mathcal{C}(X,Y)_*$ (Example 2.5.3.3), or equivalently with a morphism from $X$ to $Y$ in the underlying category $\mathcal{C}^\circ$ of Construction 2.5.2.4.

We now explain how to describe the homotopy category of $N^\bullet_{dg}(\mathcal{C})$ directly in terms of the differential graded category $\mathcal{C}$ (Proposition 2.5.4.10).

**Definition 2.5.4.1.** Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, and let $f$ and $f'$ be 0-cycles of the chain complex $\text{Hom}_\mathcal{C}(X,Y)_*$. A homotopy from $f$ to $f'$ is a 1-chain $h \in \text{Hom}_\mathcal{C}(X,Y)_1$ satisfying $\partial(h) = f' - f$. We will say that $f$ and $f'$ are homotopic if there exists a homotopy from $f$ to $f'$: that is, if we have an equality $[f] = [f']$ in the homology group $H_0(\text{Hom}_\mathcal{C}(X,Y))$.

**Example 2.5.4.2.** Let $\mathcal{A}$ be an additive category, let $C_\ast$ and $D_\ast$ be chain complexes with values in $\mathcal{A}$, and let $f, f' : C_\ast \to D_\ast$ be chain maps, which we regard as 0-cycles in the mapping complex $\text{Hom}_{\mathcal{Ch}(\mathcal{A})}(C_\ast, D_\ast)_*$ in the differential graded category $\mathcal{Ch}(\mathcal{A})$ of Example 2.5.2.5. Let $h = \{h_n : C_n \to D_{n+1}\}_{n \in \mathbb{Z}}$ be a collection of morphisms, which we regard as a 1-chain of $\text{Hom}_{\mathcal{Ch}(\mathcal{A})}(C_\ast, D_\ast)_*$. Then $h$ is a homotopy from $f$ to $f'$ (in the sense of Definition 2.5.4.1) if and only if it is a chain homotopy from $f$ to $f'$ (in the sense of Definition 2.5.0.5). In particular, $f$ and $f'$ are homotopic morphisms of the differential graded category $\mathcal{Ch}(\mathcal{A})$ (in the sense of Definition 2.5.4.1) if and only if they are chain homotopic (in the sense of Definition 2.5.0.5).

**Remark 2.5.4.3.** Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, and let $f$ and $g$ be 0-cycles of the chain complex $\text{Hom}_\mathcal{C}(X,Y)_*$. Then giving a homotopy from $h$ to $g$ in the sense of Definition 2.5.4.1 is equivalent to giving a homotopy from $g$ to $f$ as morphisms in the $\infty$-category $N^\bullet_{dg}(\mathcal{C})$ (Definition 1.3.3.1): this follows from Example 2.5.3.4. In particular, $f$ and $g$ are homotopic in the sense of Definition 2.5.4.1 if and only if they are homotopic in the sense of Definition 1.3.3.1.

**Remark 2.5.4.4.** Let $\mathcal{C}$ be a differential graded category containing objects $X, Y,$ and $Z$, and suppose we are given 0-cycles $f \in \text{Hom}_\mathcal{C}(X,Y)_0, g \in \text{Hom}_\mathcal{C}(Y,Z)_0,$ and $h \in \text{Hom}_\mathcal{C}(X,Z)_0$. Then Example 2.5.3.4 supplies an equivalence between the following data:

- The datum of a homotopy from $h$ to $g \circ f$, in the sense of Definition 2.5.4.1.
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- The datum of a 2-simplex of $N_{dg}(\mathcal{C})$ witnessing $h$ as a composition of $f$ and $g$, in the sense of Definition 1.3.4.1.

In particular, $h$ is homotopic to the composition $g \circ f$ (in the differential graded category $\mathcal{C}$) if and only if it is a composition of $g$ and $f$ (in the $\infty$-category $N_{dg}(\mathcal{C})$).

00Q3 Proposition 2.5.4.5. Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \text{Ob}(\mathcal{C})$. Let $f$ and $g$ be 0-cycles of the chain complex $\text{Hom}_C(X, Y)_*$ which are homotopic. Then:

(a) For any object $W \in \text{Ob}(\mathcal{C})$ and any 0-cycle $u \in \text{Hom}_C(W, X)_0$, the composite cycles $f \circ u$ and $g \circ u$ are homotopic.

(b) For any object $Z \in \text{Ob}(\mathcal{C})$ and any 0-cycle $v \in \text{Hom}_C(Y, Z)_0$, the composite cycles $v \circ f$ and $v \circ g$ are homotopic.

Proof. By virtue of Remarks 2.5.4.3 and 2.5.4.4, we can regard Proposition 2.5.4.5 as a special case of Proposition 1.3.4.7. However, it is easy to prove directly. If $h \in \text{Hom}_C(X, Y)_1$ is a homotopy from $f$ to $g$ and $u$ is a 0-cycle in $\text{Hom}_C(W, X)_0$, then the calculation

$$\partial(h \circ u) = ((\partial h) \circ u) - (h \circ (\partial u))$$

$$= (\partial h) \circ u$$

$$= (f - g) \circ u$$

$$= (f \circ u) - (g \circ u)$$

shows that $(h \circ u) \in \text{Hom}_C(W, Y)_1$ is a homotopy from $f \circ u$ to $g \circ u$. This proves (a), and (b) follows from a similar argument.

00Q4 Construction 2.5.4.6 (The Homotopy Category of a Differential Graded Category). Let $\mathcal{C}$ be a differential graded category. We define a category $\text{h} \mathcal{C}$ as follows:

- The objects of $\text{h} \mathcal{C}$ are the objects of $\mathcal{C}$.

- For every pair of objects $X, Y \in \text{Ob}(\text{h} \mathcal{C}) = \text{Ob}(\mathcal{C})$, we define

$$\text{Hom}_{\text{h} \mathcal{C}}(X, Y) = \text{H}_0(\text{Hom}_{\mathcal{C}}(X, Y)).$$

If $f$ is a 0-cycle of the chain complex $\text{Hom}_{\mathcal{C}}(X, Y)_*$, let $[f]$ denote its image in the homology group $\text{H}_0(\text{Hom}_{\mathcal{C}}(X, Y)) = \text{Hom}_{\text{h} \mathcal{C}}(X, Y)$.

- For each object $X \in \text{Ob}(\text{h} \mathcal{C}) = \text{Ob}(\mathcal{C})$, the identity morphism from $X$ to itself in the category $\text{h} \mathcal{C}$ is given by $[\text{id}_X]$, where $\text{id}_X$ is the identity morphism from $X$ to itself in $\mathcal{C}$. 
• For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C})$, the composition law

$$\text{Hom}_{h\mathcal{C}}(Y, Z) \times \text{Hom}_{h\mathcal{C}}(X, Y) \to \text{Hom}_{h\mathcal{C}}(X, Z)$$

is characterized by the formula $[g] \circ [f] = [g \circ f]$ for $f \in Z_0(\text{Hom}_{\mathcal{C}}(X, Y))$ and $g \in Z_0(\text{Hom}_{\mathcal{C}}(Y, Z))$ (this composition law is well-defined by virtue of Proposition 2.5.4.5).

We will refer to $h\mathcal{C}$ as the \textit{homotopy category} of the differential graded category $\mathcal{C}$.

\textbf{Remark 2.5.4.7}. Passage from a differential graded category $\mathcal{C}$ to its homotopy category $h\mathcal{C}$ can be regarded as a special case of Remark 2.1.7.4, applied to the lax monoidal functor $\text{Ch}(\mathcal{Z}^\bullet) \to \text{Set (}C^\bullet, d\text{)} \mapsto H_0(C^\bullet)$ with tensor constraints given by $\mu_{C,D} : H_0(C) \times H_0(D) \to H_0(C \otimes D) \quad ([x], [y]) \mapsto [x \otimes y]$.

\textbf{Remark 2.5.4.8}. Let $\mathcal{C}$ be a differential graded category, with underlying category $\mathcal{C}^\circ$ (Construction 2.5.2.4) and homotopy category $h\mathcal{C}$ (Construction 2.5.4.6). There is an evident functor $\mathcal{C}^\circ \to h\mathcal{C}$ which is the identity on objects, given on morphisms by the construction

$$\text{Hom}_{\mathcal{C}^\circ}(X, Y) = Z_0(\text{Hom}_{\mathcal{C}}(X, Y)) \to H_0(\text{Hom}_{\mathcal{C}}(X, Y)) = \text{Hom}_{h\mathcal{C}}(X, Y) \quad f \mapsto [f].$$

\textbf{Example 2.5.4.9} (The Homotopy Category of Chain Complexes). Let $\mathcal{A}$ be an additive category, and let $\text{Ch}(\mathcal{A})$ be the differential graded category of chain complexes with values in $\mathcal{A}$ (Example 2.5.2.5). Then the homotopy category of $\text{Ch}(\mathcal{A})$ in the sense of Construction 2.5.4.6 agrees with the homotopy category $h\text{Ch}(\mathcal{A})$ introduced in Construction 2.5.0.8.

\textbf{Proposition 2.5.4.10}. Let $\mathcal{C}$ be a differential graded category and let $N_{dg}(\mathcal{C})$ denote the differential graded nerve of $\mathcal{C}$. Then the homotopy category $hN_{dg}(\mathcal{C})$ (Definition 1.3.5.3) is canonically isomorphic to the homotopy category $h\mathcal{C}$ (Construction 2.5.4.6).

\textit{Proof}. Combine Remarks 2.5.4.3 and 2.5.4.4. \hfill \Box

\subsection{2.5.5 Digression: The Homology of Simplicial Sets}

Among the most useful invariants studied in algebraic topology are the \textit{singular homology groups} $H_\ast(X; \mathbb{Z})$ of a topological space $X$. These are defined as the homology groups of the \textit{singular chain complex}

$$\cdots \to C_3(X; \mathbb{Z}) \xrightarrow{\partial} C_2(X; \mathbb{Z}) \xrightarrow{\partial} C_1(X; \mathbb{Z}) \xrightarrow{\partial} C_0(X; \mathbb{Z}),$$
where $C_n(X;\mathbb{Z})$ denotes the free abelian group generated by the set $\text{Hom}_{\text{Top}}(|\Delta^n|,X)$ of singular $n$-simplices of $X$, and the boundary operator $\partial$ is given by the formula

$$\partial : C_n(X;\mathbb{Z}) \to C_{n-1}(X;\mathbb{Z}) \quad \partial(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma).$$

We can therefore view the passage from the topological space $X$ to its homology $H_*(X;\mathbb{Z})$ as proceeding in four stages:

- We first extract from the topological space $X$ its singular simplicial set $\text{Sing}_*(X)$ (Construction 1.1.7.1).
- We then replace $\text{Sing}_*(X)$ by the simplicial abelian group $\mathbb{Z}[\text{Sing}_*(X)]$, carrying each object $[n] \in \Delta^{\text{op}}$ to the free abelian group $\mathbb{Z}[\text{Sing}_n(X)]$ generated by the set $\text{Sing}_n(X)$.
- We next regard the abelian groups $\{\mathbb{Z}[\text{Sing}_n(X)]\}_{n \geq 0}$ as the terms of a chain complex $(C_*(X;\mathbb{Z}), \partial)$, where the differential $\partial$ is given by the alternating sum of the face maps of the simplicial abelian group $\mathbb{Z}[\text{Sing}_*(X)]$.
- For each integer $n$, we define $H_n(X;\mathbb{Z})$ to be the $n$th homology group of the chain complex $(C_*(X;\mathbb{Z}), \partial)$ (Definition 2.5.1.4).

In other words, the functor $X \mapsto H_*(X;\mathbb{Z})$ factors as a composition

$$\text{Top} \xrightarrow{\text{Sing}_*} \text{Set}_\Delta \xrightarrow{\mathbb{Z}[-]} \text{Ab}_\Delta \xrightarrow{C_*} \text{Ch}(\mathbb{Z}) \xrightarrow{H_*} \text{Ab},$$

where $\text{Ab}_\Delta$ denotes the category of simplicial abelian groups and $C_* : \text{Ab}_\Delta \to \text{Ch}(\mathbb{Z})$ is given by the following:

**Construction 2.5.5.1 (The Moore Complex).** Let $A_\bullet$ be a semisimplicial abelian group (Variant 1.1.6). For each $n \geq 1$, we define a group homomorphism $\partial : A_n \to A_{n-1}$ by the formula

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma),$$

where $d_i : A_n \to A_{n-1}$ is the $i$th face map (Notation 1.1.8). For $n \geq 2$ and $\sigma \in A_n$, we compute

$$\partial^2(\sigma) = \partial(\sum_{i=0}^{n} (-1)^i d_i(\sigma)) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (d_j d_i)(\sigma) = 0$$
where the final equality follows from the identity $d_i \circ d_j = d_{j-1} \circ d_i$ for $0 \leq i < j \leq n$ (see Exercise 1.1.1.10). We let $C_\ast(A)$ denote the chain complex of abelian groups given by

$$C_n(A) = \begin{cases} A_n & \text{if } n \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the differential is given by $\partial$. We will refer to $C_\ast(A)$ as the *Moore complex* of the semisimplicial abelian group $A_\ast$.

If $A_\ast$ is a simplicial abelian group, we let $C_\ast(A)$ denote the Moore complex of the semisimplicial abelian group underlying $A_\ast$ (Remark 1.1.1.7).

**Definition 2.5.5.2** (Homology of Simplicial Sets). Let $S_\ast$ be a simplicial set and let $\mathbb{Z}[S_\ast]$ denote the simplicial abelian group freely generated by $S_\ast$. We let $C_\ast(S; \mathbb{Z})$ denote the Moore complex of $\mathbb{Z}[S_\ast]$. We will refer to $C_\ast(S; \mathbb{Z})$ as the *chain complex of $S_\ast$* for each integer $n$, we denote the $n$th homology group of $C_\ast(S; \mathbb{Z})$ by $H_n(S; \mathbb{Z})$ and refer to it as the $n$th homology group of $X$ (with coefficients in $\mathbb{Z}$).

**Example 2.5.5.3.** Let $X$ be a topological space. Then the singular chain complex $C_\ast(X; \mathbb{Z})$ is the chain complex of the singular simplicial set $\text{Sing}_\ast(X)$. In particular, the homology groups of the simplicial set $\text{Sing}_\ast(X)$ are the usual singular homology groups of the topological space $X$.

**Example 2.5.5.4.** Let $S_\ast = \Delta^0$ be the standard 0-simplex. Then $S_\ast$ is a simplicial set having a single simplex of each dimension. Consequently, the chain complex $C_\ast(S; \mathbb{Z})$ is given by $\mathbb{Z}$ in each nonnegative degree. For $n > 0$, the differential $\mathbb{Z} \cong C_n(S; \mathbb{Z}) \xrightarrow{\partial} C_{n-1}(S; \mathbb{Z}) \cong \mathbb{Z}$ is given by multiplication by the integer

$$\sum_{i=0}^{n} (-1)^i = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

as indicated in the diagram

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$ 

It follows that the homology groups of $S_\ast$ are given by

$$H_n(S; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that although the *homology* of the simplicial set $S_\ast = \Delta^0$ is concentrated in degree zero, the chain complex $C_\ast(S; \mathbb{Z})$ is not. Essentially, this is because $S_\ast$ has degenerate simplices in each dimension $n > 0$ which do not contribute to its homology. This is a special case of a more general phenomenon.
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**Notation 2.5.5.5.** Let $A_\bullet$ be a simplicial abelian group. For each $n \geq 0$, let $D_n(A)$ denote the subgroup of $C_n(A) = A_n$ generated by the images of the degeneracy operators \( \{ s_i : A_{n-1} \to A_n \} \) \( 0 \leq i \leq n-1 \). By convention, we set $D_n(A) = 0$ for $n < 0$.

**Proposition 2.5.5.6.** Let $A_\bullet$ be a simplicial abelian group. For every positive integer $n$, the boundary operator $\partial : C_n(A) \to C_{n-1}(A)$ carries the subgroup $D_n(A)$ into $D_{n-1}(A)$. Consequently, we can regard $D_\bullet(A)$ as a subcomplex of the Moore complex $C_\bullet(A)$.

**Proof.** Choose an element $\sigma \in D_n(A)$; we wish to show that $\partial(\sigma)$ belongs to $D_{n-1}(A)$. Without loss of generality, we may assume that $\sigma = s_i(\tau)$ for some $0 \leq i \leq n - 1$ and some $\tau \in A_{n-1}$. We now compute

$$\partial(\sigma) = \sum_{j=0}^{n} (-1)^j d_j(\sigma)$$

$$= \sum_{j=0}^{i-1} (-1)^j d_j s_i \tau + (-1)^i d_i s_i \tau + (-1)^{i+1} d_{i+1} s_i \tau + \left( \sum_{j=i+2}^{n} (-1)^j d_j s_i \tau \right)$$

$$= \sum_{j<i} (-1)^j s_{i-1} d_j \tau + (-1)^i \tau + (-1)^{i+1} \tau + \left( \sum_{j=i+2}^{n} (-1)^j s_i d_{j-1} \tau \right)$$

$$\in \text{im}(s_{i-1}) + \text{im}(s_i) \subseteq D_{n-1}(A).$$

**Construction 2.5.5.7 (The Normalized Moore Complex: First Construction).** Let $A_\bullet$ be a simplicial abelian group. We let $N_s(A)$ denote the chain complex given by the quotient $C_s(A)/D_s(A)$, where $C_s(A)$ is the Moore complex of Construction 2.5.5.1 and $D_s(A) \subseteq C_s(A)$ is the subcomplex of Proposition 2.5.5.6. We will refer to $N_s(A)$ as the normalized Moore complex of the simplicial abelian group $A_\bullet$.

Put more informally, the normalized Moore complex $N_s(A)$ of a simplicial abelian group $A_\bullet$ is obtained the Moore complex $C_s(A)$ by forming the quotient by degenerate simplices of $A_\bullet$.

**Remark 2.5.5.8.** By taking Construction 2.5.5.7 as our definition of the chain complex $N_s(A)$, we have adopted the perspective that $N_s(A)$ is a quotient of the Moore complex $C_s(A)$. However, it can also be realized as a subcomplex of the Moore complex $C_s(A)$: see Construction 2.5.6.16 and Proposition 2.5.6.19.

**Construction 2.5.5.9 (The Normalized Chain Complex of a Simplicial Set).** Let $S_\bullet$ be a simplicial set and let $\mathbb{Z}[S_\bullet]$ be the simplicial abelian group freely generated by $S_\bullet$. We let $N_s(S; \mathbb{Z})$ denote the normalized Moore complex of $\mathbb{Z}[S_\bullet]$. This chain complex can be described more concretely as follows:
For each integer $n \geq 0$, we can identify $N_n(S)$ with the free abelian group generated by the set $S_n^{nd}$ of nondegenerate $n$-simplices of $S_\bullet$.

The boundary map $\partial : N_n(S) \to N_{n-1}(S)$ is given by the formula

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \begin{cases} d_i(\sigma) & \text{if } d_i(\sigma) \text{ is nondegenerate} \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to $N_\ast(S; \mathbb{Z})$ as the normalized chain complex of the simplicial set $S_\bullet$.

**Example 2.5.5.10.** Let $S_\bullet = \Delta^0$ be the standard 0-simplex. Then the normalized chain complex $N_\ast(S; \mathbb{Z})$ can be identified with abelian group $\mathbb{Z}$, regarded as a chain complex concentrated in degree zero. Note that the calculation of Example 2.5.5.4 shows that the quotient map $C_\ast(S; \mathbb{Z}) \twoheadrightarrow N_\ast(S; \mathbb{Z})$ induces an isomorphism on homology.

Example 2.5.5.10 is a special case of the following:

**Proposition 2.5.5.11.** For every simplicial abelian group $A_\bullet$, the quotient map $C_\ast(A) \twoheadrightarrow N_\ast(A)$ is a quasi-isomorphism of chain complexes: that is, it induces an isomorphism on homology groups.

**Remark 2.5.5.12.** In the situation of Proposition 2.5.5.11, an even stronger statement holds: the quotient map $C_\ast(A) \twoheadrightarrow N_\ast(A)$ is a chain homotopy equivalence (Definition 2.5.0.5).

We will give the proof of Proposition 2.5.5.11 in §2.5.6 (see Proposition 2.5.6.22).

**Example 2.5.5.13.** Let $S_\bullet$ be a simplicial set. It follows from Proposition 2.5.5.11 that the quotient map $C_\ast(S; \mathbb{Z}) \twoheadrightarrow N_\ast(S; \mathbb{Z})$ induces an isomorphism on homology. In particular, the homology groups $H_\ast(S; \mathbb{Z})$ of the simplicial set $S_\bullet$ (in the sense of Definition 2.5.5.2) can be computed by means of the normalized chain complex $N_\ast(S; \mathbb{Z})$. This has various practical advantages. For example, if $S_\bullet$ is a simplicial set of dimension $\leq d$, then the chain complex $N_\ast(S; \mathbb{Z})$ is concentrated in degrees $\leq d$. It follows that the homology groups $H_\ast(S; \mathbb{Z})$ are also concentrated in degrees $\leq d$, which is not immediately obvious from the definition (note that the chain complex $C_\ast(S; \mathbb{Z})$ is never concentrated in degrees $\leq d$, except in the trivial case where $S_\bullet$ is empty).

**Example 2.5.5.14.** Let $S_\bullet = N_\ast(Q)$ be the nerve of a partially ordered set $Q$. Suppose that $Q$ has a least element $e$, which determines a map of simplicial sets $i : \Delta^0 \to S_\bullet$ which is right inverse to the projection map $q : S_\bullet \to \Delta^0$. Passing to normalized chain complexes, we obtain chain maps

$$\widehat{i} : \mathbb{Z}[0] \simeq N_\ast(\Delta^0; \mathbb{Z}) \hookrightarrow N_\ast(S_\bullet; \mathbb{Z}) \quad \widehat{q} : N_\ast(S_\bullet; \mathbb{Z}) \twoheadrightarrow N_\ast(\Delta^0; \mathbb{Z}) \simeq \mathbb{Z}[0].$$
We claim that \( \hat{\iota} \) and \( \hat{q} \) are chain homotopy inverse to one another. In one direction, this is clear: the composition \( \hat{q} \circ \hat{\iota} \) is equal to the identity. We complete the proof by constructing a chain homotopy from the composite map \( \hat{\iota} \circ \hat{q} \) to the identity \( \text{id} \) on \( N_*(S; \mathbb{Z}) \). This chain homotopy is given by a collection of maps \( h_m : N_m(S; \mathbb{Z}) \to N_{m+1}(S; \mathbb{Z}) \), given on nondegenerate simplices by the construction

\[
(q_0 < q_1 < \cdots < q_m) \mapsto \begin{cases} 
0 & \text{if } q_0 = e \\
(e < q_0 < q_1 < \cdots < q_m) & \text{otherwise.}
\end{cases}
\]

In particular, if \( Q \) is a partially ordered set with a least element, then the homology groups of the nerve \( S_* = N_*(Q) \) are given by

\[
H_*(S; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } * = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

**Variant 2.5.5.15** (Relative Chain Complexes). Let \( S_* \) be a simplicial set and let \( S'_* \subseteq S_* \) be a simplicial subset. Then we can identify the free simplicial abelian group \( \mathbb{Z}[S'_*] \) with a simplicial subgroup of \( \mathbb{Z}[S_*] \). We let \( C_*(S, S'; \mathbb{Z}) \) and \( N_*(S, S'; \mathbb{Z}) \) denote the Moore complex and normalized Moore complex of the simplicial abelian group \( \mathbb{Z}[S_*]/\mathbb{Z}[S'_*]. \) By virtue of Proposition 2.5.5.11, these complexes have the same homology groups, which we denote by \( H_*(S, S'; \mathbb{Z}) \) and refer to as the relative homology groups of the pair \((S'_* \subseteq S_*)\).

### 2.5.6 The Dold-Kan Correspondence

Let \( \text{Ab} \) denote the category of abelian groups, and \( \text{Ab}_{\Delta} = \text{Fun}(\Delta^{\text{op}}, \text{Ab}) \) the category of simplicial abelian groups. The formation of normalized Moore complexes (Construction 2.5.5.7) determines a functor \( N_* : \text{Ab}_{\Delta} \to \text{Ch}(\mathbb{Z}) \). Our goal in this section is to prove the following fundamental result, which was discovered independently by Dold ([8]) and Kan ([24]):

**Theorem 2.5.6.1** (The Dold-Kan Correspondence). The normalized Moore complex functor determines an equivalence of categories \( N_* : \text{Ab}_{\Delta} \to \text{Ch}(\mathbb{Z})_{\geq 0} \).

**Remark 2.5.6.2.** Theorem 2.5.6.1 admits many generalizations. For example, if \( \mathcal{A} \) is an abelian category (Definition [?]), then a variant of Construction 2.5.5.9 supplies an equivalence of categories

\[
N_* : \{ \text{Simplicial objects of } \mathcal{A} \} \to \text{Ch}(\mathcal{A})_{\geq 0},
\]

where \( \text{Ch}(\mathcal{A})_{\geq 0} \) denotes the category of (nonnegatively graded) chain complexes with values in \( \mathcal{A} \) (see Theorem [?]). For more general categories \( \mathcal{A} \), one can think of the category of simplicial objects \( \mathcal{A}_{\Delta} = \text{Fun}(\Delta^{\text{op}}, \mathcal{A}) \) as a replacement for the category of chain complexes \( \text{Ch}(\mathcal{A})_{\geq 0} \), which is better behaved in “non-additive” situations.
We begin by constructing a right adjoint to the normalized Moore complex functor.

**Construction 2.5.6.3 (The Eilenberg-MacLane Functor).** Let \( n \) be a nonnegative integer and let \( N_*(\Delta^n; \mathbb{Z}) \) denote the normalized chain complex of the standard \( n \)-simplex (Construction 2.5.5.9). For every chain complex \( M_* \), we let \( K_n(M_*) \) denote the collection of chain maps from \( N_*(\Delta^n; \mathbb{Z}) \) into \( M_* \) (which we regard as an abelian group under addition). Note that the construction \([n] \mapsto N_*(\Delta^n; \mathbb{Z})\) determines a functor from the simplex category \( \Delta \) to the category of chain complexes, so we can regard \([n] \mapsto K_n(M_*)\) as a functor from \( \Delta^{op} \) to the category of abelian groups. We denote this simplicial abelian group by \( K(M_*) \), and refer to it as the *Eilenberg-MacLane space associated to \( M_* \).*

**Remark 2.5.6.4.** Let \( M_* \) be a chain complex. We will generally not distinguish in notation between the simplicial abelian group \( K(M_*) \) and its underlying simplicial set. Note that \( K(M_*) \) is automatically a Kan complex (Proposition 1.1.9.9), which motivates our usage of the term “space”.

**Example 2.5.6.5.** Let \( M_* \) be a chain complex. Then we have canonical isomorphisms

\[
K_0(M_*) = \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(\Delta^0; \mathbb{Z}), M_*) = \text{Hom}_{\text{Ch}(\mathbb{Z})}(\mathbb{Z}[0], M_*) = Z_0(M)
\]

In other words, we can identify vertices of the simplicial set \( K(M_*) \) with 0-cycles of the chain complex \( M_* \).

**Example 2.5.6.6.** Let \( M_* \) be a chain complex, and let \( x, y \in M_0 \) be a pair of 0-cycles, which we identify with vertices of the simplicial set \( K(M_*) \). The following conditions are equivalent:

(a) The vertices \( x \) and \( y \) belong to the same connected component of the simplicial set \( K(M_*) \) (Definition 1.1.6.8).

(b) There exists an edge \( e \) of the simplicial set \( K(M_*) \) connecting \( x \) to \( y \) (so that \( d_1(e) = x \) and \( d_0(e) = y \)).

(c) The cycles \( x \) and \( y \) are homologous: that is, there exists an element \( u \in M_1 \) satisfying \( \partial(u) = x - y \).

The equivalence of \((a) \Leftrightarrow (b)\) follows from the fact that \( K(M_*) \) is a Kan complex (see Remark 1.3.6.14), while the equivalence \((b) \Leftrightarrow (c)\) follows immediately from the construction of the simplicial set \( K(M_*) \). It follows that the set of connected components \( \pi_0(K(M_*)) \) can be identified with the 0th homology group \( H_0(M) \).

We now describe a particularly important special case of Construction 2.5.6.3.
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Construction 2.5.6.7 (Eilenberg-MacLane Spaces). Let $A$ be an abelian group, let $n$ be an integer, and let $A[n]$ denote the chain complex consisting of the single abelian group $A$, concentrated in degree $n$ (Example 2.5.1.2). We will denote the simplicial abelian group $K(A[n])$ by $K(A, n)$ and refer to it as the $n$th Eilenberg-MacLane space of $A$.

Remark 2.5.6.8. The formation of Eilenberg-MacLane spaces $A \mapsto K(A, n)$ is defined for every integer $n$. However, it is only interesting for $n \geq 0$: if $n$ is negative, then the simplicial abelian group $K(A, n)$ is trivial (that is, it is isomorphic to $\Delta^0$ as a simplicial set).

Example 2.5.6.9. Let $A$ be an abelian group. To supply an $n$-simplex of the simplicial set $K(A, 0)$, one must give a chain map $\sigma : N_\ast(\Delta^n; \mathbb{Z}) \to A[0]$. By definition, a homomorphism of graded abelian groups from $N_\ast(\Delta^n; \mathbb{Z})$ to $A[0]$ is given by a tuple $\{a_i\}_{0 \leq i \leq n}$ of elements of $A$, indexed by the set $[n] = \{0 < 1 < \cdots < n\}$ of vertices of $\Delta^n$. Under this identification, the chain maps can be identified with those tuples $\{a_i\}_{0 \leq i \leq n}$ which are constant: that is, which satisfy $a_i = a_j$ for all $i, j \in [n]$. It follows that the Eilenberg-MacLane space $K(A, 0)$ can be identified with the constant simplicial abelian group taking the value $A$.

Example 2.5.6.10. Let $A$ be an abelian group. To supply an $n$-simplex of the simplicial set $K(A, 1)$, one must give a chain map $\sigma : N_\ast(\Delta^n; \mathbb{Z}) \to A[1]$. By definition, a homomorphism of graded abelian groups from $N_\ast(\Delta^n; \mathbb{Z})$ to $A[1]$ is given by a system $\{a_{i,j}\}_{0 \leq i < j \leq n}$ of elements of $A$, indexed by the set of all nondegenerate edges of $\Delta^n$. Under this identification, the chain maps can be identified with those systems $\{a_{i,j}\}_{0 \leq i < j \leq n}$ satisfying $a_{i,j} + a_{j,k} = a_{i,k}$ for $0 \leq i < j < k \leq n$. It follows that the Eilenberg-MacLane space $K(A, 1)$ can be identified with the the Milnor construction $B_\ast A$ (Example 1.2.4.3).

Notation 2.5.6.11. Let $M_\ast$ be a chain complex. Then every $n$-simplex $\sigma$ of the simplicial set $K(M_\ast)$ can be identified with a map of chain complexes $N_\ast(\Delta^n; \mathbb{Z}) \to M_\ast$, which carries the generator of $N_n(\Delta^n; \mathbb{Z})$ to an $n$-chain $\bar{v}(\sigma) \in M_n$. Moreover:

- Since $\sigma$ is a map of chain complexes, we have
  \[ \partial(\bar{v}(\sigma)) = \sum_{i=0}^{n} (-1)^i \bar{v}(d_i\sigma)). \]
  In other words, the construction $\sigma \mapsto \bar{v}(\sigma)$ determines a chain map from the Moore complex $C_\ast(K(M_\ast))$ to the chain complex $M_\ast$.

- If $\sigma$ is a degenerate $n$-simplex of $K(M_\ast)$, then the map of chain complexes $\sigma : N_\ast(\Delta^n; \mathbb{Z}) \to M_\ast$ factors through $N_\ast(\Delta^m; \mathbb{Z})$ for some $m < n$, and therefore annihilates the generator of $N_n(\Delta^n; \mathbb{Z})$. It follows that $\bar{v}$ factors (uniquely) as a composition
  \[ C_\ast(K(M_\ast)) \to N_\ast(K(M_\ast)) \xrightarrow{v} M_\ast. \]
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We will refer to the chain map \( v : N_*(K(M_*)) \to M_* \) as the counit map.

**Proposition 2.5.6.12.** Let \( M_* \) be a chain complex and let \( v : N_*(K(M_*)) \to M_* \) be the counit map of Notation 2.5.6.11. Then, for any simplicial abelian group \( A_* \), the composite map

\[
\theta : \text{Hom}_{\text{Ab}_\Delta}(A_*, K(M_*)) \to \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(A), N_*(K(M_*))) \xrightarrow{\psi} \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(A), M_*)
\]

is an isomorphism of abelian groups.

**Proof.** Let us say that a simplicial abelian group \( A_* \) is free if it can be written as a (possibly infinite) direct sum of simplicial abelian groups of the form \( \mathbb{Z}[\Delta^n] \). Note that every simplicial abelian group \( A_* \) admits a surjection \( P_* \to A_* \), where \( P_* \) is free (for example, we can take \( P_* \) to be the direct sum \( \bigoplus_\sigma \mathbb{Z}[\Delta^{\dim(\sigma)}] \) where \( \sigma \) ranges over all the simplices of \( A_* \)). Applying this observation twice, we observe that every simplicial abelian group \( A_* \) admits a resolution \( Q_* \to P_* \to A_* \to 0 \), which determines a commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_{\text{Ab}_\Delta}(A_*, K(M_*)) & \to & \text{Hom}_{\text{Ab}_\Delta}(P_*, K(M_*)) & \to & \text{Hom}_{\text{Ab}_\Delta}(Q_*, K(M_*)) & \to & 0 \\
& & \downarrow{\theta} & & \downarrow{\theta'} & & \downarrow{\theta''} & & \\
0 & \to & \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(A), M_*) & \to & \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(P), M_*) & \to & \text{Hom}_{\text{Ch}(\mathbb{Z})}(N_*(Q), M_*) & & \\
\end{array}
\]

Consequently, to prove that \( \theta \) is an isomorphism, it will suffice to show that \( \theta' \) and \( \theta'' \) are isomorphisms. In other words, we may assume without loss of generality that the simplicial abelian group \( A_* \) is free. Decomposing \( A_* \) as a direct sum, we can further reduce to the case \( A_* = \mathbb{Z}[\Delta^n] \), in which case the result follows immediately from the definitions.

**Corollary 2.5.6.13.** The normalized Moore complex functor \( N_* : \text{Ab}_\Delta \to \text{Ch}(\mathbb{Z}) \) admits a right adjoint \( K : \text{Ch}(\mathbb{Z}) \to \text{Ab}_\Delta \), given on objects by Construction 2.5.6.3.

Note that we can also regard \( M_* \mapsto K(M_*) \) as a functor from chain complexes to simplicial sets (by neglecting the group structure on \( K(M_*) \)). This simplicial set also has a universal property:

**Corollary 2.5.6.14.** The normalized chain complex functor

\[
N_*(- ; \mathbb{Z}) : \text{Set}_\Delta \to \text{Ch}(\mathbb{Z})
\]

admits a right adjoint, given on objects by the functor \( M_* \mapsto K(M_*) \) of Construction 2.5.6.3.
Remark 2.5.6.15. When regarded as a functor from $\text{Ch}(\mathbb{Z})$ to the category of simplicial sets, the functor $M_* \mapsto K(M_*)$ fits into the paradigm of Variant 1.1.7.6: it is the functor $\text{Sing}_Q^*$ associated to the cosimplicial chain complex $Q : \Delta \to \text{Ch}(\mathbb{Z})$ $[n] \mapsto N_*(\Delta^n; \mathbb{Z})$.

To deduce Theorem 2.5.6.1, it is convenient to use a different description of the normalized Moore complex.

Construction 2.5.6.16 (The Normalized Moore Complex: Second Construction). Let $A_\bullet$ be a simplicial abelian group. For each $n \geq 0$, we let $\tilde{N}_n(A)$ denote the subgroup of $C_n(A) = A_n$ consisting of those elements $x$ which satisfy $d_i(x) = 0$ for $1 \leq i \leq n$. Note that if $x$ satisfies this condition, then we have $\partial(x) = \sum_{i=0}^{n} (-1)^i d_i(x) = d_0(x)$.

Moreover, the identity $d_i d_0(x) = d_0 d_{i+1}(x) = 0$ shows that $\partial(x) = d_0$ belongs to the subgroup $\tilde{N}_{n-1}(A) \subseteq C_{n-1} = A_{n-1}$. We can therefore regard $\tilde{N}_*(A)$ as a subcomplex of the Moore complex $C_*(A)$.

In the situation of Construction 2.5.6.16, we will abuse terminology by referring to the chain complex $\tilde{N}_*(A)$ as the normalized Moore complex of $A_\bullet$. This abuse is justified by the observation that the chain complexes $\tilde{N}_*(A)$ is canonically isomorphic to the normalized Moore complex $N_*(A)$ of Construction 2.5.5.7 (Proposition 2.5.6.19 below). We will deduce this from the following more precise statement:

Lemma 2.5.6.17. Let $A_\bullet$ be a simplicial abelian group and let $n$ be a nonnegative integer. Then the map

$$f : \bigoplus_{\alpha : [n] \to [m]} \tilde{N}_m(A) \to A_n \quad \{x_\alpha\} \mapsto \sum \alpha^*(x_\alpha)$$

is an isomorphism of abelian groups. Here the direct sum is indexed by surjective nondecreasing maps $\alpha : [n] \to [m]$ for $0 \leq m \leq n$, and $\alpha^* : A_m \to A_n$ denotes the associated group homomorphism.

Proof. We first prove that $f$ is surjective. The proof proceeds by induction on $n$. By virtue of our inductive hypothesis, the image of $f$ contains the subgroups $\tilde{N}_n(A), D_n(A) \subseteq C_n(A) = A_n$.

It will therefore suffice to show that the composite map

$$\tilde{N}_n(A) \hookrightarrow C_n(A) \to C_n(A)/D_n(A)$$

is surjective. Fix an element $x \in C_n(A)/D_n(A)$. For each $x \in C_n(A)$ representing $x$, let $i_x$ be the smallest nonnegative integer such that $d_j(x)$ vanishes for $i_x < j \leq n$. Without
loss of generality, we may assume that \( x \) is chosen so that \( i = i_x \) is as small as possible. We wish to prove that \( i = 0 \) (so that \( x \) belongs to \( N_n(A) \)). Assume otherwise, and set \( y = x - (s_{i-1} \circ d_i)(x) \). Then \( y \) is congruent to \( x \) modulo \( D_n(A) \), and for \( i \leq j \leq n \) we have

\[
\begin{align*}
d_j(y) &= d_j(x) - (d_j \circ (s_{i-1} \circ d_i))(x) \\
&= d_j(x) - \begin{cases} 
  d_i(x) & \text{if } i = j \\
  (s_{i-1} \circ d_{j-1} \circ d_i)(x) & \text{if } i < j.
\end{cases}
\end{align*}
\]

It follows that \( i_y < i = i_x \), contradicting our choice of \( x \).

We now prove that \( f \) is injective. Suppose otherwise, so that there exists a nonzero element

\[ \{x_\alpha \} \in \bigoplus_{\alpha: [n] \to [m]} \tilde{N}_m(A) \]

which is annihilated by \( f \). Then there exists some surjective map \( \beta : [n] \to [k] \) such that \( x_\beta \) is nonzero. Assume that \( k \) has been chosen as small as possible. Moreover, we may assume that \( \beta \) is maximal among nondecreasing maps \( [n] \to [k] \) such that \( x_\beta \neq 0 \): in other words, that for any other map \( \alpha : [n] \to [k] \) satisfying \( \beta(i) \leq \alpha(i) \) for \( 0 \leq i \leq n \), we either have \( \beta = \alpha \) or \( x_\alpha = 0 \). Let \( \gamma : [k] \to [n] \) be the map given by \( \gamma(j) = \min\{i \in [n] : \beta(i) = j\} \). Then \( \gamma \) is a nondecreasing map satisfying \( \beta \circ \gamma = \text{id}_{[k]} \) and \( \gamma(0) = 0 \). We then have

\[
\gamma^*f(\{x_\alpha\}) = \gamma^* \left( \sum_{\alpha: [n] \to [m]} \alpha^*(x_\alpha) \right) = \sum_{\alpha: [n] \to [m]} (\alpha \circ \gamma)^*(x_\alpha).
\]

We now inspect the summands appearing on the right hand side:

- Let \( \alpha : [n] \to [m] \) be a surjective nondecreasing function, and suppose that the composite map \( [k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m] \) is not surjective. Then we can choose \( 0 \leq i \leq m \) such that \( i \) does not belong to the image of \( \alpha \circ \gamma \). Then the homomorphism \((\alpha \circ \gamma)^* : A_m \to A_k\) factors through the face map \( d_i : A_m \to A_{m-1} \). Note that we must have \( i > 0 \) (since \( \gamma(0) = 0 \) and \( \alpha(0) = 0 \)), so that \( x_\alpha \) is annihilated by \( d_i \) (by virtue of our assumption that \( x_\alpha \) belongs to the subgroup \( N_m(A) \subseteq A_m \)) and therefore also by \((\alpha \circ \gamma)^* \).

- Let \( \alpha : [n] \to [m] \) be a surjective nondecreasing function, and suppose that the composite map \( [k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m] \) is surjective but not injective. In this case, we must have \( m < k \), so that \( x_\alpha \) vanishes by virtue of the minimality assumption on \( k \).
• Let $\alpha : [n] \to [m]$ be a surjective map, and suppose that the composite map $[k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m]$ is bijective, so that $m = k$ and $\alpha \circ \gamma$ is the identity on $[k]$. For $0 \leq i \leq n$, we have $(\gamma \circ \beta)(i) \leq i$ (by the definition of $\gamma$), so that $\beta(i) = ((\alpha \circ \gamma) \circ \beta)(i) = (\alpha \circ (\gamma \circ \beta))(i) \leq \alpha(i)$.

Invoking our maximality assumption on $\beta$, we conclude that either $\alpha = \beta$ or $x_\alpha$ vanishes.

Combining these observations, we obtain an equality $x_\beta = \sum_{\alpha : [n] \to [m]} (\alpha \circ \gamma)^* (x_\alpha) = \gamma^* f(\{x_\alpha\}) = 0$, contradicting our choice of $\beta$. \hfill \Box

**Remark 2.5.6.18.** Let $f : A_\bullet \to B_\bullet$ be a morphism of simplicial abelian groups. By virtue of Lemma 2.5.6.17, the following assertions are equivalent:

• For every integer $n \geq 0$, the map of abelian groups $A_n \to B_n$ is surjective (respectively split surjective, injective, split injective).

• For every integer $n \geq 0$, the map of abelian groups $N_n(A) \to N_n(B)$ is surjective (respectively split surjective, injective, split injective).

**Proposition 2.5.6.19.** Let $A_\bullet$ be a simplicial abelian group. Then the composite map $\tilde{N}_*(A) \hookrightarrow C_*(A) \to N_*(A)$ is an isomorphism of chain complexes. In other words, the Moore complex $C_*(A)$ splits as a direct sum of the subcomplex $\tilde{N}_*(A)$ of Construction 2.5.6.16 and the subcomplex $D_*(A)$ of Proposition 2.5.5.6.

**Proof.** The surjectivity of the composite map $\tilde{N}_*(A) \hookrightarrow C_*(A) \to N_*(A)$ follows from Lemma 2.5.6.17. Moreover, it follows by induction that the subgroup $D_n(A) \subseteq A_n$ is generated by the images of the maps $\tilde{N}_m(A) \hookrightarrow A_m \xrightarrow{\alpha^*} A_n$ where $\alpha : [n] \to [m]$ is a nondecreasing surjection and $m < n$, so that the injectivity of $\rho$ also follows from Lemma 2.5.6.17. \hfill \Box

**Warning 2.5.6.20.** Let $A_\bullet$ be a simplicial abelian group, and let $A_\bullet^{\text{op}}$ be the opposite simplicial abelian group (obtained by precomposing the functor $A_\bullet : \Delta^{\text{op}} \to \text{Ab}$ with the order-reversal involution $\text{Op} : \Delta^{\text{op}} \to \Delta^{\text{op}}$ of Notation 1.3.2.1). Then there is a canonical isomorphism of Moore complexes $\psi : C_*(A^{\text{op}}) \simeq C_*(A)$, given by $\psi(x) = (-1)^m x$ for $x \in A_n$. This isomorphism carries the subcomplex $D_*(A^{\text{op}})$ generated by the degenerate simplices of $A_\bullet^{\text{op}}$ to the subcomplex $D_*(A)$ generated by the degenerate simplices of $A_\bullet$, and therefore
descends to an isomorphism of normalized Moore complexes $N_* (A^{\text{op}}) \simeq N_* (A)$, where we view $N_* (A)$ and $N_* (A^{\text{op}})$ as quotients of $C_* (A)$ and $C_* (A^{\text{op}})$ (as in Construction 2.5.5.7). Beware that the isomorphism $\psi$ does not carry the subcomplex $\tilde{N}_*(A^{\text{op}}) \subseteq C_*(A^{\text{op}})$ of Construction 2.5.6.16 to the subcomplex $\tilde{N}_*(A) \subseteq C_*(A)$. Instead, it carries it carries $\tilde{N}_*(A^{\text{op}})$ to a different subcomplex of $C_*(A)$, given in degree $n$ by those elements $x \in C_n (A) = A_n$ satisfying $d_i (x)$ for $0 \leq i < n$, and with differential given by $x \mapsto (-1)^n d_n (x)$. This subcomplex is yet another incarnation of the normalized Moore complex of $A_\bullet$, which is canonically isomorphic to $\tilde{N}_*(A)$ but not identical as a subcomplex of $C_*(A)$.

More informally: the definition of the normalized Moore complex $N_* (A)$ as a quotient of $C_* (A)$ (via Construction 2.5.5.7) is compatible with passage from a simplicial abelian group $A_\bullet$ to its opposite $A_\bullet^{\text{op}}$, but the realization as a subcomplex of $C_*(A)$ (via Construction 2.5.6.16) is not.

Remark 2.5.6.21. Let $A_\bullet$ be a simplicial abelian group. Then Warning 2.5.6.20 supplies a canonical isomorphism of normalized Moore complexes $N_* (A) \simeq N_* (A^{\text{op}})$. By virtue of Theorem 2.5.6.1, this isomorphism can be lifted uniquely to an isomorphism of simplicial abelian groups $\varphi : A_\bullet \simeq A_\bullet^{\text{op}}$. The isomorphism $\varphi$ is characterized by the requirement that for every $n$-simplex $x \in A_n$, we have $\varphi (x) \equiv (-1)^n x$ modulo degenerate simplices of $A_\bullet$.

We now use Proposition 2.5.6.19 to deduce Proposition 2.5.5.11, which was stated without proof in §2.5.5. The statement can be reformulated as follows:

Proposition 2.5.6.22. Let $A_\bullet$ be a simplicial abelian group. Then:

(a) The quotient map $C_* (A) \twoheadrightarrow N_* (A)$ induces an isomorphism on homology.

(b) The inclusion map $\tilde{N}_*(A) \hookrightarrow C_*(A)$ induces an isomorphism on homology.

(c) The subcomplex $D_*(A) \subseteq C_*(A)$ of Notation 2.5.5.5 is acyclic: that is, its homology groups are trivial.

Proof. By virtue of Proposition 2.5.6.19 assertions (a), (b), and (c) are equivalent. It will therefore suffice to prove (b). Note that the map $\tilde{N}_*(A) \hookrightarrow C_*(A)$ is the inclusion of a direct summand (Proposition 2.5.6.19) and is therefore automatically injective on homology. To show that it also induces a surjective map, it will suffice to show that every $n$-cycle $x \in C_n (A)$ is homologous to an element of the subgroup $\tilde{N}_n (A)$. Let $i$ denote the smallest nonnegative integer for which the faces $d_j (x)$ vanish for $i < j \leq n$; our proof will proceed by induction on $i$. If $i = 0$, then $x$ belongs to $\tilde{N}_n (A)$, and there is nothing to prove. Otherwise,
let $y \in C_n(A)$ denote the boundary given by $\partial(s_i(x))$. We then compute

\begin{align*}
y &= \partial(s_i(x)) \\
    &= \sum_{j=0}^{n+1} (-1)^j (d_j \circ s_i)(x) \\
    &= \sum_{j=0}^{i-1} (-1)^j (s_{i-1} \circ d_j)(x) + (-1)^i x + (-1)^{i+1} x + \left( \sum_{j=i+2}^{n} (s_i \circ d_{j-1})(x) \right) \\
    &= s_{i-1} \left( \sum_{j=0}^{i-1} (-1)^j d_j(x) \right) \\
    &= s_{i-1} \left( i \sum_{j=0}^{i-1} (-1)^j d_j(x) + \sum_{j=i+1}^{n} (-1)^j d_j(x) \right) \\
    &= s_{i-1}(\partial(x) - (-1)^i d_i(x)) \\
    &= (-1)^{i+1} (s_{i-1} \circ d_i)(x).
\end{align*}

Set $x' = x + (-1)^i y$. For $j \geq i$ we compute

\begin{align*}
d_j(x') &= d_j(x) + (-1)^i d_j(y) \\
    &= d_j(x) - (d_j \circ s_{i-1} \circ d_i)(x) \\
    &= \begin{cases} 
        d_j(x) - d_i(x) & \text{if } j = i \\
        d_j(x) - (s_{i-1} \circ d_i \circ d_j)(x) & \text{if } j > i \\
    \end{cases} \\
    &= 0.
\end{align*}

Our inductive hypothesis then guarantees that $x'$ is homologous to an element of the subgroup $\tilde{\mathcal{N}}_n(A)$. Since $x$ is homologous to $x'$, it follows that $x$ is also homologous to an element of the subgroup $\tilde{\mathcal{N}}_n(A)$.

**Warning 2.5.6.23.** Let $A_\bullet$ be a semisimplicial abelian group. Then we can still apply Construction 2.5.6.16 to define a subcomplex $\tilde{\mathcal{N}}_\bullet(A)$ of the Moore complex $C_\bullet(A)$ (note that the definition of $\tilde{\mathcal{N}}_\bullet(A)$ refers only to the face maps of $A_\bullet$). However, it is generally not true that the inclusion map $\tilde{\mathcal{N}}_\bullet(A) \hookrightarrow C_\bullet(A)$ induces an isomorphism on homology unless $A_\bullet$ can be promoted to a simplicial abelian group.

We now turn to the proof of the Dold-Kan correspondence. The main ingredient is the following consequence of Proposition 2.5.6.19:

**Proposition 2.5.6.24.** Let $M_\bullet$ be a chain complex and let $v : N_\bullet(K(M_\bullet)) \to M_\bullet$ be the counit map of Notation 2.5.6.11. Then:
• The map \( v_0 : N_0(K(M_*)) \to M_0 \) is a monomorphism, whose image is the set \( Z_0(M) \) of 0-cycles in \( M_* \).

• For \( n > 0 \), the map \( v_n : N_n(K(M_*)) \to M_n \) is an isomorphism.

Proof. The first assertion follows from Example 2.5.6.5. To prove the second, fix \( n > 0 \) and let \( f \) denote the composite map

\[
N_n(K(M_*)) \hookrightarrow C_n(K(M_*)) \to N_n(K(M_*)) \xrightarrow{v_n} M_n.
\]

By virtue of Proposition 2.5.6.19 it will suffice to show that \( f \) is an isomorphism. By definition, we can identify \( C_n(K(M_*)) = K_n(M_* \Delta^n) \) with the set of all chain maps \( \sigma : N_*(\Delta^n; Z) \to M_* \). Unwinding the definitions, we see that \( \sigma \) belongs to the subgroup \( \tilde{N}_n(K(M_*)) \subseteq C_n(K(M_*)) \) if and only if it annihilates the subcomplex \( N_*(\Lambda^n_0; Z) \), where \( \Lambda^n_0 \subseteq \Delta^n \) is the 0-horn defined in Construction 1.1.2.9. We can therefore identify \( \tilde{N}_n(K(M_*)) \) with the abelian group \( \text{Hom}_{\text{Ch}(Z)}(K_*; M_*) \), where \( K_* \) denotes the quotient of \( N_*(\Delta^n; Z) \) by the subcomplex \( N_*(\Lambda^n_0; Z) \). Note that there are exactly two nondegenerate simplices of \( \Delta^n \) which do not belong to \( \Lambda^n_0 \); let us denote them by \( \tau \) and \( \tau' \) (where \( \tau \) is of dimension \( n \) and \( \tau' \) of dimension \( n - 1 \)). Moreover, the differential on \( N_*(\Delta^n; Z) \) satisfies \( \partial(\tau) \equiv \tau' \pmod{N_*(\Lambda^n_0; Z)} \). We conclude by observing that, under the preceding identification, the homomorphism \( f : \text{Hom}_{\text{Ch}(Z)}(K_*; M_*) \to M_n \) is given by evaluation on \( \tau \), and is therefore an isomorphism.

Proof of Theorem 2.5.6.1. By virtue of Corollary 2.5.6.13 it will suffice to show that the construction \( M_* \mapsto K(M_*) \) induces an equivalence of categories \( K : \text{Ch}(Z)_{\geq 0} \to \text{Ab}_\Delta \). We first show that the functor \( K \) is fully faithful when restricted to \( \text{Ch}(Z)_{\geq 0} \). Let \( M_* \) and \( M'_* \) be chain complexes which are concentrated in degrees \( \geq 0 \); we wish to show that the canonical map

\[
\varphi : \text{Hom}_{\text{Ch}(Z)}(M_*; M'_*) \to \text{Hom}_{\text{Ab}_\Delta}(K(M_*), K(M'_*))
\]

is an isomorphism. Let \( \theta : \text{Hom}_{\text{Ab}_\Delta}(K(M_*), K(M'_*)) \simeq \text{Hom}_{\text{Ch}(Z)}(N_*(K(M_*)), M'_*) \) be the isomorphism of Proposition 2.5.6.12. Unwinding the definitions, we see that \( \theta \circ \varphi \) is given by precomposition with the counit map \( v : N_*(K(M_*)) \to M_* \) of Notation 2.5.6.11 and is therefore an isomorphism by virtue of Proposition 2.5.6.24 (together with our assumption that \( M_* \) is concentrated in degrees \( \geq 0 \)). It follows that \( \varphi \) is also an isomorphism, as desired.

We now prove that the functor \( K : \text{Ch}(Z)_{\geq 0} \to \text{Ab}_\Delta \) is essentially surjective. Let \( A_* \) be a simplicial abelian group and let \( M_* = N_*(A) \) be its normalized Moore complex. Then there is a unique map of simplicial abelian groups \( u : A_* \to K(M_*) \) for which the isomorphism

\[
\theta : \text{Hom}_{\text{Ab}_\Delta}(A_*, K(M_*)) \to \text{Hom}_{\text{Ch}(Z)}(N_*(A), M_*)
\]

of Proposition 2.5.6.12 carries \( u \) to the identity map \( \text{id} : N_*(A) \to M_* \). By construction, the induced map of normalized Moore complexes \( N_*(u) : N_*(A) \to N_*(K(M_*)) \) is right inverse
to the counit map \( \nu : N_\ast(K(M_\ast)) \to M_\ast \), which is an isomorphism by virtue of Proposition \[2.5.6.24.\] Combining this observation with Proposition \[2.5.6.19.\] we deduce that \( u \) induces an isomorphism of chain complexes \( \tilde{N}_\ast(A) \to \tilde{N}_\ast(K(M_\ast)) \), and is therefore an isomorphism by virtue of Lemma \[2.5.6.17.\] It follows that \( A_\ast \simeq K(M_\ast) \) belongs to the essential image of the functor \( K \), as desired. \( \square \)

### 2.5.7 The Shuffle Product

Let \( \text{Ab}_\Delta = \text{Fun}(\Delta^{\text{op}}, \text{Ab}) \) denote the category of simplicial abelian groups. We will regard \( \text{Ab}_\Delta \) as a monoidal category with respect to the “levelwise” tensor product of (Example \[2.1.2.16.\]): if \( A_\ast \) and \( B_\ast \) are simplicial abelian groups, then their tensor product \( A_\ast \otimes B_\ast \) is the simplicial abelian group given by the construction \( ([n] \in \Delta^{\text{op}}) \mapsto A_n \otimes B_n \). The category of chain complexes \( \text{Ch}(\mathbb{Z}) \) is also equipped with a monoidal structure (Construction \[2.5.1.17.\]): we denote the tensor product of chain complexes \( X_\ast \) and \( Y_\ast \) by \( X_\ast \boxtimes Y_\ast \) or \( (X \boxtimes Y)_\ast \); given chains \( x \in X_p \) and \( y \in Y_q \), we will write \( x \boxtimes y \) for the image of \( (x, y) \) in the abelian group \( (X \boxtimes Y)_{p+q} \). According to Theorem \[2.5.6.1.\] the normalized Moore complex functor \( A_\ast \mapsto N_\ast(A) \) determines a fully faithful embedding \( N_\ast : \text{Ab}_\Delta \hookrightarrow \text{Ch}(\mathbb{Z}) \). Beware that this functor does not commute with the formation of tensor products. Nevertheless, we have the following result:

#### Proposition 2.5.7.1.

There exists a collection of maps

\[
\triangledown : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B) \quad (a, b) \mapsto a \triangledown b,
\]

defined for every pair of simplicial abelian groups \( A_\ast \) and \( B_\ast \) and every pair of integers \( p, q \in \mathbb{Z} \), and uniquely determined by the following properties:

- Each of the maps \( \triangledown : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B) \) is bilinear and satisfies the Leibniz rule \( \partial(a \triangledown b) = (\partial a) \triangledown b + (-1)^p a \triangledown (\partial b) \) (and therefore induces a chain map \( N_\ast(A) \boxtimes N_\ast(B) \to N_\ast(A \otimes B) \); see Exercise \[2.5.1.15.\]).

- The operation \( \triangledown \) depends functorially on \( A_\ast \) and \( B_\ast \). That is, if \( f : A_\ast \to A'_\ast \) and \( g : B_\ast \to B'_\ast \) are homomorphisms of simplicial abelian groups, then the diagram

\[
\begin{array}{ccc}
N_p(A) \times N_q(B) & \xrightarrow{\triangledown} & N_{p+q}(A \otimes B) \\
\downarrow_{N_\ast(f) \times N_\ast(g)} & & \downarrow_{N_\ast(f \otimes g)} \\
N_\ast(A') \times N_\ast(B') & \xrightarrow{\triangledown} & N_\ast(A' \otimes B')
\end{array}
\]

commutes.

- For \( a \in A_0 \) and \( b \in B_0 \), we have \( a \triangledown b = a \otimes b \) (where we identify \( a \), \( b \), and \( a \otimes b \) with the corresponding elements of \( N_0(A) \), \( N_0(B) \), and \( N_0(A \otimes B) \), respectively).
For simplicial abelian groups $A_\bullet$ and $B_\bullet$ and integer $p, q \in \mathbb{Z}$, we will refer to the map

$$\nabla : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B)$$

of Proposition 2.5.7.1 as the *shuffle product*. We begin by giving an explicit construction of this map, following Eilenberg and MacLane (see [12]).

**Notation 2.5.7.2** ($(p, q)$-Shuffles). Let $p$ and $q$ be nonnegative integers. A $(p, q)$-shuffle is a strictly increasing map of partially ordered sets $\sigma : [p + q] \to [p] \times [q]$, which we will often identify with a nondegenerate $(p + q)$-simplex of the Cartesian product $\Delta^p \times \Delta^q$.

If $\sigma$ is a $(p, q)$-shuffle, we let $\sigma_- : [p + q] \to [p]$ and $\sigma_+ : [p + q] \to [q]$ denote the nondecreasing maps given by the components of $\sigma$ (so that $\sigma(i) = (\sigma_-(i), \sigma_+(i))$ for $0 \leq i \leq p + q$). Let $I_-$ denote the set of integers $1 \leq i \leq p + q$ satisfying $\sigma_-(i - 1) < \sigma_-(i)$ (or equivalently $\sigma_+(i - 1) = \sigma_+(i)$), and let $I_+$ the set of integers $1 \leq i \leq p + q$ satisfying $\sigma_+(i - 1) < \sigma_+(i)$ (or equivalently $\sigma_-(i - 1) = \sigma_-(i)$). We let $(-1)^\sigma$ denote the product

$$\prod_{(i, j) \in I_- \times I_+} \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i > j. \end{cases}$$

We will refer to $(-1)^\sigma$ as the *sign* of the $(p, q)$-shuffle $\sigma$.

**Construction 2.5.7.3** (The Unnormalized Shuffle Product). Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups, and suppose we are given elements $a \in A_p$ and $b \in B_q$. We let $a \nabla b$ denote the sum

$$\sum_\sigma (-1)^\sigma \sigma_-^*(a) \otimes \sigma_+^*(b) \in (A \otimes B)_{p+q}$$

Here the sum is taken over all $(p, q)$-shuffles $\sigma = (\sigma_-, \sigma_+)$ (Notation 2.5.7.2), and we write $\sigma_-^* : A_p \to A_{p+q}$ and $\sigma_+^* : B_q \to B_{p+q}$ for the structure morphisms of the simplicial abelian groups $A_\bullet$ and $B_\bullet$, respectively. We will refer to $a \nabla b$ as the *unnormalized shuffle product* of $a$ and $b$.

We now summarize some essential properties of Construction 2.5.7.3.

**Remark 2.5.7.4** (Unitality of the Shuffle Product). Let $\mathbf{Z}[\Delta^0]$ be the constant simplicial abelian group taking the value $\mathbf{Z}$, and let us identify the integer 1 with the corresponding 0-simplex of $\mathbf{Z}[\Delta^0]$. Then, for any simplicial abelian group $A_\bullet$, the canonical isomorphisms $A_\bullet \simeq (A \otimes \mathbf{Z}[\Delta^0])_\bullet$ and $A_\bullet \simeq (\mathbf{Z}[\Delta^0] \otimes A_\bullet)$ are given by $a \mapsto a \nabla 1$ and $a \mapsto 1 \nabla a$, respectively.

**Remark 2.5.7.5** (Commutativity of the Shuffle Product). Let $\sigma : [p + q] \to [p] \times [q]$ be a $(p, q)$-shuffle, and let $\sigma' : [p + q] \to [q] \times [p]$ denote the composition of $\sigma$ with the isomorphism $[p] \times [q] \simeq [q] \times [p]$ given by permuting the factors. Then $\sigma'$ is a $(q, p)$-shuffle, whose sign is given by $(-1)^{\sigma'} = (-1)^{pq} \cdot (-1)^\sigma$. Consequently, if $A_\bullet$ and $B_\bullet$ are simplicial abelian groups containing simplices $a \in A_p$ and $b \in B_q$, then the canonical isomorphism $(A \otimes B)_{p+q} \simeq (B \otimes A)_{p+q}$ carries $a \nabla b$ to $(-1)^{pq}(b \nabla a)$. 
Remark 2.5.7.6 (Associativity of the Shuffle Product). Let $A_\bullet$, $B_\bullet$, and $C_\bullet$ be simplicial abelian groups containing simplices $a \in A_p$, $b \in B_q$, and $c \in C_r$. Then the canonical isomorphism $(A \otimes (B \otimes C))_{p+q+r} \simeq ((A \otimes B) \otimes C)_{p+q+r}$ carries $a \bar{\nabla}(b \bar{\nabla}c)$ to $(a \bar{\nabla}b) \bar{\nabla}c$. Both of these iterated shuffle products can be described concretely as the sum

$$
\sum_{\sigma} (-1)^{\bar{\sigma}} \sigma^-(a) \otimes \sigma^0(b) \otimes \sigma^+(c),
$$

where the sum is taken over all strictly increasing maps $\sigma = (\sigma_-, \sigma_0, \sigma_+) : [p+q+r] \to [p] \times [q] \times [r]$, and $(-1)^{\sigma}$ denotes the product

$$\prod_{1 \leq i < j \leq p+q+r} \begin{cases} 
-1 & \text{if } \sigma_-(j-1) < \sigma_-(j) \text{ and } \sigma_-(i-1) = \sigma_-(i) \\
-1 & \text{if } \sigma_+(j-1) = \sigma_+(j) \text{ and } \sigma_+(i-1) < \sigma_+(i) \\
1 & \text{otherwise.}
\end{cases}
$$

Proposition 2.5.7.7. Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized shuffle product $\nabla : A_p \times B_q \to (A \otimes B)_{p+q}$ satisfies the Leibniz rule

$$\partial(a \bar{\nabla}b) = (\partial a) \bar{\nabla}b + (-1)^p a \bar{\nabla}(\partial b).$$

Proof. Without loss of generality, we may assume that $(p, q) \neq (0, 0)$ and that the simplicial abelian groups $A_\bullet \simeq \mathbb{Z}[\Delta^p]$ and $B_\bullet \simeq \mathbb{Z}[\Delta^q]$ are freely generated by $a$ and $b$, respectively. In this case, we can identify $(A \otimes B)_{p+q}$ with the free abelian group generated by the set of $(p+q-1)$-simplices of $\Delta^p \times \Delta^q$, which we view as nondecreasing functions $\tau : [p+q-1] \to [p] \times [q]$. For every such simplex $\tau$, let $c_-, c_-$, and $c_+$ denote the coefficients of $\tau$ appearing in $\partial(a \bar{\nabla}b)$, $(\partial a) \bar{\nabla}b$, and $a \bar{\nabla}(\partial b)$, respectively. We wish to prove that $c = c_- + (-1)^p c_+$. We may assume without loss of generality that the map $\tau$ is injective (otherwise, we have $c = c_- = c_+ = 0$). Let us identify $\tau$ with a pair $(\tau_-, \tau_+)$, where $\tau_- : [p+q-1] \to [p]$ and $\tau_+ : [p+q-1] \to [q]$ are nondecreasing functions. We now distinguish three cases:

1. Suppose that the map $\tau_- : [p+q-1] \to [p]$ is not surjective (that is, $\tau$ belongs to the simplicial subset $(\partial \Delta^p) \times \Delta^q \subseteq \Delta^p \times \Delta^q$). Then $p > 0$ and there exists a unique integer $0 \leq i < p$ which does not belong to the image of $\tau_-$. We proceed under the assumption that $i < p$ (the case $i > 0$ follows by a similar argument, with minor changes in notation). We then make the following observations:

- There is a unique $(p, q)$-shuffle $\sigma$ and integer $0 \leq j \leq p+q$ satisfying $\tau = d_j(\sigma)$. Here $j$ is the smallest integer satisfying $\tau_-(j) = i + 1$, and $\sigma$ is given by the formula

$$
\sigma(k) = \begin{cases} 
(\tau_-(k), \tau_+(k)) & \text{if } k < j \\
(i, \tau_+(j)) & \text{if } k = j \\
(\tau_-(k-1), \tau_+(k-1)) & \text{if } k > j.
\end{cases}
$$

2. Suppose that the map $\tau_+ : [p+q-1] \to [q]$ is not surjective (that is, $\tau$ belongs to the simplicial subset $\Delta^p \times (\partial \Delta^q) \subseteq \Delta^p \times \Delta^q$). Then $q > 0$ and there exists a unique integer $0 \leq i < q$ which does not belong to the image of $\tau_+$. We proceed under the assumption that $i < q$ (the case $i > 0$ follows by a similar argument, with minor changes in notation). We then make the following observations:

- There is a unique $(p, q)$-shuffle $\sigma$ and integer $0 \leq j \leq p+q$ satisfying $\tau = d_j(\sigma)$. Here $j$ is the smallest integer satisfying $\tau_+(j) = i + 1$, and $\sigma$ is given by the formula

$$
\sigma(k) = \begin{cases} 
(\tau_-(k), \tau_+(k)) & \text{if } k < j \\
(\tau_-(j), \tau_+(k)) & \text{if } k = j \\
(\tau_-(k-1), \tau_+(k-1)) & \text{if } k > j.
\end{cases}
$$

3. Suppose that the map $\tau_+ : [p+q-1] \to [q]$ is surjective (that is, $\tau$ belongs to the simplicial subset $\Delta^p \times (\partial \Delta^q) \subseteq \Delta^p \times \Delta^q$). Then $q > 0$ and there exists a unique integer $0 \leq i < q$ which does not belong to the image of $\tau_+$ and $\tau_-$. We proceed under the assumption that $i < q$ (the case $i > 0$ follows by a similar argument, with minor changes in notation). We then make the following observations:

- There is a unique $(p, q)$-shuffle $\sigma$ and integer $0 \leq j \leq p+q$ satisfying $\tau = d_j(\sigma)$. Here $j$ is the smallest integer satisfying $\tau_+(j) = i + 1$, and $\sigma$ is given by the formula

$$
\sigma(k) = \begin{cases} 
(\tau_-(k), \tau_+(k)) & \text{if } k < j \\
(\tau_-(j), \tau_+(k)) & \text{if } k = j \\
(\tau_-(k-1), \tau_+(k-1)) & \text{if } k > j.
\end{cases}
$$

4. Suppose that the map $\tau_- : [p+q-1] \to [p]$ is surjective (that is, $\tau$ belongs to the simplicial subset $(\partial \Delta^p) \times \Delta^q \subseteq \Delta^p \times \Delta^q$). Then $p > 0$ and there exists a unique integer $0 \leq i < p$ which does not belong to the image of $\tau_-$. We proceed under the assumption that $i < p$ (the case $i > 0$ follows by a similar argument, with minor changes in notation). We then make the following observations:

- There is a unique $(p, q)$-shuffle $\sigma$ and integer $0 \leq j \leq p+q$ satisfying $\tau = d_j(\sigma)$. Here $j$ is the smallest integer satisfying $\tau_-(j) = i + 1$, and $\sigma$ is given by the formula

$$
\sigma(k) = \begin{cases} 
(\tau_-(k), \tau_+(k)) & \text{if } k < j \\
(\tau_-(j), \tau_+(k)) & \text{if } k = j \\
(\tau_-(k-1), \tau_+(k-1)) & \text{if } k > j.
\end{cases}
$$
It follows that \( c = (-1)^j \cdot (-1)^\sigma \).

- There is a unique \((p - 1, q)\)-shuffle \( \sigma' \) and integer \( 0 \leq a \leq p \) such that \( \tau \) is given by the composition

\[
[p + q - 1] \xrightarrow{\sigma'} [p - 1] \times [q] \xrightarrow{\delta^a \times \text{id}} [p] \times [q];
\]

here \( \delta^a : [p - 1] \hookrightarrow [p] \) denotes the unique monomorphism whose image does not contain \( a \) (Notation 1.1.1.8). These conditions guarantee that \( a = i \) and that \( \sigma' \) is given by the formula

\[
\sigma'(k) = \begin{cases} (\tau_-(k), \tau_+(k)) & \text{if } k < j \\ (\tau_-(k) - 1, \tau_+(k)) & \text{if } k \geq j. \end{cases}
\]

Consequently, we have \( c' = (-1)^j \cdot (-1)^\sigma' \).

- There does not exist a \((p, q - 1)\)-shuffle \( \sigma'' \) and an integer \( 0 \leq b \leq q \) for which \( \tau \) is equal to the composition

\[
[p + q - 1] \xrightarrow{\sigma''} [p] \times [q - 1] \xrightarrow{\text{id} \times \delta^b} [p] \times [q].
\]

Consequently, the coefficient \( c'' \) vanishes.

We are therefore reduced to verifying the identity \((-1)^j \cdot (-1)^\sigma = (-1)^j \cdot (-1)^\sigma'\), which is an immediate consequence of the definitions.

(2) Suppose that the map \( \tau_+ : [p + q - 1] \to [q] \) is not surjective (that is, \( \tau \) belongs to the simplicial subset \( \Delta^q \times (\partial \Delta^q) \subseteq \Delta^p \times \Delta^q \)). The argument in this case proceeds as in (1), with minor adjustments in notation.

(3) The functions \( \tau_+ \) and \( \tau_- \) are both surjective. In this case, we have \( c_- = c_+ = 0 \). Note that there is a unique integer \( 1 \leq j \leq p + q - 1 \) satisfying \( \tau_-(j - 1) < \tau_-(j) \) and \( \tau_+(j - 1) < \tau_+(j) \). From this, it is easy to see that if \( \sigma \) is a \((p, q)\)-shuffle satisfying \( d_k(\sigma) = \tau \) for some \( 0 \leq k \leq p + q \), then we must have \( k = j \). Moreover, there are exactly two \((p, q)\)-shuffles \( \sigma \) satisfying \( d_j(\sigma) = \tau \), given by the formulae

\[
\sigma(i) = \begin{cases} \tau(i) & \text{if } i < j \\ (\tau_-(j - 1), \tau_+(j)) & \text{if } i = j \\ \tau(i - 1) & \text{if } i > j \end{cases} \quad \text{and} \quad \sigma(i) = \begin{cases} \tau(i) & \text{if } i < j \\ (\tau_-(j), \tau_+(j - 1)) & \text{if } i = j \\ \tau(i - 1) & \text{if } i > j. \end{cases}
\]

Since these \((p, q)\)-shuffles have opposite sign, we conclude that \( c = 0 = c_- + (-1)^p c_+ \), as desired.
We now adapt the shuffle product to the setting of normalized Moore complexes. For every simplicial abelian group $A_\bullet$, let $D_\bullet(A) \subseteq C_\bullet(A)$ be the subcomplex generated by the degenerate simplices of $A_\bullet$ (see Proposition 2.5.5.6).

**Proposition 2.5.7.8.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized shuffle product
\[ \bar{\nabla} : C_p(A) \times C_q(B) \to C_{p+q}(A \otimes B) \]
carries the subsets $D_p(A) \times C_q(B)$ and $C_p(A) \otimes D_q(B)$ into the subgroup $D_{p+q}(A \otimes B) \subseteq C_{p+q}(A \otimes B)$.

**Proof.** Let $a \in A_p$ and $b \in B_q$ be simplices of $A_\bullet$ and $B_\bullet$, respectively. We wish to show that if either $a$ belongs to $D_p(A)$ or $b$ belongs to $D_q(B)$, then the unnormalized shuffle product $a \bar{\nabla} b$ belongs to $D_{p+q}(A \otimes B)$. Without loss of generality, we may assume that $a$ belongs to $D_p(A)$. Decomposing $a$ into summands, we can further assume that $a = s_i(a')$ for some $0 \leq i \leq p - 1$ and some $a' \in A_{p-1}$. Let $\sigma = (\sigma_-, \sigma_+)$ be a $(p, q)$-shuffle. Then there exists a unique integer $0 \leq j < p + q$ satisfying $\sigma_-(j) = i$ and $\sigma_-(j + 1) = i + 1$. It then follows that both $\sigma_+(a)$ and $\sigma_+(b)$ are fixed points of the composite maps
\[ A_{p+q} \xrightarrow{d_j} A_{p+1} \xrightarrow{s_i} A_{p+q} \quad B_{p+q} \xrightarrow{d_j} B_{p+1} \xrightarrow{s_i} B_{p+q}, \]
so that $\sigma_+(a) \otimes \sigma_+(b)$ is a degenerate simplex of $(A \otimes B)_\bullet$. Allowing $\sigma$ to vary, we deduce that the shuffle product
\[ \sum_\sigma (-1)^\sigma \sigma_+(a) \otimes \sigma_+(b) \]
belongs to $D_{p+q}(A \otimes B)$. \qed

**Construction 2.5.7.9 (The Shuffle Product).** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. It follows from Proposition 2.5.7.8 that for every pair of integers $p, q \in \mathbb{Z}$, there is a unique bilinear map $\nabla : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B)$ for which the diagram
\[
\begin{array}{ccc}
C_p(A) \otimes C_q(B) & \xrightarrow{\bar{\nabla}} & C_{p+q}(A \otimes B) \\
\downarrow & & \downarrow \\
N_p(A) \otimes N_q(B) & \xrightarrow{\nabla} & N_{p+q}(A \otimes B)
\end{array}
\]
commutes. We will refer to $\nabla : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B)$ as the **shuffle product map**. Given elements $a \in N_p(A)$ and $b \in N_q(B)$, we will write $a \nabla b$ for the image of the pair $(a, b)$ under the shuffle product map, which we refer to as the **shuffle product of $a$ and $b$**.
We now summarize some properties of the properties of Construction 2.5.7.9, which follow immediately from the corresponding results for the unnormalized shuffle product (Remarks 2.5.7.4, 2.5.7.5, 2.5.7.6, and Proposition 2.5.7.7).

**Proposition 2.5.7.10.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then:

1. The canonical isomorphisms $N_\ast(A) \simeq N_\ast(A \otimes \mathbb{Z}[\Delta^0])$ and $N_\ast(A) \simeq N_\ast(\mathbb{Z}[\Delta^0] \otimes A)$ are given by $a \mapsto a \triangledown 1$ and $a \mapsto 1 \triangledown a$, respectively; here we identify the integer 1 with its image in $N_\ast(\Delta^0; \mathbb{Z}) \simeq \mathbb{Z}$.

2. For $a \in N_p(A)$ and $b \in N_q(B)$, we have $a \triangledown b = (-1)^{pq}(b \triangledown a)$; here we abuse notation by identifying $a \triangledown b$ with its image under the canonical isomorphism $N_{p+q}(A \otimes B) \simeq N_{p+q}(B \otimes A)$.

3. Let $C_\bullet$ be another simplicial abelian group, and suppose we are given elements $a \in N_p(A)$, $b \in N_q(B)$, and $c \in N_r(C)$. Then $a \triangledown (b \triangledown c) = (a \triangledown b) \triangledown c$; here we abuse notation by identifying $a \triangledown (b \triangledown c)$ with its image under the canonical isomorphism $N_{p+q+r}(A \otimes (B \otimes C)) \simeq N_{p+q+r}((A \otimes B) \otimes C)$.

4. The shuffle product $\triangledown : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B)$ satisfies the Leibniz rule

$$\partial(a \triangledown b) = (\partial a) \triangledown b + (-1)^p a \triangledown (\partial b).$$

**Notation 2.5.7.11 (The Eilenberg-Zilber Homomorphism).** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. It follows from assertion (4) of Proposition 2.5.7.10 that there is a unique chain map

$$\text{EZ} : N_\ast(A) \boxtimes N_\ast(B) \to N_\ast(A \otimes B)$$

satisfying $\text{EZ}(a \boxtimes b) = a \triangledown b$ (see Exercise 2.5.1.15). We will refer to $\text{EZ}$ as the *Eilenberg-Zilber homomorphism* (see Remark 2.5.7.16). It follows from assertions (1) and (3) of Proposition 2.5.7.10 that the collection of chain maps

$$\{\text{EZ} : N_\ast(A) \boxtimes N_\ast(B) \to N_\ast(A \otimes B)\}_{A_\bullet, B_\bullet \in \text{Ab}_\Delta}$$

determine a lax monoidal structure (Definition 2.1.5.8) on the normalized Moore complex functor $N_\ast : \text{Ab}_\Delta \to \text{Ch}(\mathbb{Z})$, with unit given by the canonical isomorphism of chain complexes $\mathbb{Z}[0] \simeq N_\ast(\mathbb{Z}[\Delta^0])$ (in fact, it is even a lax *symmetric* monoidal structure in the sense of Definition [?]: this follows from assertion (2) of Proposition 2.5.7.10).

**Example 2.5.7.12.** Let $S_\bullet$ and $T_\bullet$ be simplicial sets, and let $\mathbb{Z}[S_\bullet]$ and $\mathbb{Z}[T_\bullet]$ denote the free simplicial abelian groups generated by $S_\bullet$ and $T_\bullet$, respectively. Then the tensor product $\mathbb{Z}[S_\bullet] \otimes \mathbb{Z}[T_\bullet]$ can be identified with the free simplicial abelian group $\mathbb{Z}[S_\bullet \times T_\bullet]$ generated...
by the Cartesian product $S_* \times T_*$. Invoking Construction [2.5.7.9] we obtain shuffle product maps

$$\forall : N_p(S; \mathbb{Z}) \times N_q(T; \mathbb{Z}) \to N_{p+q}(S \times T; \mathbb{Z})$$

which induce a map of chain complexes $EZ : N_*(S; \mathbb{Z}) \boxtimes N_*(T; \mathbb{Z}) \to N_*(S \times T; \mathbb{Z})$. Allowing $S_*$ and $T_*$ to vary, these chain maps furnish a lax (symmetric) monoidal structure on the functor

$$N_*(-; \mathbb{Z}) : \text{Set} \to \text{Ch}(\mathbb{Z}) \quad S_* \mapsto N_*(S; \mathbb{Z}).$$

**Remark 2.5.7.13.** The Eilenberg-Zilber homomorphism of Example 2.5.7.12 admits a topological interpretation. Recall that, for every simplicial set $S_*$, the topological space $|S_*|$ is a CW complex (Remark 1.1.8.14). More precisely, $|S_*|$ admits a CW decomposition with one cell $e_\sigma$ for each nondegenerate simplex $\sigma : \Delta^n \to S_*$, where $e_\sigma$ is defined as the image of the composite map

$$|\Delta^n| \hookrightarrow |\Delta^n| \xrightarrow{[\sigma]} |S_*|,$$

here $|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}_{>0} : t_0 + \cdots + t_n = 1\}$ denotes the interior of the topological $n$-simplex. The chain complex $N_*(S; \mathbb{Z})$ of Construction 2.5.5.9 can then be identified with the cellular chain complex associated to this cell decomposition of $|S_*|$.

When $S_* = S'_* \times S''_*$ factors a product of two other simplicial sets $S'_*$ and $S''_*$, the topological space $|S_*|$ admits a different CW structure, whose cells are given by $\varphi^{-1}(e_{\sigma'} \times e_{\sigma''})$; here $\varphi$ denotes the canonical map $|S_*| \to |S'_*| \times |S''_*|$, and $\sigma'$ and $\sigma''$ range over the collection of nondegenerate simplices of $S'_*$ and $S''_*$, respectively. The cellular chain complex associated to this cell decomposition can be identified with the tensor product complex $N_*(S'_*; \mathbb{Z}) \boxtimes N_*(S''_*; \mathbb{Z})$.

It is not difficult to see that if $\sigma' \in S'_p$ and $\sigma'' \in S''_q$ are nondegenerate simplices of $S'_*$ and $S''_*$, respectively, then the subset $\varphi^{-1}(e_{\sigma'} \times e_{\sigma''}) \subseteq |S_*|$ can be written as a finite union of cells of the form $e_\sigma$ (where $\sigma$ is a nondegenerate simplex of $S_*$). Writing $[\sigma']$ and $[\sigma'']$ for the corresponding generators of $N_p(S'_*; \mathbb{Z})$ and $N_q(S''_*; \mathbb{Z})$, the shuffle product is given by

$$[\sigma'] \triangledown [\sigma''] = \sum_{\sigma} \pm [\sigma] \in N_{p+q}(S),$$

where the sum is taken over all nondegenerate $(p+q)$-simplices $\sigma$ of $S_*$ satisfying $e_\sigma \subseteq \varphi^{-1}(e_{\sigma'} \times e_{\sigma''})$; note that every such simplex $\sigma$ can be written uniquely as a composition

$$\Delta^{p+q} \xrightarrow{\tau} \Delta^p \times \Delta^q \xrightarrow{\sigma' \times \sigma''} S'_* \times S''_* = S_*$$

where $\tau$ is a $(p, q)$-shuffle in the sense of Notation 2.5.7.2. Moreover, the sign $(-1)^\tau$ also admits a topological interpretation: it is the degree of the open embedding $\varphi|_{e_\sigma} : e_\sigma \hookrightarrow e_{\sigma'} \times e_{\sigma''}$ (with respect to certain standard orientations of the cells $e_\sigma$, $e_{\sigma'}$, and $e_{\sigma''}$).
Theorem 2.5.7.14. Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the Eilenberg-Zilber homomorphism

$$EZ : N_*(A) \boxtimes N_*(B) \to N_*(A \otimes B)$$

is a quasi-isomorphism: that is, it induces an isomorphism on homology.

Corollary 2.5.7.15. Let $S_\bullet$ and $T_\bullet$ be simplicial sets. Then the Eilenberg-Zilber homomorphism

$$EZ : N_*(S; \mathbb{Z}) \boxtimes N_*(T; \mathbb{Z}) \to N_*(S \times T; \mathbb{Z})$$

is a quasi-isomorphism.

Remark 2.5.7.16. Corollary 2.5.7.15 is essentially due to Eilenberg and Zilber. More precisely, in [14], Eilenberg and Zilber proved that there exists a collection of quasi-isomorphisms $G_{S,T} : N_*(S; \mathbb{Z}) \boxtimes N_*(T; \mathbb{Z}) \to N_*(S \times T; \mathbb{Z})$ depending functorially on the simplicial sets $S_\bullet$ and $T_\bullet$. The proof given in [14] uses the method of acyclic models and does not provide a concrete description of the maps $G_{S,T}$. However, it is not difficult to see that such a collection of chain maps $\{G_{S,T}\}$ must coincide up to sign with the Eilenberg-Zilber homomorphisms of Example 2.5.7.12 (see Exercise 2.5.7.18 below).

Variant 2.5.7.17. Let $S_\bullet$ and $T_\bullet$ be simplicial sets containing simplicial subsets $S'_\bullet$ and $T'_\bullet$, respectively. Applying Theorem 2.5.7.14 to the simplicial abelian groups $\mathbb{Z}[S_\bullet]/\mathbb{Z}[S'_\bullet]$ and $\mathbb{Z}[T_\bullet]/\mathbb{Z}[T'_\bullet]$, we obtain a quasi-isomorphism

$$EZ : N_*(S, S'; \mathbb{Z}) \boxtimes N_*(T, T'; \mathbb{Z}) \to N_*(S \times T, (S' \times T) \cup (S \times T'); \mathbb{Z})$$

Proof of Theorem 2.5.7.14. Let us first regard the simplicial abelian group $A_\bullet$ as fixed. Let $M_* \in \text{Ch}(\mathbb{Z})_{\geq 0}$ be a chain complex of abelian groups which is concentrated in degrees $\geq 0$, and let $K(M_*)$ be the associated Eilenberg-MacLane space (Construction 2.5.6.7). We will say that $M_*$ is good if the Eilenberg-Zilber map

$$N_*(A) \boxtimes M_* \simeq N_*(A) \boxtimes N_*(K(M_*)) \xrightarrow{EZ} N_*(A \otimes K(M_*))$$

is a quasi-isomorphism. By virtue of Theorem 2.5.6.1 it will suffice to show that every object $M_* \in \text{Ch}(\mathbb{Z})_{\geq 0}$ is good. Writing $M_*$ as a filtered direct limit of bounded subcomplexes, we may assume that $M_*$ is concentrated in degrees $\leq n$ for some integer $n \geq 0$. We proceed by induction on $n$. Let $T$ denote the abelian group $M_n$, so that we have a short exact sequence of chain complexes

$$0 \to M'_* \to M_* \to T[n] \to 0,$$

where $M'_*$ is concentrated in degrees $\leq n - 1$. Note that this sequence is degreewise split, so that the associated exact sequence of simplicial abelian groups

$$0 \to K(M'_*) \to K(M_*) \to K(T[n]) \to 0$$
is also degreewise split (see Remark 2.5.6.18). We therefore have a commutative diagram of short exact sequences

\[
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
\rightarrow
\begin{array}{c}
N_*(A) \otimes M' \\
\downarrow \\
N_*(A \otimes K(M'))
\end{array}
\rightarrow
\begin{array}{c}
N_*(A) \otimes M_* \\
\downarrow \\
N_*(A \otimes K(M_*))
\end{array}
\rightarrow
\begin{array}{c}
N_*(A) \otimes T[n] \\
\downarrow \\
N_*(A \otimes K(T[n]))
\end{array}
\rightarrow
0
\]

where the left vertical map is a quasi-isomorphism by virtue of our inductive hypothesis. Invoking Remark 2.5.1.7, we see that \( M_* \) is good if and only if the chain complex \( T[n] \) is good. In particular, the condition that \( M_* \) is good depends only on the abelian group \( T = M_* \).

We may therefore assume without loss of generality that \( M_* \) factors as a tensor product

\[
\begin{array}{c}
N_*(A \otimes K(M_*)) \\
\downarrow \\
N_*(A \otimes K(T[n]))
\end{array}
\rightarrow
\begin{array}{c}
N_*(A \otimes K(M_*)) \\
\downarrow \\
N_*(A \otimes K(T[n]))
\end{array}
\rightarrow
0,
\]

and

\[
\begin{array}{c}
N_*(A \otimes K(M'_*)） \\
\downarrow \\
N_*(A \otimes K(T[0]))
\end{array}
\rightarrow
\begin{array}{c}
N_*(A \otimes K(M'_*)) \\
\downarrow \\
N_*(A \otimes K(T'[0]))
\end{array}
\rightarrow
0.
\]

Applying the same argument with the roles of \( A_* \) and \( B_* \) reversed, we can also assume that \( A_* \) factors as the tensor product of \( Z[\Delta^m] \) with another abelian group \( T' \). In this case, we are reduced to proving that the Eilenberg-Zilber map

\[
EZ : N_*(\Delta^m; Z) \otimes N_*(\Delta^n; Z) \rightarrow N_*(\Delta^m \otimes \Delta^n; Z)
\]

becomes a quasi-isomorphism after tensoring both sides with the abelian group \( T' \otimes T \). In fact, we claim that this map is chain homotopy equivalence. To prove this, let \( u \) and \( v \) denote the initial vertices of \( \Delta^m \) and \( \Delta^n \), respectively, and write \([u]\) and \([v]\) for the corresponding generators of \( N_0(\Delta^m; Z) \) and \( N_0(\Delta^n; Z) \). Then the shuffle product \([u] \triangledown [v]\) is given by \([w]\), where \( w = (u, v) \) is the vertex of \( \Delta^m \times \Delta^n \) corresponding to the least element of the partially ordered set \([m] \times [n]\). We have a commutative diagram of chain complexes

\[
\begin{array}{c}
Z[0] \otimes Z[0] \\
\downarrow \sim \\
N_*(\Delta^m; Z) \otimes N_*(\Delta^n; Z)
\end{array}
\rightarrow
\begin{array}{c}
Z[0] \\
\downarrow \sim \\
N_*(\Delta^m \otimes \Delta^n; Z)
\end{array}
\]

where the vertical maps are chain homotopy equivalences (Example 2.5.5.14) and the upper horizontal map is an isomorphism, so the lower horizontal map is a chain homotopy equivalence as well.

Proof of Proposition 2.5.7.1: It follows immediately from the definitions that the shuffle product maps

\[
\triangledown : N_p(A) \times N_q(B) \rightarrow N_{p+q}(A \otimes B)
\]

depend functorially on \( A_* \) and \( B_* \) and satisfy \( a \triangledown b = a \otimes b \) when \( p = q = 0 \), and the Leibniz rule follows from Proposition 2.5.7.10. To complete the proof of Proposition 2.5.7.1, we will
show that the shuffle product is the unique operation with these properties. To this end, suppose we are given another collection of bilinear maps

\[ \nabla' : N_p(A) \times N_q(B) \to N_{p+q}(A \otimes B) \]

which depend functorially on \( A \) and \( B \) and satisfy the Leibniz rule. In the special case \( A = B = \mathbb{Z}[\Delta^0] \), we can identify \( N_0(A), N_0(B), \) and \( N_0(A \otimes B) \) with the group \( \mathbb{Z} \) of integers, so that \( 1 \nabla' 1 = n \) for some integer \( n \). We will complete the proof by showing that for every pair of simplicial abelian groups \( A_\bullet \) and \( B_\bullet \) and every pair of elements \( a \in N_p(A), b \in N_q(B) \), we have \( a \nabla' b = n(a \nabla b) \) (in particular, if \( a \nabla' b = a \otimes b \) whenever \( p = q = 0 \), we must have \( n = 1 \) and therefore \( \nabla' = \nabla \)).

Without loss of generality, we may assume that \( p, q \geq 0 \). We will proceed by induction on \( p + q \). Choose a lift of \( a \) to an element of \( C_p(A) \), which we identify with a map of simplicial abelian groups \( \mathbb{Z}[\Delta^p] \to A_\bullet \). Invoking our assumption that \( \nabla' \) is functorial, we can assume without loss of generality that \( A_\bullet = \mathbb{Z}[\Delta^p] \) and that \( a \) is the generator of \( N_p(\Delta^p; \mathbb{Z}) \) corresponding to the unique nondegenerate \( p \)-simplex of \( \Delta^p \). Similarly, we may assume that \( B_\bullet = \mathbb{Z}[\Delta^q] \) and that \( b \in N_q(\Delta^q; \mathbb{Z}) \) is the generator given by the unique nondegenerate \( q \)-simplex of \( \Delta^q \).

Let \( \alpha \) and \( \beta \) denote the images of \( a \) and \( b \) in the relative chain complexes \( N_\bullet(\Delta^p, \partial \Delta^p; \mathbb{Z}) \simeq \mathbb{Z}[p] \) and \( N_\bullet(\Delta^q, \partial \Delta^q; \mathbb{Z}) \simeq \mathbb{Z}[q] \). Let \( \partial(\Delta^p \times \Delta^q) \subseteq \Delta^p \times \Delta^q \) denote the union of the simplicial subsets \( (\partial \Delta^p) \times \Delta^q \) and \( \Delta^p \times (\partial \Delta^q) \), so that we have an isomorphism of simplicial abelian groups

\[ (\mathbb{Z}[\Delta^p]/\mathbb{Z}[\partial \Delta^p]) \otimes (\mathbb{Z}[\Delta^q]/\mathbb{Z}[\partial \Delta^q]) \simeq \mathbb{Z}[\Delta^p \times \Delta^q]/\mathbb{Z}[\partial(\Delta^p \times \Delta^q)]. \]

By virtue of Theorem 2.5.7.14, the Eilenberg-Zilber homomorphism

\[ \text{EZ} : N_\bullet(\Delta^p, \partial \Delta^p; \mathbb{Z}) \boxtimes N_\bullet(\Delta^q, \partial \Delta^q; \mathbb{Z}) \to N_\bullet(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbb{Z}) \]

is a quasi-isomorphism. In particular, the \((p + q)\)-cycles of the chain complex \( N_\bullet(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbb{Z}) \) form a cyclic group generated by the shuffle product \( \pi \nabla \beta \). Since the operation \( \nabla' \) satisfies the Leibniz rule, the chain \( \pi \nabla \beta \in N_{p+q}(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbb{Z}) \) is a cycle, and therefore satisfies \( \pi \nabla \beta = m(\pi \nabla \beta) \) for some integer \( m \). Using the commutativity of the diagram

\[ \begin{array}{ccc}
N_p(\Delta^p; \mathbb{Z}) \times N_q(\Delta^q; \mathbb{Z}) & \xrightarrow{\nabla'} & N_{p+q}(\Delta^p \times \Delta^q; \mathbb{Z}) \\
\sim & & \sim \\
N_p(\Delta^p, \partial \Delta^p; \mathbb{Z}) \times N_q(\Delta^q, \partial \Delta^q; \mathbb{Z}) & \xrightarrow{\nabla} & N_{p+q}(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbb{Z})
\end{array} \]

and the observation that the vertical maps are isomorphisms, we conclude that \( a \nabla' b = m(a \nabla b) \). We will complete the proof by showing that \( m = n \). In the case \( p = q = 0 \), this
follows from the definition of the integer \( n \). If \( p + q > 0 \), we invoke our inductive hypothesis to compute

\[
\begin{align*}
  m\partial(a \triangledown b) &= \partial(a \triangledown' b) \\
  &= (\partial a) \triangledown' b + (-1)^p a \triangledown' (\partial b) \\
  &= n((\partial a) \triangledown b + (-1)^p a \triangledown (\partial b)) \\
  &= n\partial(a \triangledown b).
\end{align*}
\]

Since \( \partial(a \triangledown b) \) is a nonzero element of the free abelian group \( N_{p+q-1}(\Delta^p \times \Delta^q; \mathbb{Z}) \), we must have \( m = n \) as desired.

\[\square\]

\textbf{Exercise 2.5.7.18.} For every pair of simplicial sets \( S \) and \( T \), let

\[
G_{S,T} : N_*(S; \mathbb{Z}) \boxtimes N_*(T; \mathbb{Z}) \rightarrow N(S \times T; \mathbb{Z})
\]

be a chain map. Assume that the maps \( G_{S,T} \) depend functorially on \( S \) and \( T \); that is, for all maps of simplicial sets \( f : S \rightarrow S' \) and \( g : T \rightarrow T' \), the diagram of chain complexes

\[
\begin{array}{ccc}
N_*(S; \mathbb{Z}) \boxtimes N_*(T; \mathbb{Z}) & \xrightarrow{G_{S,T}} & N_*(S \times T; \mathbb{Z}) \\
\downarrow N_*(f; \mathbb{Z}) \boxtimes N_*(g; \mathbb{Z}) & & \downarrow N_*(f \times g; \mathbb{Z}) \\
N_*(S'; \mathbb{Z}) \boxtimes N_*(T'; \mathbb{Z}) & \xrightarrow{G_{S',T'}} & N_*(S' \times T'; \mathbb{Z})
\end{array}
\]

is commutative. Adapt the proof Proposition 2.5.7.1 to show that there exists an integer \( n \) (not depending on \( S \) and \( T \)) such that \( G_{S,T} = nEZ \), where \( EZ \) is the Eilenberg-Zilber homomorphism of Example 2.5.7.12.

\textbf{2.5.8 The Alexander-Whitney Construction}

Let \( A \) and \( B \) be simplicial abelian groups, having normalized Moore complexes \( N_*(A) \) and \( N_*(B) \) (Construction 2.5.5.7). In 2.5.7, we introduced the Eilenberg-Zilber homomorphism

\[
EZ : N_*(A) \boxtimes N_*(B) \rightarrow N_*(A \otimes B)
\]

and showed that it induces an isomorphism on homology groups (Theorem 2.5.7.14). The Eilenberg-Zilber homomorphism is usually not an isomorphism of chain complexes. However, it always exhibits the tensor product complex \( N_*(A) \boxtimes N_*(B) \) as a \textit{direct summand} of the normalized Moore complex \( N_*(A \otimes B) \). More precisely, there exist chain maps \( AW : N_*(A \otimes B) \rightarrow N_*(A) \boxtimes N_*(B) \), depending functorially on \( A \) and \( B \), for which the composite map

\[
N_*(A) \boxtimes N_*(B) \xrightarrow{EZ} N_*(A \otimes B) \xrightarrow{AW} N_*(A) \boxtimes N_*(B)
\]

is equal to the identity. Our goal in this section is to construct these maps and to establish their basic properties.
**Notation 2.5.8.1.** Let \( n \) be a nonnegative integer. For \( 0 \leq p \leq n \), we define strictly increasing functions

\[
\iota_{\leq p} : [p] \hookrightarrow [n] \quad \iota_{\geq p} : [n-p] \hookrightarrow [n]
\]

by the formulae \( \iota_{\leq p}(i) = i \) and \( \iota_{\geq p}(j) = j + p \). If \( A_{\bullet} \) is a simplicial abelian group, we let \( \iota_{\leq p} : A_{n} \rightarrow A_{p} \) and \( \iota_{\geq p} : A_{n} \rightarrow A_{n-p} \) denote the associated group homomorphisms.

**Construction 2.5.8.2** (The Alexander-Whitney Construction: Unnormalized Version). Let \( A_{\bullet} \) and \( B_{\bullet} \) be simplicial abelian groups with Moore complexes \( C_{\ast}(A) \) and \( C_{\ast}(B) \), respectively. We define a map of graded abelian groups \( \overline{\text{AW}} : C_{\ast}(A \otimes B) \rightarrow C_{\ast}(A) \boxtimes C_{\ast}(B) \) by the formula

\[
\overline{\text{AW}}(a \otimes b) = \sum_{0 \leq p \leq n} \iota_{\leq p}^{\ast}(a) \boxtimes \iota_{\geq p}^{\ast}(b)
\]

for \( a \in A_{n} \) and \( b \in B_{n} \). We will refer to \( \overline{\text{AW}} \) as the unnormalized Alexander-Whitney homomorphism.

**Proposition 2.5.8.3.** Let \( A_{\bullet} \) and \( B_{\bullet} \) be simplicial abelian groups. Then the unnormalized Alexander-Whitney homomorphism \( \overline{\text{AW}} : C_{\ast}(A \otimes B) \rightarrow C_{\ast}(A) \boxtimes C_{\ast}(B) \) is a chain map.

**Proof.** Let \( x \) be an element of the abelian group \( C_{n}(A \otimes B) = A_{n} \otimes B_{n} \); we wish to show that \( \partial(\overline{\text{AW}}(x)) = \text{AW}^{\text{un}}(\partial x) \). Without loss of generality, we may assume that \( n > 0 \) and that \( x \) has the form \( a \otimes b \), for some elements \( a \in A_{n} \) and \( b \in B_{n} \). In this case, we compute

\[
\overline{\text{AW}}(\partial(a \otimes b)) = \sum_{i=0}^{n} (-1)^{i} \overline{\text{AW}}(d_{i}a \otimes d_{i}b)
\]

\[
= \sum_{i=0}^{n} \sum_{p=0}^{n-1} (-1)^{i} \iota_{\leq p}^{\ast}(d_{i}a) \boxtimes \iota_{\geq p}^{\ast}(d_{i}b)
\]

\[
= \sum_{i=0}^{n} \sum_{p=0}^{i-1} (-1)^{i} \iota_{\leq p}^{\ast}(d_{i}a) \boxtimes \iota_{\geq p}^{\ast}(d_{i}b) + \sum_{i=0}^{n} \sum_{p=i}^{i-1} (-1)^{i} \iota_{\leq p}^{\ast}(d_{i}a) \boxtimes \iota_{\geq p}^{\ast}(d_{i}b)
\]

\[
= \sum_{i=0}^{n} \sum_{p=0}^{i-1} (-1)^{i} \iota_{\leq p}^{\ast}(a) \boxtimes d_{i-p} \iota_{\geq p}^{\ast}(b) + \sum_{i=0}^{n} \sum_{q=i+1}^{n} (-1)^{i} d_{i} \iota_{\leq q}^{\ast}(a) \boxtimes \iota_{\geq q}^{\ast}(b)
\]

\[
= \sum_{i=0}^{n} \sum_{p=0}^{i-1} (-1)^{i} \iota_{\leq p}^{\ast}(a) \boxtimes d_{i-p} \iota_{\geq p}^{\ast}(b) + \sum_{i=0}^{n} \sum_{q=i}^{n} (-1)^{i} d_{i} \iota_{\leq q}^{\ast}(a) \boxtimes \iota_{\geq q}^{\ast}(b)
\]

\[
= \sum_{p=0}^{n} (-1)^{p} \iota_{\leq p}^{\ast}(a) \boxtimes (\sum_{j=0}^{n-p} (-1)^{j} d_{j} \iota_{\geq p}^{\ast}(b)) + \sum_{q=0}^{n} (-1)^{q} d_{q} \iota_{\leq q}^{\ast}(a) \boxtimes \iota_{\geq q}^{\ast}(b)
\]

\[
= \sum_{p=0}^{n} (-1)^{p} \iota_{\leq p}^{\ast}(a) \boxtimes \iota_{\geq p}^{\ast}(b) + \sum_{q=0}^{n} \partial \iota_{\leq q}^{\ast}(a) \boxtimes \iota_{\geq q}^{\ast}(b)
\]
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\[ = \partial \left( \sum_{p=0}^{n} \iota_{\leq p}^* (a) \boxtimes \iota_{\geq p}^* (b) \right) \]

\[ = \partial (\Lambda W (a \otimes b)). \]

\[ Prop. 2.5.8.4. \] The collection of unnormalized Alexander-Whitney homomorphisms $\Lambda W : C_\ast (A \otimes B) \to C_\ast (A) \boxtimes C_\ast (B)$ determine a colax monoidal structure on the Moore complex functor $C_\ast : Ab_\Delta \to Ch(Z)$ (see Variant 2.1.5.11).

Proof. We first show that the unnormalized Alexander-Whitney homomorphisms determine a nonunital colax monoidal structure on the functor $C_\ast$ (Variant 2.1.4.16). By construction, the homomorphism $\Lambda W : C_\ast (A \otimes B) \to C_\ast (A) \boxtimes C_\ast (B)$ is natural in $A_\ast$ and $B_\ast$. It will therefore suffice to show that, for every triple of simplicial abelian groups $A_\ast$, $B_\ast$, and $C_\ast$, the diagram of chain complexes

\[ C_\ast (A \otimes (B \otimes C)) \xrightarrow{\sim} C_\ast ((A \otimes B) \otimes C) \]

\[ \Lambda W \downarrow \quad \Lambda W \downarrow \]

\[ C_\ast (A) \boxtimes C_\ast (B \otimes C) \quad C_\ast (A \otimes B) \boxtimes C_\ast (C) \]

\[ \text{id} \otimes \Lambda W \quad \Lambda W \otimes \text{id} \]

\[ C_\ast (A) \boxtimes (C_\ast (B) \boxtimes C_\ast (C)) \xrightarrow{\sim} (C_\ast (A) \boxtimes C_\ast (B)) \boxtimes C_\ast (C) \]

commutes, where the horizontal maps are given by the associativity constraints of the monoidal categories $Ab_\Delta$ and $Ch(Z)$, respectively. Unwinding the definitions, we see that both the clockwise and counterclockwise composition are given by the construction

\[ a \otimes (b \otimes c) \mapsto \sum_{0 \leq p \leq q \leq n} (\iota_{\leq p}^* (a) \boxtimes \rho^* (b)) \boxtimes \iota_{\geq q}^* (c) \]

for $a \in A_n$, $b \in B_n$, and $c \in C_n$, where $\rho$ denotes the nondecreasing map $[q - p] \hookrightarrow [n]$ given by $\rho(i) = i + p$.

Note that the unit object of the category of simplicial abelian groups is the constant functor $\Delta^0 \to Ab$ taking the value $Z$, which we can identify with the free simplicial abelian group $Z[\Delta^0]$ generated by the simplicial set $\Delta^0$. The image of this object under the functor $\Lambda W$ is the unnormalized chain complex $C_\ast (\Delta^0; Z)$. On the other hand, the unit object of $Ch(Z)$ is the chain complex $Z[0]$, which we will identify with the normalized chain complex $N_\ast (\Delta^0; Z)$. We will complete the proof of Proposition 2.5.8.4 by showing that the quotient map $\epsilon : C_\ast (\Delta^0; Z) \to N_\ast (\Delta^0; Z)$ is a counit for the nonunital colax monoidal structure constructed above (in the sense of Variant 2.1.5.11). To prove this, we must show that for every simplicial abelian group $A_\ast$, both of the composite maps

\[ C_\ast (A) \simeq C_\ast (A \otimes Z[\Delta^0]) \xrightarrow{\Lambda W} C_\ast (A) \boxtimes C_\ast (\Delta^0; Z) \xrightarrow{\text{id} \otimes \epsilon} C_\ast (A) \boxtimes Z[0] \simeq C_\ast (A) \]
The subcomplex generated by $C$ image of the composite map

We will refer to $AW$ as the Alexander-Whitney homomorphism. This is clear: the simplex $\iota$ degenerate for $\epsilon$ vanishes on every element of $C_s(\Delta^0; Z)$ of positive degree. □

We now adapt the Alexander-Whitney construction to the setting of normalized Moore complexes. Recall that, for every simplicial abelian group $A\bullet$, the degenerate simplices of $A\bullet$ generate a subcomplex $D_s(A) \subseteq C_s(A)$ (Proposition 2.5.5.6), which is a direct summand of $C_s(A)$ (Proposition 2.5.6.19). It follows that, if $B\bullet$ is another simplicial abelian group, then we can view $C_s(A) \boxtimes D_s(B)$ and $D_s(A) \boxtimes C_s(B)$ as direct summands of $C_s(A) \boxtimes C_s(B)$.

**Proposition 2.5.8.5.** Let $A\bullet$ and $B\bullet$ be simplicial abelian groups, and let $K\bullet \subseteq C_s(A \otimes B)$ be the subcomplex generated by $C_s(A) \boxtimes D_s(B)$ and $D_s(A) \boxtimes C_s(B)$. Then $K\bullet$ contains the image of the composite map

$$D_s(A \otimes B) \hookrightarrow C_s(A \otimes B) \xrightarrow{AW} C_s(A) \boxtimes C_s(B).$$

**Proof.** Let $x$ be an $n$-simplex of the tensor product $A\bullet \otimes B\bullet$, let $0 \leq i \leq n$, and let $s_i(x)$ denote the associated degenerate $(n+1)$-simplex of $A\bullet \otimes B\bullet$. We wish to show that $\overline{AW}(s_i(x))$ belongs to $K\bullet$. Without loss of generality, we may assume that $x = a \otimes b$ for $n$-simplices $a \in A_n$ and $b \in B_n$. In this case, we have

$$\overline{AW}(s_i(x)) = \overline{AW}(s_i(a) \otimes s_i(b)) = \sum_{p=0}^{n+1} \iota^{\ast}_{\leq p}(s_i(a)) \boxtimes \iota^{\ast}_{\geq p}(s_i(b)).$$

It will therefore suffice to show that each summand $\iota_{p}^{\ast}(s_i(a)) \boxtimes \iota_{p}^{\ast}(s_i(b))$ belongs to $K\bullet$. This is clear: the simplex $\iota_{p}^{\ast}(s_i(a))$ is degenerate if $p > i$, and the simplex $\iota_{p}^{\ast}(s_i(b))$ is degenerate for $p \leq i$. □

**Construction 2.5.8.6** (The Alexander-Whitney Construction: Normalized Version). Let $A\bullet$ and $B\bullet$ be simplicial abelian groups. It follows from Proposition 2.5.8.5 that there is a unique chain map $\overline{AW} : N_s(A \otimes B) \rightarrow N_s(A) \boxtimes N_s(B)$ for which the diagram

$$\begin{align*}
C_s(A \otimes B) &\xrightarrow{AW} C_s(A) \boxtimes C_s(B) \\
N_s(A \otimes B) &\xrightarrow{AW} N_s(A) \boxtimes N_s(B).
\end{align*}$$

We will refer to $\overline{AW}$ as the *Alexander-Whitney homomorphism.*

We have the following normalized variant of Proposition 2.5.8.4 (which follows immediately from Proposition 2.5.8.4 itself):
Proposition 2.5.8.7. The collection of Alexander-Whitney homomorphisms

\[ \text{AW} : N_*(A \otimes B) \to N_*(A) \boxtimes N_*(B) \]

determine a colax monoidal structure on the normalized Moore complex functor \( N_* : \text{Ab}_\Delta \to \text{Ch}(\mathbb{Z}) \).

Warning 2.5.8.8. Let \( A_* \) and \( B_* \) be simplicial abelian groups. Then we have a canonical isomorphism of simplicial abelian groups \( A_* \otimes B_* \cong B_* \otimes A_* \), given degreewise by the construction \( a \otimes b \mapsto b \otimes a \). Likewise, there is a canonical isomorphism of chain complexes \( N_*(A) \boxtimes N_*(B) \cong N_*(B) \boxtimes N_*(A) \) given by the Koszul sign rule (see Warning 2.5.1.14). Beware that these isomorphisms are not compatible with the Alexander-Whitney construction: that is, the diagram

\[
\begin{array}{ccc}
N_*(A \otimes B) & \xrightarrow{\text{AW}} & N_*(B \otimes A) \\
\downarrow \text{AW} & & \downarrow \text{AW} \\
N_*(A) \boxtimes N_*(B) & \xrightarrow{\text{AW}} & N_*(B) \boxtimes N_*(A)
\end{array}
\]

usually does not commute. Instead, the composite map

\[ N_*(A \otimes B) \cong N_*(B \otimes A) \xrightarrow{\text{AW}} N_*(B) \boxtimes N_*(A) \cong N_*(A) \boxtimes N_*(B) \]

can be identified with the Alexander-Whitney homomorphism associated to the opposite simplicial abelian groups \( A_*^{op} \) and \( B_*^{op} \). In other words, the colax monoidal structure of Proposition 2.5.8.7 is not a colax symmetric monoidal structure (see Definition [?]). The same remark applies to the unnormalized Alexander-Whitney construction \( \overline{\text{AW}} \) of Construction 2.5.8.2.

Proposition 2.5.8.9. Let \( A_* \) and \( B_* \) be simplicial abelian groups. Then the composition

\[ N_*(A) \boxtimes N_*(B) \xrightarrow{\text{EZ}} N_*(A \otimes B) \xrightarrow{\text{AW}} N_*(A) \boxtimes N_*(B) \]

is the identity map.

Proof. Fix elements \( a \in N_p(A) \) and \( b \in N_q(B) \) having shuffle product \( a \vee b \in N_{p+q}(A \otimes B) \). We wish to show that the Alexander-Whitney homomorphism \( \text{AW} \) satisfies \( \text{AW}(a \vee b) = a \boxtimes b \). Lift \( a \) and \( b \) to elements \( \overline{a} \in C_p(A) = A_p \) and \( \overline{b} \in C_q(B) = B_q \), respectively. Unwinding the definitions, we see that \( \text{AW}(a \vee b) \) is given by the image of

\[
\text{AW}(\overline{a} \vee \overline{b}) = \text{AW}(\sum_{\sigma} (-1)^{\sigma} (\sigma_+^* \overline{a}) \otimes (\sigma_-^* \overline{b}))
\]

\[ = \sum_{r=0}^{p+q} \sum_{\sigma} (-1)^{\sigma} (\iota_{\geq r}^* \sigma_-^*) (\overline{a}) \boxtimes (\iota_{\geq r}^* \sigma_+^*) (\overline{b}) \]
under the quotient map \( C_\bullet(A) \boxtimes C_\bullet(B) \to N_\bullet(A) \boxtimes N_\bullet(B) \); here the sum is taken over all \((p, q)\)-shuffles \( \sigma = (\sigma_-, \sigma_+) \) (see Notation 2.5.7.2). Note that the simplex \( (\iota^\sigma_\leq, \iota^\sigma_+) \) is degenerate unless \( \sigma_-(r) = r \) (which implies that \( r \leq p \)). Similarly, the simplex \( (\iota^\sigma_\geq, \iota^\sigma_-) \) is degenerate unless \( \sigma_+(r) = r - p \) (which guarantees that \( r \geq p \)). We may therefore ignore every term in the sum except for the one with \( r = p \) and \( \sigma(i) = \begin{cases} (i, 0) & \text{if } i \leq p \\ (p, i - p) & \text{if } i \geq p, \end{cases} \) for which the corresponding summand is equal to \( a \boxtimes b \) (and therefore has image \( a \boxtimes b \) in \( N_\bullet(A) \boxtimes N_\bullet(B) \)).

**Warning 2.5.8.10.** Let \( A_\bullet \) and \( B_\bullet \) be simplicial abelian groups. Then the unnormalized shuffle product \( \bar{\nabla} \) of Construction 2.5.7.3 induces a chain map \( EZ : C_\bullet(A) \boxtimes C_\bullet(B) \to C_\bullet(A \otimes B) \). However, the analogue of Proposition 2.5.8.9 for unnormalized Moore complexes is false: that is, the composite map

\[
EZ : C_\bullet(A) \boxtimes C_\bullet(B) \to C_\bullet(A \otimes B) \xrightarrow{AW} C_\bullet(A) \boxtimes C_\bullet(B)
\]

is usually not equal to the identity.

**Corollary 2.5.8.11.** Let \( A_\bullet \) and \( B_\bullet \) be simplicial abelian groups. Then the Alexander-Whitney homomorphism

\[
AW : N_\bullet(A \otimes B) \to N_\bullet(A) \boxtimes N_\bullet(B)
\]

is a quasi-isomorphism: that is, it induces an isomorphism on homology.

**Proof.** By virtue of Proposition 2.5.8.9, the Alexander-Whitney homomorphism is a left inverse to the Eilenberg-Zilber map \( EZ : N_\bullet(A) \boxtimes N_\bullet(B) \to N_\bullet(A \otimes B) \), which is a quasi-isomorphism by virtue of Theorem 2.5.7.14.

### 2.5.9 Comparison with the Homotopy Coherent Nerve

Throughout this section, we maintain the notational convention of \( \otimes \) denoting the tensor product of chain complexes \( X_\bullet \) and \( Y_\bullet \) by \( X_\bullet \boxtimes Y_\bullet \). According to Proposition 2.5.8.7, the Alexander-Whitney homomorphisms

\[
AW : N_\bullet(A \otimes B) \to N_\bullet(A) \boxtimes N_\bullet(B)
\]

determine a colax monoidal structure on the normalized Moore complex functor \( N_\bullet : \text{Ab}_\Delta \to \text{Ch}(\mathbb{Z}) \). Applying Remark 2.1.5.12, we deduce that the right adjoint functor \( K : \text{Ch}(\mathbb{Z}) \to \text{Ab}_\Delta \) inherits the structure of a lax monoidal functor. Composing with the (lax monoidal) forgetful functor \( \text{Ab}_\Delta \to \text{Set}_\Delta \), we obtain the following:
Proposition 2.5.9.1. The functor $K : \text{Ch}(\mathbb{Z}) \to \text{Set}_\Delta$ admits a lax monoidal structure, which associates to each pair of chain complexes $X_*$ and $Y_*$ a map of simplicial sets

$$\mu_{X_*, Y_*} : K(X_*) \times K(Y_*) \to K(X_* \boxtimes Y_*)$$

which can be described concretely as follows:

- Let $\sigma$ and $\tau$ be $n$-simplices of $K(X_*)$ and $K(Y_*)$, respectively, which we identify with chain maps
  $$\sigma : N_*(\Delta^n; \mathbb{Z}) \to X_* \quad \tau : N_*(\Delta^n; \mathbb{Z}) \to Y_*.$$

  Then $\mu_{X_*, Y_*}(\sigma, \tau) \in K_n(X_* \boxtimes Y_*)$ is the composite map
  $$N_*(\Delta^n; \mathbb{Z}) \hookrightarrow N_*(\Delta^n \times \Delta^n; \mathbb{Z}) \xrightarrow{\text{AW}} N_*(\Delta^n; \mathbb{Z}) \boxtimes N_*(\Delta^n; \mathbb{Z}) \xrightarrow{\sigma \boxtimes \tau} X_* \boxtimes Y_*.$$

Applying the general construction described in Remark 2.1.7.4 to the lax monoidal functor $K : \text{Ch}(\mathbb{Z}) \to \text{Set}_\Delta$, we obtain the following:

Construction 2.5.9.2. Let $C$ be a differential graded category. We define a simplicial category $C^\Delta_\bullet$ as follows:

- The objects of $C^\Delta_\bullet$ are the objects of $C$.

- For every pair of objects $X, Y \in \text{Ob}(C^\Delta_\bullet) = \text{Ob}(C)$, the simplicial set $\text{Hom}_{C^\Delta}(X, Y)_\bullet$ is the generalized Eilenberg-MacLane space $K(\text{Hom}_C(X, Y)_*)$. More concretely, the $n$-simplices of $\text{Hom}_{C^\Delta}(X, Y)_\bullet$ are chain maps $\sigma : N_*(\Delta^n; \mathbb{Z}) \to \text{Hom}_C(X, Y)_*$.

- For every triple of objects $X, Y, Z \in \text{Ob}(C^\Delta_\bullet) = \text{Ob}(C)$ and every nonnegative integer $n \geq 0$, the composition law
  $$\text{Hom}_{C^\Delta}(Y, Z)_n \times \text{Hom}_{C^\Delta}(X, Y)_n \to \text{Hom}_{C^\Delta}(X, Z)_n$$
  carries a pair $(\sigma, \tau)$ to the $n$-simplex of $K(\text{Hom}_C(X, Z)_*)$ given by the composite map
  $$N_*(\Delta^n; \mathbb{Z}) \hookrightarrow N_*(\Delta^n \times \Delta^n; \mathbb{Z}) \xrightarrow{\text{AW}} N_*(\Delta^n; \mathbb{Z}) \boxtimes N_*(\Delta^n; \mathbb{Z}) \xrightarrow{\sigma \boxtimes \tau} X_* \boxtimes Y_*.$$

We will refer to $C^\Delta_\bullet$ as the underlying simplicial category of the differential graded category $C$.

Remark 2.5.9.3. Let $C$ be a differential graded category and let $C^\circ$ denote its underlying category (in the sense of Construction 2.5.2.4). Then $C^\circ$ is isomorphic to the underlying ordinary category $C^\Delta_0$ of the simplicial category $C^\Delta$ (in the sense of Example 2.4.1.4). Both of these categories can be described concretely as follows:
The objects of $C^\circ \simeq C_0^\Delta$ are the objects of $C$.

For objects $X, Y \in C$, the morphisms from $X$ to $Y$ in the category $C^\circ \simeq C_0^\Delta$ are given by 0-cycles in the chain complex $\text{Hom}_C(X, Y)_*$. 

**Remark 2.5.9.4.** Let $C$ be a differential graded category. Then the underlying simplicial category $C^\Delta_\bullet$ is locally Kan (Definition 2.4.1.8). This follows from the observation that each of the simplicial sets $\text{Hom}_{C^\Delta}(X, Y)_\bullet = K(\text{Hom}_C(X, Y)_*)$ has the structure of a simplicial abelian group, and is therefore automatically a Kan complex (Proposition 1.1.9.9).

**Remark 2.5.9.5.** Let $C$ be a differential graded category, let $X$ and $Y$ be objects of $C$, and let $f, g : X \to Y$ be morphisms from $X$ to $Y$ in the underlying category $C^\circ$ (that is, 0-cycles of the chain complex $\text{Hom}_C(X, Y)_*$). Then $f$ and $g$ are homotopic as morphisms of the differential graded category $C$ (in the sense of Definition 2.5.4.1) if and only if they are homotopic as morphisms of the simplicial category $C^\Delta_\bullet$ (Remark 2.4.1.9); see Example 2.5.6.6. 

It follows that the isomorphism of underlying categories $C^\circ \simeq C_0^\Delta$ of Remark 2.5.9.3 induces an isomorphism from the homotopy $hC$ (given by Construction 2.5.4.6) to the homotopy category $hC^\Delta$ (given by Construction 2.4.6.1).

Our goal in this section is to establish a refinement of Remark 2.5.9.5. Let $C$ be a differential graded category and let $C^\Delta_\bullet$ denote the underlying simplicial category. Then $C^\Delta_\bullet$ is locally Kan (Remark 2.5.9.4), so the homotopy coherent nerve $N^\text{hc}(C^\Delta)$ is an $\infty$-category (Theorem 2.4.5.1). Similarly, the differential graded nerve $N^\text{dg}(C)$ is an $\infty$-category (Theorem 2.5.3.10). The $\infty$-categories $N^\text{hc}(C^\Delta)$ and $N^\text{dg}(C)$ are generally not isomorphic as simplicial sets. However, we will construct a comparison map $N^\text{hc}(C^\Delta) \to N^\text{dg}(C)$ and show that it is a trivial Kan fibration (and therefore an equivalence of $\infty$-categories; see Proposition 4.5.2.9). 

We begin with some auxiliary remarks.

**Construction 2.5.9.6** (The Fundamental Chain of a Cube). Let $I$ be a finite set of cardinality $n$, and let $\square^n = \prod_{i \in I} \Delta^1$ denote the associated cube (Notation 2.4.5.2), which we will identify with the nerve of the partially ordered set of all subsets of $I$. Using this identification, we obtain a bijective correspondence

$$
\{\text{Linear orderings of } I\} \simeq \{\text{Nondegenerate } n\text{-simplices of } \square^n\},
$$

which carries a linear ordering $\{i_1 < i_2 < \cdots < i_n\}$ to the chain of subsets

$$
\emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset I.
$$

In particular, the symmetric group $\Sigma_I$ of permutations of $I$ acts simply transitively on the set of nondegenerate $n$-simplices of $\square^n$.

Fix a linear ordering of $I$, corresponding to a nondegenerate $n$-simplex $\sigma : \Delta^n \to \square^n$. We let $[\square^n]$ denote the alternating sum $\sum_{\pi \in \Sigma_I} (-1)^\pi \pi(\sigma)$, which we regard as an $n$-chain of
the normalized chain complex $N_\ast(\square^I; \mathbb{Z})$. We will refer to $[\square^I]$ as the *fundamental chain* of the cube $\square^I$. We will be particularly interested in the special case where $I$ is the set $\{1, 2, \cdots, n\}$, endowed with its usual ordering; in this case, we denote the cube $\square^I$ by $\square^n$ and its fundamental chain $[\square^I]$ by $[\square^n]$.

**Remark 2.5.9.7.** Let $n$ be a nonnegative integer. Then the fundamental chain $[\square^n]$ of Construction 2.5.9.6 is given by the iterated shuffle product

$$[\Delta^1] \triangledown [\Delta^1] \triangledown \cdots \triangledown [\Delta^1] \in N_n(\Delta^1 \times \Delta^1 \times \cdots \times \Delta^1; \mathbb{Z}) \simeq N_n(\square^n; \mathbb{Z})$$

(see (2.5.7); here $[\Delta^1]$ denotes the generator of the group $N_1(\Delta^1; \mathbb{Z}) \simeq \mathbb{Z}$ (which is also the fundamental chain of the 1-dimensional cube $\square^1$).

**Warning 2.5.9.8.** The simplicial set $\square^I$ and its normalized chain complex $N_\ast(\square^I; \mathbb{Z})$ depend only on the choice of the finite set $I$. However, the fundamental chain $[\square^I]$ of Construction 2.5.9.6 is *a priori* ambiguous up to a sign. One can resolve this ambiguity by choosing a linear ordering on the set $I$ (as in Construction 2.5.9.6), which will be sufficient for our purposes in this section. However, less is needed: one needs only an *orientation* on the set $I$ (or equivalently an orientation of the topological manifold-with-boundary $|\square^I| \simeq [0, 1]^I$).

**Notation 2.5.9.9.** Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}^\Delta_\ast$ denote the underlying simplicial category (Construction 2.5.9.2). Let $n \geq 0$ be a nonnegative integer and let $\sigma$ be a nondegenerate $(n+1)$-simplex of the homotopy coherent nerve $N^\text{hc}_\ast(\mathcal{C}^\Delta)$, which we will identify with a simplicial functor $\sigma : \text{Path}[n+1]_\ast \to \mathcal{C}^\Delta_\ast$. Set $X = \sigma(0)$ and $Y = \sigma(n+1)$, so that $\sigma$ induces a map of simplicial sets

$$\square^n \simeq \text{Hom}_{\text{Path}[n+1]}(0, n+1)_\ast \to \text{Hom}_{\mathcal{C}^\Delta}(X, Y)_\ast = K(\text{Hom}_{\mathcal{C}}(X, Y)_\ast),$$

which we can identify with a chain map $N_\ast(\square^n; \mathbb{Z}) \to \text{Hom}_{\mathcal{C}}(X, Y)_\ast$. This map carries the fundamental chain $[\square^n]$ of Construction 2.5.9.6 to an element of the abelian group $\text{Hom}_{\mathcal{C}}(X, Y)_n$, which we will denote by $\sigma([\square^n])$.

**Proposition 2.5.9.10.** Let $\mathcal{C}$ be a differential graded category. Then there is a unique functor of $\infty$-categories $\mathcal{F} : N^\text{hc}_\ast(\mathcal{C}^\Delta) \to N^d_\ast(\mathcal{C})$ with the following properties:

- On 0-simplices the functor $\mathcal{F}$ is the identity: that is, it carries each object of the simplicial category $\mathcal{C}^\Delta$ to the corresponding object of the differential graded category $\mathcal{C}$.

- Let $n \geq 0$ and let $\sigma$ be an $(n+1)$-simplex of $N^\text{hc}_\ast(\mathcal{C}^\Delta)$. Set $X = \sigma(0)$ and $Y = \sigma(n+1)$. Then the value of $\mathcal{F}(\sigma)$ on $\{n+1 > n > \cdots > 0\}$ is the chain $\sigma([\square^n]) \in \text{Hom}_{\mathcal{C}}(X, Y)_n$ of Notation 2.5.9.9.

The proof of Proposition 2.5.9.10 will require an elementary property of Construction 2.5.9.6.
Notation 2.5.9.11. Let $I$ be a finite linearly ordered set of cardinality $n > 0$ and let $\square^I$ denote the corresponding simplicial cube. For each element $i \in I$, the linear ordering on $I$ restricts to linear ordering on the subset $I \setminus \{i\}$, which determines a fundamental chain
\[ [\square^I \setminus \{i\}] \in N_{n-1}(\square^I; \mathbb{Z}). \]
We will write $[[0] \times \square^I \setminus \{i\}] \in N_{n-1}(\square^I; \mathbb{Z})$ for the image of the fundamental chain $[\square^I \setminus \{i\}]$ under the inclusion of simplicial sets
\[ \square^I \setminus \{i\} \simeq \{0\} \times \square^I \setminus \{i\} \hookrightarrow \Delta^1 \times \square^I \setminus \{i\} \simeq \square^I. \]
Similarly, we write $[[1] \times \square^I \setminus \{i\}] \in N_{n-1}(\square^I; \mathbb{Z})$ for the image of the fundamental chain $[\square^I \setminus \{i\}]$ under the inclusion
\[ \square^I \setminus \{i\} \simeq \{1\} \times \square^I \setminus \{i\} \hookrightarrow \Delta^1 \times \square^I \setminus \{i\} \simeq \square^I. \]

Lemma 2.5.9.12. Let $n$ be a nonnegative integer and let $I$ denote the linearly ordered set $\{1 < 2 < \cdots < n\}$. Then we have an equality
\[ \partial[\square^I] = \sum_{i=1}^{n} (-1)^i(([[0] \times \square^I \setminus \{i\}] - [[1] \times \square^I \setminus \{i\}]) \]
in the abelian group $N_{n-1}(\square^I; \mathbb{Z})$.

Remark 2.5.9.13. Lemma 2.5.9.12 is a homological incarnation of the following topological assertion: the geometric realization $|\square^I| \simeq [0, 1]^I$ is a manifold, whose boundary can be written as a union of the faces $\{0\} \times [0, 1]^I \setminus \{i\}$ and $\{1\} \times [0, 1]^I \setminus \{i\}$.

Proof of Lemma 2.5.9.12. Using the description of $[\square^I]$ as a shuffle product (Remark 2.5.9.7) and the fact that the shuffle product satisfies the Leibniz rule (Proposition 2.5.7.10), we compute
\[
\partial[\square^I] = \partial([\Delta^1] \triangledown \cdots \triangledown [\Delta^1]) = \sum_{i=1}^{n} (-1)^{i-1} [\square^{i-1}] \triangledown \partial([\Delta^1]) \triangledown [\square^{n-i}] = \sum_{i=1}^{n} (-1)^i [\square^{i-1}] \triangledown (d_1[\Delta^1] - d_0[\Delta^1]) \triangledown [\square^{n-i}] = \sum_{i=1}^{n} (-1)^i ([0] \times \square^I \setminus \{i\}] - ([1] \times \square^I \setminus \{i\}]).
\]
Remark 2.5.9.14. Let \( n \) be a nonnegative integer. It follows from Lemma 2.5.9.12 that the boundary \( \partial [n] \) belongs to the subcomplex \( N_*(\partial [n]; Z) \subset N_*([n]; Z) \). In other words, the image of the fundamental chain \([n]\) in the relative chain complex
\[
N_*([n], \partial [n]; Z) = N_*([n]; Z) / N_*(\partial [n]; Z)
\]
is a cycle. In fact, one can be more precise: the construction \( 1 \mapsto [n] \) determines a quasi-isomorphism of chain complexes \( u_n : Z[n] \to N_*([n], \partial [n]; Z) \). To prove this, we proceed by induction on \( n \): the case \( n = 0 \) is trivial, and the inductive step follows by identifying \( u_n \) with the composition
\[
Z[n] \xrightarrow{\id \otimes u_{n-1}} Z[1] \otimes N_*(\partial^{n-1}, \partial \partial^{n-1}; Z) \cong N_*([1], \partial [1]; Z) N_*(\partial^{n-1}, \partial \partial^{n-1}; Z)
\]
where \( EZ \) denotes the Eilenberg-Zilber map of Variant 2.5.7.17 (which is a quasi-isomorphism by virtue of Theorem 2.5.7.14. Note that this property characterizes the fundamental chain \([n]\) up to sign (since the quotient map \( N_*([n]; Z) \to N_*([n], \partial [n]; Z) \) is an isomorphism in degree \( n \)).

Lemma 2.5.9.15. Let \( I \) be a finite linearly ordered set which is a union of disjoint subsets \( I_-, I_+ \subseteq I \). Then the Alexander-Whitney homomorphism \( AW : N_*([I]; Z) \to N_*([I_-]; Z) \times N_*([I_+]; Z) \) satisfies
\[
AW([I]) = (-1)^d [I_-] \otimes [I_+],
\]
where \( d \) denotes the cardinality of the set \( \{(i, j) \in I_- \times I_+ : i > j\} \).

Proof. Using Remark 2.5.7.9 (and the graded-commutativity of the shuffle product; see Proposition 2.5.8.9), we observe that the shuffle product map
\[
\vee : N_*([I_-]; Z) \times N_*([I_+]; Z) \to N_*([I_-] \times [I_+]; Z) \cong N_*([I]; Z)
\]
satisfies \([I] = (-1)^d [I_-] \vee [I_+]\). Applying the Alexander-Whitney homomorphism and invoking Proposition 2.5.8.9, we obtain the identity
\[
AW([I]) = (-1)^d AW([I_-] \vee [I_+]) = [I_-] \otimes [I_+].
\]

Proof of Proposition 2.5.9.10. Fix an integer \( n \geq 0 \), and let \( \sigma \) be an \((n+1)\)-simplex of the homotopy coherent nerve \( N_{hc}(\mathcal{C}^\Delta) \), which we will identify with a simplicial functor \( \sigma : \text{Path}[n+1] \to \mathcal{C}^\Delta \). Set \( X = \sigma(0) \) and \( Y = \sigma(n+1) \), and let us identify the simplicial set \( \text{Hom}_{\text{Path}[n+1]}(0, n+1) \) with the cube \([n] \). By virtue of Remark 2.5.3.9 it will suffice to verify the following three assertions:
(a) If \( n = 0 \) and \( \sigma \) is the degenerate edge of \( \mathbf{N}^\bullet_{hc}(C^\Delta) \) determined by the object \( X \in C \), then 
\[
\sigma([\square^n]) = \text{id}_X.
\]

(b) If \( n > 0 \) and \( \sigma \) is degenerate, then \( \sigma([\square^n]) = 0. \)

(c) If \( n \geq 0 \), then 
\[
\partial \sigma([\square^n]) = \sum_{i=1}^{n} (-1)^i d_i(\sigma)([\square^{n-1}]) + (-1)^{(n+1)(i+1)} \sigma_{\geq i}([\square^{n-i}] \circ \sigma_{\leq i}([\square^{i-1}]));
\]

Assertion (a) is immediate from the definition. To prove (b), we observe that \( \sigma \) determines a map of simplicial sets 
\[
\text{Hom}_{\text{Path}[n+1]}(0,n+1) \rightarrow \text{Hom}_{C^\Delta}(X,Y) \simeq K(\text{Hom}_C(X,Y)),
\]
which we can identify with a chain map \( u : N_*(\text{Hom}_{\text{Path}[n+1]}(0,n+1); \mathbb{Z}) \rightarrow \text{Hom}_C(X,Y)_* \). If \( \sigma \) is degenerate, then (as a simplicial functor) it factors as a composition 
\[
\text{Path}[n+1] \rightarrow \text{Path}[n] \rightarrow C^\Delta,
\]
where \( \rho \) is a simplicial functor satisfying \( \rho(0) = 0 \) and \( \rho(n+1) = n \). For \( n > 0 \), it follows that the chain map \( u \) factors through the complex \( N_*(\text{Hom}_{\text{Path}[n]}(0,n); \mathbb{Z}) \simeq N_*(\square^{n-1}; \mathbb{Z}) \). Since \( \square^{n-1} \) is a simplicial set of dimension \( \leq n-1 \), the chain complex \( N_*(\square^{n-1}; \mathbb{Z}) \) vanishes in degrees \( \geq n \) (see Example 2.5.5.13). In particular, the map \( u \) vanishes in degree \( n \), so that \( \sigma([\square^n]) = 0. \)

We now prove (c). Set \( I = \{1, 2, \ldots, n\} \). Using Lemma 2.5.9.12, we obtain the identity 
\[
\partial \sigma([\square^I]) = \sum_{i=1}^{n} (-1)^i (\sigma([\{0\} \times \square^{I \setminus \{i\}}]) - \sigma([\{1\} \times \square^{I \setminus \{i\}}]).
\]

It will therefore suffice to show that, for each \( 1 \leq i \leq n \), we have equalities 
\[
\sigma([\{0\} \times \square^{I \setminus \{i\}}]) = d_i(\sigma)([\square^{n-1}])
\]
\[
\sigma([\{1\} \times \square^{I \setminus \{i\}}]) = (-1)^{(i-1)(n-i)} \sigma_{\geq i}([\square^{n-i}] \circ \sigma_{\leq i}([\square^{i-1}]))
\]
in the abelian group \( \text{Hom}_C(X,Y)_{n-1} \). The first of these identities follows immediately from the definition of \( d_i(\sigma) \). To prove the second, we note that the inclusion \( \{1\} \times \square^{I \setminus \{i\}} \hookrightarrow \square^I \simeq \text{Hom}_{\text{Path}[n+1]}(0,n+1)_* \) factors as a composition 
\[
\{1\} \times \square^{I \setminus \{i\}} \simeq \square^{n-i} \times \square^{-1}
\]
\[
\simeq \text{Hom}_{\text{Path}[n+1]}(i,n+1)_* \times \text{Hom}_{\text{Path}[n+1]}(0,i)_* 
\]
\[
\circ \text{Hom}_{\text{Path}[n+1]}(0,n+1)_*.
\]
Set $Z = \sigma(i)$. Using the fact that $\sigma$ is a simplicial functor (and the definition of the simplicial category $\mathcal{C}_{\Delta}$), we see that $\sigma([\{1\} \times \Box^{i-1}])$ is the image of the fundamental chain $\Box^{i-1}$ under the composite map

$$\begin{align*}
\sigma([\{1\} \times \Box^{i-1}]) &\xrightarrow{\Delta W} \sigma([\{1\} \times \Box^{n-i}]) \\
&\xrightarrow{\sigma > [0] \sigma < 1} \sigma([\{1\} \times \Box^{n-i}]) \\
&\xrightarrow{\circ} \Hom_{\mathcal{C}}(Z,Y)_* \boxtimes \Hom_{\mathcal{C}}(X,Z)_* \xrightarrow{\circ} \Hom_{\mathcal{C}}(X,Z)_*.
\end{align*}$$

The desired result now follows from the identity $\Delta W([\{1\} \times \Box^{i-1}]) = (-1)^{(i-1)(n-i)} \Box^{n-i} \boxtimes \Box^{i-1}$ supplied by Lemma 2.5.9.15.

**Exercise 2.5.9.16.** Let $\mathcal{C}$ be a differential graded category, and let $\mathcal{Z} : N^{hc}_{\Delta}(\mathcal{C}) \to N^{dg}_{\Delta}(\mathcal{C})$ be the functor of $\infty$-categories supplied by Proposition 2.5.9.10. Show that $\mathcal{Z}$ is bijective on simplices of dimension $n \leq 2$ (for the case $n = 2$, this is essentially the content of Remark 2.5.4.4).

The functor $\mathcal{Z} : N^{hc}_{\Delta}(\mathcal{C}) \to N^{dg}_{\Delta}(\mathcal{C})$ is generally not bijective on simplices of dimension $n \geq 3$. Nevertheless, we have the following:

**Theorem 2.5.9.17.** Let $\mathcal{C}$ be a differential graded category and let $\mathcal{Z} : N^{hc}_{\Delta}(\mathcal{C}) \to N^{dg}_{\Delta}(\mathcal{C})$ be the functor of $\infty$-categories supplied by Proposition 2.5.9.10. Then $\mathcal{Z}$ is a trivial Kan fibration of simplicial sets.

**Proof.** Fix an integer $n \geq 0$ and a diagram of simplicial sets

$$\begin{align*}
\partial \Delta^{n+1} &\xrightarrow{\sigma_0} N^{hc}_{\Delta}(\mathcal{C}) \\
\Delta^{n+1} &\xrightarrow{\sigma} N^{dg}_{\Delta}(\mathcal{C});
\end{align*}$$

we wish to show that the map $\sigma_0$ admits an extension $\sigma : \Delta^n \to N^{hc}_{\Delta}(\mathcal{C})$ as indicated, rendering the diagram commutative. Let us abuse notation by identifying $\sigma_0$ with a simplicial functor from $\text{Path}([\partial \Delta^{n+1}]_\bullet)$ to $\mathcal{C}^{\Delta}$. Set $X = \sigma_0(0)$ and $Y = \sigma_0(n + 1)$, so that $\sigma_0$ determines a map of simplicial sets

$$u_0 : \partial \Box^n \simeq \text{Hom}_{\text{Path}([\partial \Delta^{n+1}]_\bullet)}(0, n + 1) \to \Hom_{\mathcal{C}^{\Delta}}(X,Y)_\bullet = K(\Hom_{\mathcal{C}}(X,Y))$$

(see Proposition 2.4.6.11), which we will identify with a chain map $f_0 : N_*(\partial \Box^n; \mathbb{Z}) \to \Hom_{\mathcal{C}}(X,Y)_*$. By virtue of Corollary 2.4.6.12, choosing an extension of $\sigma_0$ to a map $\sigma : \Delta^{n+1} \to N^{hc}_{\Delta}(\mathcal{C})$ is equivalent to choosing an extension of $u_0$ to a map of simplicial sets $u : \Box^n \to K(\Hom_{\mathcal{C}}(X,Y))$, or an extension of $f_0$ to a chain map $f : N_*(\Box^n; \mathbb{Z}) \to \Hom_{\mathcal{C}}(X,Y)_*$. 


Note that the boundary $\partial[\square^n]$ belongs to the subcomplex $N_*(\partial[\square^n]; \mathbb{Z}) \subset N_*(\square^n; \mathbb{Z})$ (see Lemma 2.5.9.12). Unwinding the definitions, we see that $\tau$ supplies a chain $z \in \text{Hom}_C(X,Y)_n$ satisfying $\partial(z) = f_0(\partial[\square^n]) \in \text{Hom}_C(X,Y)_{n-1}$. Let $M_*$ denote the subcomplex of $N_*(\square^n; \mathbb{Z})$ generated by $N_*(\partial[\square^n]; \mathbb{Z})$ together with the fundamental chain $[\square^n]$, so that $f_0$ extends uniquely to a chain map $f_1 : M_* \to \text{Hom}_C(X,Y)_*$ satisfying $f_1([\square^n]) = z$. Unwinding the definitions, we see that if $f : N_*(\square^n; \mathbb{Z}) \to \text{Hom}_C(X,Y)_*$ is a map of chain complexes extending $f_0$, then the corresponding extension $\sigma : \Delta^{n+1} \to N^{\text{hc}}_*(C^{\Delta})$ of $\sigma_0$ satisfies $3 \circ \sigma = \tau$ if and only if $f|_{M_*} = f_1$. We will complete the proof by showing that $M_*$ is a direct summand of $N_*(\square^n; \mathbb{Z})$ (so that any map $f_1 : M_* \to \text{Hom}_C(X,Y)_*$ can be extended to $N_*(\square^n; \mathbb{Z})$). To prove this, note that we have an exact sequence of chain complexes

$$0 \to \mathbb{Z}[n] \xrightarrow{[\square^n]} N_*(\square^n, \partial[\square^n]; \mathbb{Z}) \to N_*(\square^n; \mathbb{Z})/M_* \to 0,$$

where the first map is a quasi-isomorphism (Variant 2.5.7.17). It follows that the chain complex $N_*(\square^n; \mathbb{Z})/M_*$ is acyclic and free in each degree, so that the exact sequence

$$0 \to M_* \to N_*(\square^n; \mathbb{Z}) \to N_*(\square^n; \mathbb{Z})/M_* \to 0$$

splits by virtue of Proposition 2.5.1.10. \qed
Chapter 3

Kan Complexes

Recall that a Kan complex is a simplicial set $X$ with the property that, for $n \geq 0$ and $0 \leq i \leq n$, any morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ can be extended to an $n$-simplex of $X$ (Definition 1.1.9.1). Kan complexes play an important role in the theory of $\infty$-categories, for three different (but closely related) reasons:

(a) Every Kan complex is an $\infty$-category (Example 1.3.0.3). Conversely, every $\infty$-category $\mathcal{C}$ contains a largest Kan complex $\mathcal{C}^\simeq \subseteq \mathcal{C}$ (obtained from $\mathcal{C}$ by removing all non-invertible morphisms; see Construction 4.4.3.1), which is an important invariant of $\mathcal{C}$. Consequently, understanding the homotopy theory of Kan complexes can be regarded as a first step towards understanding $\infty$-categories in general.

(b) Let $\mathcal{C}$ be an $\infty$-category. To every pair of objects $X, Y \in \mathcal{C}$, one can associate a Kan complex $\text{Hom}_\mathcal{C}(X,Y)$ which we will refer to as the space of maps from $X$ to $Y$ (see Construction [?]). These mapping spaces are essential to the structure of $\mathcal{C}$. For example, we will see later that a functor of $\infty$-categories $F : \mathcal{C} \to \mathcal{D}$ admits a homotopy inverse if and only if it is essentially surjective at the level of homotopy categories and induces a homotopy equivalence $\text{Hom}_\mathcal{C}(X,Y) \to \text{Hom}_\mathcal{D}(F(X),F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$ (see Theorem [?]).

(c) The collection of all Kan complexes can be organized into an $\infty$-category, which we will denote by $\mathcal{S}$ and refer to as the $\infty$-category of spaces (Construction 3.1.4.12). The $\infty$-category $\mathcal{S}$ plays a central role in the general theory of $\infty$-categories, analogous to the role of Set in classical category theory. This can be articulated in several different ways:

- To any $\infty$-category $\mathcal{C}$, one can associate a functor $h : \mathcal{C} \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ called the Yoneda embedding, which is given informally (and up to homotopy equivalence) by the construction $C \mapsto \text{Hom}_\mathcal{C}(\bullet, C)$ (see Construction [?]). Like the classical
Yoneda embedding, the functor $h$ is fully faithful: that is, it induces an equivalence on mapping spaces (Theorem [?]).

- The $\infty$-category $S$ has a pointed variant $S_*$, whose objects are pointed Kan complexes (Construction [?]). This $\infty$-category is equipped with a forgetful functor $S_* \rightarrow S$, given on objects by the construction $(X, x) \mapsto X$. This forgetful functor is an example of a left fibration of $\infty$-categories (see Definition 4.2.1.1). In fact, it is a universal left fibration in the following sense: for any $\infty$-category $C$, the construction
  $$(F : C \rightarrow S) \mapsto (u : C \times_S S_* \rightarrow C)$$
  induces a bijection from the set of equivalence classes of functors $F : C \rightarrow S$ to the set of equivalence classes of left fibrations $C \rightarrow C$ having essentially small fibers (Theorem [?]).

- The $\infty$-category $S$ admits small colimits (Proposition [?]). Moreover, if $C$ is any other $\infty$-category which admits small colimits, then evaluation on the Kan complex $\Delta^0 \in S$ induces an equivalence of $\infty$-categories
  $$\text{LFun}(C, S) \rightarrow C \quad F \mapsto F(\Delta^0),$$
  where $\text{LFun}(C, S)$ denotes the full subcategory of $\text{Fun}(C, S)$ spanned by those functors which preserve small colimits (Theorem [?]). In other words, the $\infty$-category $S$ is freely generated under small colimits by the Kan complex $\Delta^0$.

Our goal in this chapter is to give an exposition of the homotopy theory of Kan complexes. We begin in §3.1 by developing the basic vocabulary of simplicial homotopy theory. In particular, we introduce the notions of Kan fibration (Definition 3.1.1.1), anodyne morphism (Definition 3.1.2.1), and (weak) homotopy equivalence between simplicial sets (Definitions 3.1.5.1 and 3.1.5.11), and establish some of their basic formal properties.

Recall that, to any Kan complex $X$, we can associate a set $\pi_0(X)$ of connected components of $X$ (Definition 1.1.6.8). In §3.2 we associate to each base point $x \in X$ a sequence of groups $\{\pi_n(X, x)\}_{n>0}$, which we refer to as the homotopy groups of $X$ (Construction 3.2.2.4 and Theorem 3.2.2.10), and establish some of their essential properties. In particular, we prove a simplicial analogue of Whitehead’s theorem: a morphism of Kan complexes $f : X \rightarrow Y$ is a homotopy equivalence if and only if it induces a bijection $\pi_0(X) \rightarrow \pi_0(Y)$ and isomorphisms $\pi_n(X, x) \rightarrow \pi_n(Y, f(x))$, for every choice of base point $x \in X$ and every positive integer $n$ (Theorem 3.2.6.1).

A general simplicial set $X$ need not be a Kan complex. However, one can always find a weak homotopy equivalence $f : X \rightarrow Y$, where $Y$ is a Kan complex; in this case, we refer to $Y$ as a fibrant replacement for $X$ (in the case where $X$ is an $\infty$-category, one can think of $Y$...
as another ∞-category obtained from $X$ by formally adjoining inverses of all morphisms: see Remark [?]). The existence of fibrant replacements has an easy formal proof (a special case of Quillen’s small object argument; see §3.1.6), which gives very little information about the structure of the Kan complex $Y$. In §3.3, we outline another approach (due to Kan) which associates to each simplicial set $X$ a Kan complex $\text{Ex}^{\infty}(X) = \lim_{n \to \infty} \text{Ex}^n(X)$ which is defined using combinatorics of iterated subdivision (Construction 3.3.6.1). The functor $X \mapsto \text{Ex}^{\infty}(X)$ has many useful properties: for example, it preserves Kan fibrations (Proposition 3.3.6.6) and commutes with finite limits (Proposition 3.3.6.4). As an application, we show that a Kan fibration of simplicial sets $f : X \to Y$ is a weak homotopy equivalence if and only if it is a trivial Kan fibration (Proposition 3.3.7.4), and that a monomorphism of simplicial sets $i : A \hookrightarrow B$ is a weak homotopy equivalence if and only if it is anodyne (Corollary 3.3.7.5).

Let $\text{Set}_{\Delta}$ denote the category of simplicial sets, and let $\text{Kan} \subset \text{Set}_{\Delta}$ denote the full subcategory spanned by the Kan complexes. We let $h\text{Kan}$ denote the homotopy category of Kan complexes (Construction 3.1.4.10), which can be obtained from Kan by identifying morphisms which are homotopic. Beware that the category $h\text{Kan}$ is somewhat ill-behaved: for example, it admits neither pullbacks or pushouts. In §3.4 we address this point by introducing the notions of homotopy pullback and homotopy pushout diagrams of simplicial sets (which can be regarded as homotopy-theoretic counterparts for the classical categorical notion of pullback and pushout diagrams), and establishing their basic properties. We will later see that these diagrams can be interpreted as pullback and pushout squares in the ∞-category $\mathcal{S}$ (see Propositions [?], rather than its homotopy category $h\mathcal{S}$.

Recall that, for every topological space $Y$, the singular simplicial set $\text{Sing}_{\bullet}(Y)$ is a Kan complex (Proposition 1.1.9.8). In §3.5, we show that every Kan complex arises in this way, at least up to homotopy equivalence. More precisely, we show that the unit map $u_X : X \to \text{Sing}_{\bullet}(|X|)$ is a homotopy equivalence for any Kan complex $X$ (and a weak homotopy equivalence for any simplicial set $X$; see Theorem 3.5.4.1). Using this fact, we show that the geometric realization functor $X \mapsto |X|$ induces a fully faithful embedding of homotopy categories $h\text{Kan} \hookrightarrow \text{hTop}$, whose essential image consists of those topological spaces having the homotopy type of a CW complex (Theorem 3.5.0.1). In other words, the (combinatorially defined) homotopy theory of Kan complexes studied in this section is essentially equivalent to the (topologically defined) homotopy theory of CW complexes.

### 3.1 The Homotopy Theory of Kan Complexes

Let $X$ and $Y$ be simplicial sets, and suppose we are given a pair of maps $f_0, f_1 : X \to Y$. A homotopy from $f_0$ to $f_1$ is a morphism of simplicial sets $h : \Delta^1 \times X \to Y$ satisfying $f_0 = h|_{\{0\} \times X}$ and $f_1 = h|_{\{1\} \times Y}$ (Definition 3.1.4.2). Beware that, for general simplicial sets,
this terminology can be misleading: for example, the existence of a homotopy from $f_0$ to $f_1$ need not imply the existence of a homotopy from $f_1$ to $f_0$. However, the situation is better in the case if we assume that $Y_\bullet$ is a Kan complex. In general, we can identify morphisms from $X$ to $Y$ as vertices of the simplicial set $\text{Fun}(X, Y)$ of Construction 1.4.3.1 and homotopies with edges of the simplicial set $\text{Fun}(X, Y)$. In §3.1.3 we will show that when $Y$ is a Kan complex, then $\text{Fun}(X, Y)$ is also a Kan complex (Corollary 3.1.3.4). In §3.1.4, we exploit this fact to construct an $\infty$-category $\mathcal{S}$ (Construction 3.1.4.12) whose objects are Kan complexes, which will play an essential role throughout this book.

Our approach to Corollary 3.1.3.4 is somewhat indirect. We begin in §3.1.1 by introducing the notion of a Kan fibration between simplicial sets. Roughly speaking, a Kan fibration $f : X \to S$ can be viewed as a family of Kan complexes parametrized by $S$: in particular, if $f$ is a Kan fibration, then each fiber $X_s = \{s\} \times_S X$ is a Kan complex (Remark 3.1.1.9). In §3.1.3 we will deduce Corollary 3.1.3.4 as a consequence of a more general stability result for Kan fibrations under exponentiation (Theorem 3.1.3.1). Our proof of this result will make use of the Gabriel-Zisman calculus of anodyne morphisms, which we review in §3.1.2.

We say that a morphism of Kan complexes $f : X \to Y$ is a homotopy equivalence if its image in the homotopy category $\text{h Kan}$ is an isomorphism: that is, if $f$ admits a homotopy inverse $g : Y \to X$. This definition makes sense for more general simplicial sets (Definition 3.1.5.1), but is of somewhat limited utility. When working with simplicial sets which are not Kan complexes, it is often better to consider the more liberal notion of weak homotopy equivalence (Definition 3.1.5.11), which we introduce and study in §3.1.5. In §3.1.6 we show that every simplicial set $X_\bullet$ admits an anodyne morphism $f : X_\bullet \to Q_\bullet$ (Corollary 3.1.6.2), using a simple incarnation of Quillen’s “small object argument.”

### 3.1.1 Kan Fibrations

Recall that a simplicial set $X$ is said to be a Kan complex if it has the extension property with respect to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $n > 0$ (Definition 1.1.9.1). For many purposes, it is useful to consider a relative version of this notion, which applies to a morphism between simplicial sets.

**Definition 3.1.1.1.** Let $f : X \to S$ be a morphism of simplicial sets. We say that $f$ is a Kan fibration if, for each $n > 0$ and each $0 \leq i \leq n$, every lifting problem

```
\[ \begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\sigma_0} & X \\
\downarrow & & \downarrow f \\
\Delta^n & \xrightarrow{\sigma} & S \\
\end{array} \]
```

exists.
admits a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets \( \sigma_0 : \Lambda_i^n \rightarrow X \) and every \( n \)-simplex \( \sigma : \Delta^n \rightarrow S \) extending \( f \circ \sigma_0 \), we can extend \( \sigma_0 \) to an \( n \)-simplex \( \sigma : \Delta^n \rightarrow X \) satisfying \( f \circ \sigma = \sigma \).

**Example 3.1.1.2.** Let \( X \) be a simplicial set. Then the projection map \( X \rightarrow \Delta^0 \) is a Kan fibration if and only if \( X \) is a Kan complex.

**Example 3.1.1.3.** Any isomorphism of simplicial sets is a Kan fibration.

**Example 3.1.1.4.** Let \( q : X \rightarrow S \) be a morphism of simplicial sets which induces an isomorphism from \( X \) to a summand of \( S \) (Definition 1.1.6.1). Then \( q \) is a Kan fibration.

**Remark 3.1.1.5.** The collection of Kan fibrations is closed under retracts. That is, given a diagram of simplicial sets

\[
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow f & & \downarrow f' \\
S & \rightarrow & S'
\end{array}
\]

where both horizontal compositions are the identity, if \( f' \) is a Kan fibration, then so is \( f \).

**Remark 3.1.1.6.** The collection of Kan fibrations is closed under pullback. That is, given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \rightarrow & S
\end{array}
\]

where \( f \) is a Kan fibration, \( f' \) is also a Kan fibration.

**Remark 3.1.1.7.** Let \( f : X \rightarrow S \) be a map of simplicial sets. Suppose that, for every simplex \( \sigma : \Delta^n \rightarrow S \), the projection map \( \Delta^n \times_S X \rightarrow \Delta^n \) is a Kan fibre. Then \( f \) is a Kan fibration. Consequently, if we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \rightarrow & S
\end{array}
\]

where \( g \) is surjective and \( f' \) is a Kan fibration, then \( f \) is also a Kan fibration.
CHAPTER 3. KAN COMPLEXES

Remark 3.1.1.8. The collection of Kan fibrations is closed under filtered colimits. That is, if \( \{ f_\alpha : X_\alpha \to S_\alpha \} \) is any filtered diagram in the arrow category Fun([1], \text{Set}_\Delta) having colimit \( f : X \to S \), and each \( f_\alpha \) is a Kan fibration of simplicial sets, then \( f \) is also a Kan fibration of simplicial sets.

Remark 3.1.1.9. Let \( f : X \to S \) be a Kan fibration of simplicial sets. Then, for every vertex \( s \in S \), the fiber \( \{ s \} \times_S X \) is a Kan complex (this follows from Remark 3.1.1.6 and Example 3.1.1.2).

Remark 3.1.1.10. Let \( f : X \to Y \) and \( g : Y \to Z \) be Kan fibrations. Then the composite map \( (g \circ f) : X \to Z \) is a Kan fibration.

Remark 3.1.1.11. Let \( f : X \to Y \) be a Kan fibration of simplicial sets. If \( Y \) is a Kan complex, then \( X \) is also a Kan complex (this follows by applying Remark 3.1.1.10 in the case \( Z = \Delta^0 \), by virtue of Example 3.1.1.2).

3.1.2 Anodyne Morphisms

By definition, a morphism of simplicial sets \( f : X \to S \) is a Kan fibration if it has the right lifting property with respect to every horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \) for \( 0 \leq i \leq n \) and \( n > 0 \). If this condition is satisfied, then \( f \) automatically has the right lifting property with respect to a much larger class of morphisms.

Definition 3.1.2.1 (Anodyne Morphisms). Let \( T \) be the smallest collection of morphisms in the category \( \text{Set}_\Delta \) with the following properties:

- For each \( n > 0 \) and each \( 0 \leq i \leq n \), the horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \) belongs to \( T \).
- The collection \( T \) is weakly saturated (Definition 1.4.4.15). That is, \( T \) is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets \( i : A \to B \) is anodyne if it belongs to the collection \( T \).

Remark 3.1.2.2. The class of anodyne morphisms was introduced by Gabriel-Zisman in [17].

Remark 3.1.2.3. Every anodyne morphism of simplicial sets \( i : A \to B \) is a monomorphism. This follows from the observation that the collection of monomorphisms is weakly saturated (Proposition 1.4.5.12) and that every horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \) is a monomorphism.

Example 3.1.2.4. Let \( i : A \hookrightarrow B \) be an inner anodyne morphism of simplicial sets (Definition 1.4.6.4). Then \( i \) is anodyne. The converse is false in general. For example, the horn inclusions \( \Lambda^n_0 \hookrightarrow \Delta^n \) and \( \Lambda^n_n \hookrightarrow \Delta^n \) are anodyne (for \( n > 0 \)), but are not inner anodyne.
3.1. THE HOMOTOPY THEORY OF KAN COMPLEXES

Remark 3.1.2.5. By construction, the collection of anodyne morphisms is weakly saturated. In particular:

- Every isomorphism of simplicial sets is anodyne.
- If $i : A \to B$ and $j : B \to C$ are anodyne morphisms of simplicial sets, then the composition $g \circ f$ is anodyne.
- For every pushout diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow^{i} & & \downarrow^{i'} \\
B & \rightarrow & B',
\end{array}
\]

if $i$ is anodyne, then $i'$ is also anodyne.
- Suppose there exists a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & A' & \rightarrow & A \\
\downarrow^{i} & & \downarrow^{i'} & & \downarrow^{i} \\
B & \rightarrow & B' & \rightarrow & B,
\end{array}
\]

where the horizontal compositions are the identity. If $i'$ is anodyne, then $i$ is anodyne.

Remark 3.1.2.6. Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:

(a) The morphism $f$ is a Kan fibration (Definition 3.1.1.1).

(b) For every square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow^{i} & & \downarrow^{f} \\
B & \rightarrow & S
\end{array}
\]

where $i$ is anodyne, there exists a dotted arrow rendering the diagram commutative.
The implication \((b) \Rightarrow (a)\) is immediate from the definitions (since the horn inclusions \(\Lambda^n_i \hookrightarrow \Delta^n\) are anodyne for \(n > 0\)). The reverse implication follows from the fact that the collection of those morphisms of simplicial sets \(i : A \rightarrow B\) which have the left lifting property with respect to \(f\) is weakly saturated (Proposition 1.4.4.16).

We will need the following stability properties for the class of anodyne morphisms:

**Proposition 3.1.2.7.** Let \(f : A \hookrightarrow B\) and \(f' : A' \hookrightarrow B'\) be monomorphisms of simplicial sets. If either \(f\) or \(f'\) is anodyne, then the induced map
\[
(A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'
\]
is anodyne.

The proof of Proposition 3.1.2.7 will require some preliminaries.

**Lemma 3.1.2.8.** For every pair of integers \(0 < i \leq n\), the horn inclusion \(f_0 : \Lambda^n_i \hookrightarrow \Delta^n\) is a retract of the inclusion map \(f : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n\).

**Proof.** Let \(A\) denote the simplicial subset of \(\Delta^1 \times \Delta^n\) given by the union of \(\Delta^1 \times \Lambda^n_i\) with \(\{1\} \times \Delta^n\). To prove Lemma 3.1.2.8, it will suffice to show that there exists a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
0 \times \Lambda^n_i & \rightarrow & A \\
\downarrow f_0 & & \downarrow f \\
0 \times \Delta^n & \rightarrow & \Delta^1 \times \Delta^n \rightarrow \Delta^n \\
\end{array}
\]
where the left horizontal maps are given by inclusion and the horizontal compositions are the identity maps. To achieve this, it suffices to choose \(r\) to be given on vertices by the map of partially ordered sets
\[
r : [1] \times [n] \rightarrow [n] \quad r(j, k) = \begin{cases} i & \text{if } j = 1 \text{ and } k \leq i \\ k & \text{otherwise.} \end{cases}
\]

**Lemma 3.1.2.9.** Let \(n\) be a nonnegative integer. Then there exists a chain of simplicial subsets
\[
X(0) \subset X(1) \subset \cdots \subset X(n) \subset X(n + 1) = \Delta^1 \times \Delta^n
\]
with the following properties:
(a) The simplicial $X(0)$ is given by the union of $\Delta^1 \times \partial \Delta^n$ with $\{1\} \times \Delta^n$ (and can therefore be described abstractly as the pushout $(\Delta^1 \times \partial \Delta^n) \amalg_{\{1\} \times \Delta^n}(\{1\} \times \Delta^n)$).

(b) For $0 \leq i \leq n$, the inclusion map $X(i) \hookrightarrow X(i+1)$ fits into a pushout diagram

\[
\begin{array}{ccc}
\Lambda_{i+1}^{n+1} & \longrightarrow & X(i) \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \longrightarrow & X(i+1).
\end{array}
\]

Proof. For $0 \leq i \leq n$, let $\sigma_i : \Delta^{n+1} \rightarrow \Delta^1 \times \Delta^n$ denote the map of simplicial sets given on vertices by the formula $\sigma_i(j) = \begin{cases} (0, j) & \text{if } j \leq i \\ (1, j-1) & \text{if } j > i. \end{cases}$ We define simplicial subsets $X(i) \subseteq \Delta^1 \times \Delta^n$ inductively by the formulae

\[
X(0) = (\Delta^1 \times \partial \Delta^n) \amalg (\{1\} \times \Delta^n) \quad X(i+1) = X(i) \cup \text{im}(\sigma_i),
\]

where $\text{im}(\sigma_i)$ denotes the image of the morphism $\sigma_i$. Note that $\Delta^1 \times \Delta^n$ is the union of the simplicial subsets $\{\text{im}(\sigma_i)\}_{0 \leq i \leq n}$, and is therefore equal to $X(n+1)$. This definition satisfies condition (a) by construction. To verify (b), it will suffice to show that for $0 \leq i \leq n$, the inverse image $A = \sigma_i^{-1}X(i)$ is equal to $\Lambda_{i+1}^{n+1}$ (as a simplicial subset of $\Delta^{n+1}$). Regarding $\sigma_i$ as an $(n+1)$-simplex of $\Delta^1 \times \Delta^n$, we are reduced to showing that the faces $d_j(\sigma_i)$ belong to $X(i)$ if and only if $j \neq i+1$. One direction is clear: the face $d_j(\sigma_i)$ is contained in $\Delta^1 \times \partial \Delta^n$ for $j \not\in \{i, i+1\}$, the face $d_i(\sigma_i) = d_i(\sigma_{i-1})$ is contained in $\text{im}(\sigma_{i-1}) \subseteq X(i)$ for $i > 0$, and $d_0(\sigma_0)$ is contained in $\{1\} \times \Delta^n$. To complete the proof, it suffices to show that the face $d_{i+1}(\sigma_i)$ is not contained in $X(i)$, which follows by inspection.

Proof of Proposition 3.1.2.7 Let us first regard the monomorphism $f' : A' \hookrightarrow B'$ as fixed, and let $T$ be the collection of all maps $f : A \rightarrow B$ for which the induced map

\[
(A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'
\]

is anodyne. We wish to show that every anodyne morphism belongs to $T$. Since $T$ is weakly saturated, it will suffice to show that every horn inclusion $f : \Lambda^1_i \hookrightarrow \Delta^n$ belongs to $T$ (for $n > 0$). Without loss of generality, we may assume that $0 < i$, so that $f$ is a retract of the map $g : (\Delta^1 \times \Lambda^1_i) \coprod_{\{1\} \times \Lambda^1_i}(\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n$ (Lemma 3.1.2.8). It will therefore suffice to show that $g$ belongs to $T$. Replacing $f'$ by the monomorphism $(\Lambda^1_i \times B') \coprod_{\Lambda^1_i \times A'}(\Delta^n \times A')$, we are reduced to showing that the inclusion $\{1\} \hookrightarrow \Delta^1$ belongs to $T$. 


Let $T'$ denote the collection of all morphisms of simplicial sets $f'' : A'' \to B''$ for which the map $(\{1\} \times B'') \coprod_{\{1\} \times A''}(\Delta^1 \times A'') \to \Delta^1 \times B''$ is anodyne. We will complete the proof by showing that $T'$ contains all monomorphisms of simplicial sets. By virtue of Proposition 1.4.5.12, it will suffice to show that $T''$ contains the inclusion map $\partial \Delta^m \hookrightarrow \Delta^m$, for each $m > 0$. In other words, we are reduced to showing that the inclusion $(\{1\} \times \Delta^m) \coprod_{\{1\} \times \partial \Delta^m}(\Delta^1 \times \partial \Delta^m) \hookrightarrow \Delta^1 \times \Delta^m$ is anodyne, which follows from Lemma 3.1.2.9.

3.1.3 Exponentiation of Kan Fibrations

Let $B$ and $X$ be simplicial sets. In §1.4.3, we showed that if $X$ is an $\infty$-category, then the simplicial set $\text{Fun}(B, X)$ is an $\infty$-category (Theorem 1.4.3.7). If $X$ is a Kan complex, we can say more: the simplicial set $\text{Fun}(B, X)$ is also a Kan complex (Corollary 3.1.3.4). This is a consequence of the following stronger result:

**Theorem 3.1.3.1.** Let $f : X \to S$ be a Kan fibration of simplicial sets, and let $i : A \hookrightarrow B$ be any monomorphism of simplicial sets. Then the induced map

$$\text{Fun}(B, X) \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)$$

is a Kan fibration.

**Proof.** By virtue of Remark 3.1.2.6, it will suffice to show that if $i' : A' \hookrightarrow B'$ is an anodyne morphism of simplicial sets, then every lifting problem of the form

$$A' \to \text{Fun}(B, X)$$

$$\downarrow i'$$

$$B' \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)$$

admits a solution. Equivalently, we must show that every lifting problem

$$(A \times B') \coprod_{A \times A'}(B \times A') \to X$$

$$\downarrow f$$

$$B \times B' \to S$$

admits a solution. This follows from Remark 3.1.2.6, since the left vertical map is anodyne (Proposition 3.1.2.7) and the right vertical map is a Kan fibration.

Let us note some special cases of Theorem 3.1.3.1 (which can be obtained by taking the simplicial set $A$ to be empty, the simplicial set $S$ to be $\Delta^0$, or both).
Corollary 3.1.3.2. Let \( f : X \to S \) be a Kan fibration of simplicial sets. Then, for every simplicial set \( B \), composition with \( f \) induces a Kan fibration \( \text{Fun}(B, X) \to \text{Fun}(B, S) \).

Corollary 3.1.3.3. Let \( X \) be a Kan complex. Then, for every monomorphism of simplicial sets \( i : A \hookrightarrow B \), the restriction map \( \text{Fun}(B, X) \to \text{Fun}(A, X) \) is a Kan fibration.

Corollary 3.1.3.4. Let \( X \) be a Kan complex and let \( B \) be an arbitrary simplicial set. Then the simplicial set \( \text{Fun}(B, X) \) is a Kan complex.

Theorem 3.1.3.1 has an analogue for trivial Kan fibrations:

Theorem 3.1.3.5. Let \( i : A \hookrightarrow B \) be an anodyne morphism of simplicial sets and let \( f : X \to S \) be a Kan fibration. Then the induced map

\[
\text{Fun}(B, X) \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\]

is a trivial Kan fibration.

Proof. We proceed as in the proof of Theorem 3.1.3.1. Let \( i' : A' \hookrightarrow B' \) be a monomorphism of simplicial sets; we must show that every lifting problem

\[
\begin{array}{ccc}
A' & \to & \text{Fun}(B, X) \\
\downarrow i' & & \downarrow \\
B' & \to & \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\end{array}
\]

admits a solution. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \to & X \\
\downarrow & & \downarrow f \\
B \times B' & \to & S
\end{array}
\]

admits a solution. This follows from Remark 3.1.2.6, since the left vertical map is anodyne (Proposition 3.1.2.7) and the right vertical map is a Kan fibration.

Taking \( S = \Delta^0 \) in the statement of Theorem 3.1.3.5 we obtain the following:

Corollary 3.1.3.6. Let \( i : A \hookrightarrow B \) be an anodyne morphism of simplicial sets and let \( X \) be a Kan complex. Then the restriction map \( \text{Fun}(B, X) \to \text{Fun}(A, X) \) is a trivial Kan fibration.

To formulate some further consequences of Theorem 3.1.3.1 it will be convenient to introduce some notation.

Construction 3.1.3.7. Let \( B \) and \( X \) be simplicial sets, and let \( \text{Fun}(B, X) \) be the simplicial set parametrizing morphisms from \( B \) to \( X \) (Construction 1.4.3.1).
• Suppose we are given another simplicial set $A$ equipped with a pair of morphism $i : A \to B$ and $f : A \to X$. In this case, we let $\text{Fun}_{A/}(B, X) \subseteq \text{Fun}(B, X)$ denote the fiber of the precomposition morphism $\text{Fun}(B, X) \circ i \to \text{Fun}(A, X)$ over the vertex $f \in \text{Fun}(A, X)$.

• Suppose we are given another simplicial set $S$ equipped with a pair of morphism $g : B \to S$ and $q : X \to S$. We let $\text{Fun}_{/S}(B, X) \subseteq \text{Fun}(B, X)$ denote the fiber of the postcomposition morphism $\text{Fun}(B, X) \circ q \to \text{Fun}(B, S)$ over the vertex $g \in \text{Fun}(B, S)$.

• Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow q \\
B & \longrightarrow & S.
\end{array}
\]

In this case, we let $\text{Fun}_{A//S}(B, X) \subseteq \text{Fun}(B, X)$ denote the simplicial subset given by the intersection $\text{Fun}_{A/}(B, X) \cap \text{Fun}_{/S}(B, X)$.

Remark 3.1.3.8. Let $B$ and $X$ be simplicial sets, and let us identify the set of vertices of $\text{Fun}(B, X)$ with the set of morphisms $f : B \to X$ in the category of simplicial sets. Then:

• Suppose we are given another simplicial set $A$ equipped with a pair of morphism $i : A \to B$ and $f : A \to X$. Then vertices of the simplicial set $\text{Fun}_{A/}(B, X)$ can be identified with morphisms $\tilde{f} : B \to X$ satisfying $f = \tilde{f} \circ i$.

• Suppose we are given another simplicial set $S$ equipped with a pair of morphism $g : B \to S$ and $q : X \to S$. Then vertices of the simplicial set $\text{Fun}_{/S}(B, X)$ can be identified with morphisms $\tilde{f} : B \to X$ satisfying $g = q \circ \tilde{f}$.

• Suppose we are given a square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow q \\
B & \longrightarrow & S.
\end{array}
\]

Then vertices of the simplicial set $\text{Fun}_{A//S}(B, X)$ can be identified with solutions of the associated lifting problem: that is, morphisms of simplicial sets $\tilde{f} : B \to X$ satisfying $f = \tilde{f} \circ i$ and $g = q \circ \tilde{f}$.

Remark 3.1.3.9. Suppose we are given a diagram of simplicial sets

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow q \\
B & \longrightarrow & S.
\end{array}
\]
which does not commute. Then the simplicial set \( \text{Fun}_{A/S}(B, X) = \text{Fun}_{A}(B, X) \cap \text{Fun}_{S}(B, X) \) of Construction 3.1.3.7 can still be defined, but is automatically empty.

**Remark 3.1.3.10.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{i} & \overset{g}{\nearrow} & \downarrow{q} \\
B & \xrightarrow{g} & S.
\end{array}
\]

Then:

- If \( S \simeq \Delta^0 \) is a final object of the category of simplicial sets, then we have an equality \( \text{Fun}_{A/S}(B, X) = \text{Fun}_{A}(B, X) \) (as simplicial subsets of \( \text{Fun}(B, X) \)).
- If \( A \simeq \emptyset \) is an initial object of the category of simplicial sets, then we have an equality \( \text{Fun}_{A/S}(B, X) = \text{Fun}_{S}(B, X) \) (as simplicial subsets of \( \text{Fun}(B, X) \)).
- If \( S \simeq \Delta^0 \) and \( A \simeq \emptyset \) are final and initial objects, respectively, then we have an equality \( \text{Fun}_{A/S}(B, X) = \text{Fun}(B, X) \).

**Remark 3.1.3.11.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{i} & \overset{g}{\nearrow} & \downarrow{q} \\
B & \xrightarrow{g} & S.
\end{array}
\]

Then the simplicial set \( \text{Fun}_{A/S}(B, X) \) can be identified with the fiber of the induced map

\[
\text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, X)
\]

over the vertex given by the pair \((f, g)\).

**Example 3.1.3.12.** Let \( q : X \to S \) be a morphism of simplicial sets. Then, for each vertex \( s \in S \), the simplicial set \( \text{Fun}_{S}(\{s\}, X) \) can be identified with the fiber \( X_s = \{s\} \times_S X \).

**Proposition 3.1.3.13.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{i} & \overset{g}{\nearrow} & \downarrow{q} \\
B & \xrightarrow{g} & S,
\end{array}
\]

where \( i \) is a monomorphism and \( q \) is a Kan fibration. Then the simplicial set \( \text{Fun}_{A/S}(B, X) \) is a Kan complex. If \( i \) is anodyne, then the Kan complex \( \text{Fun}_{A/S}(B, X) \) is contractible.
Proof. By virtue of Remark 3.1.3.11, the simplicial set \(\text{Fun}_{A/S}(B, X)\) can be identified with a fiber of the restriction map

\[
\theta : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, X).
\]

Theorem 3.1.3.1 guarantees that \(\theta\) is a Kan fibration, so its fibers are Kan complexes by virtue of Remark 3.1.1.9. If \(i\) is anodyne, then \(\theta\) is a trivial Kan fibration (Theorem 3.1.3.5), so its fibers are contractible Kan complexes (Remark 1.4.5.9).

Corollary 3.1.3.14. Let \(B\) be a simplicial set, let \(A \subseteq B\) be a simplicial subset, and let \(f : A \to X\) be a morphism of simplicial sets. If \(X\) is a Kan complex, then the simplicial set \(\text{Fun}_{A/(B, X)}\) is a Kan complex. If the inclusion \(A \hookrightarrow B\) is anodyne, then the Kan complex \(\text{Fun}_{A/(B, X)}\) is contractible.

Proof. Apply Proposition 3.1.3.13 in the special case \(S = \Delta^0\).

Corollary 3.1.3.15. Let \(q : X \to S\) be a Kan fibration of simplicial sets, and let \(g : B \to S\) be any morphism of simplicial sets. Then the simplicial set \(\text{Fun}_{/S}(B, X)\) is a Kan complex.

Proof. Apply Proposition 3.1.3.13 in the special case \(A = \emptyset\).

3.1.4 The Homotopy Category of Kan Complexes

The category of simplicial sets is equipped with a good notion of homotopy.

Definition 3.1.4.1. Let \(X\) and \(Y\) be simplicial sets, and suppose we are given a pair of maps \(f, g : X \to Y\), which we identify with vertices of the simplicial set \(\text{Fun}(X, Y)\). We will say that \(f\) and \(g\) are \emph{homotopic} if they belong to the same connected component of the simplicial set \(\text{Fun}(X, Y)\) (Definition 1.1.6.8).

Let us now make Definition 3.1.4.1 more concrete.

Definition 3.1.4.2. Let \(X\) and \(Y\) be simplicial sets, and suppose we are given a pair of morphisms \(f_0, f_1 : X \to Y\). A \emph{homotopy} from \(f_0\) to \(f_1\) is a morphism \(h : \Delta^1 \times X \to Y\) satisfying \(f_0 = h|_{\{0\} \times X}\) and \(f_1 = h|_{\{1\} \times X}\).

Remark 3.1.4.3 (Homotopy Extension Lifting Property). Let \(f : X \to S\) be a Kan fibration of simplicial sets. Suppose we are given a morphism of simplicial sets \(u : B \to X\) and a homotopy \(\overline{h}\) from \(f \circ u\) to another map \(\overline{v} : B \to S\). Then we can choose a map of simplicial sets \(h : \Delta^1 \times B \to X\) satisfying \(f \circ h = \overline{h}\) and \(h|_{\{0\} \times B} = u\); in other words, \(\overline{h}\) can be lifted to a homotopy \(h\) from \(u\) to another map \(v = h|_{\{1\} \times B}\). Moreover, given any simplicial subset \(A \subseteq B\) and any map \(h_0 : \Delta^1 \times A \to X\) satisfying \(f \circ h_0 = \overline{h}|_{\Delta^1 \times A}\) and \(h_0|_{\{0\} \times A} = u|_A\),
we can arrange that \( h \) is an extension of \( h_0 \). This follows from Theorem \ref{3.1.3.1} which guarantees that the restriction map

\[
\text{Fun}(B, X) \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\]

is a Kan fibration (and therefore has the right lifting property with respect to the inclusion \( \{0\} \hookrightarrow \Delta^1 \)). For a partial converse, see Corollary \ref{4.2.4.2}.

**Proposition 3.1.4.4.** Let \( X \) and \( Y \) be simplicial sets, and suppose we are given a pair of morphisms \( f, g : X \to Y \). Then:

- The morphisms \( f \) and \( g \) are homotopic if and only if there exists a sequence of morphisms \( f = f_0, f_1, \ldots, f_n = g \) from \( X \) to \( Y \) having the property that, for each \( 1 \leq i \leq n \), either there exists a homotopy from \( f_{i-1} \) to \( f_i \) or a homotopy from \( f_i \) to \( f_{i-1} \).

- Suppose that \( Y \) is a Kan complex. Then \( f \) and \( g \) are homotopic if and only if there exists a homotopy from \( f \) to \( g \).

**Proof.** The first assertion follows by applying Remark \ref{1.1.6.23} to the simplicial set \( \text{Fun}(X, Y) \). If \( Y \) is a Kan complex, then \( \text{Fun}(X, Y) \) is also a Kan complex (Corollary \ref{3.1.3.4}), so the second assertion follows from Proposition \ref{1.1.9.10}.

**Example 3.1.4.5.** Let \( X \) be a simplicial set and let \( Y \) be a topological space. Suppose we are given a pair of continuous functions \( f_0, f_1 : |X| \to Y \), corresponding to morphisms of simplicial sets \( f_0', f_1' : X \to \text{Sing}_\bullet(Y) \). Let \( h : [0, 1] \times |X| \to Y \) be a continuous function satisfying \( f_0 = h|\{0\} \times |X| \) and \( f_1 = h|\{1\} \times |X| \) (that is, a homotopy from \( f_0 \) to \( f_1 \) in the category of topological spaces). Then the composite map

\[
|\Delta^1 \times X| \xrightarrow{\theta} |\Delta^1| \times |X| = [0, 1] \times |X| \xrightarrow{h} Y
\]

classifies a morphism of simplicial sets \( h' : \Delta^1 \times X \to \text{Sing}_\bullet(Y) \), which is a homotopy from \( f_0' \) to \( f_1' \) (in the sense of Definition \ref{3.1.4.2}). We will show later that \( \theta \) is a homeomorphism of topological spaces (Corollary \ref{3.5.2.2}), so every homotopy from \( f_0 \) to \( f_1 \) arises in this way. In other words, the construction \( h \mapsto h' \) induces a bijection

\[
\{(\text{Continuous}) \text{ homotopies from } f_0 \text{ to } f_1\} \simeq \{(\text{Simplicial}) \text{ homotopies from } f_0' \text{ to } f_1'\}.
\]

**Example 3.1.4.6.** Let \( X \) and \( Y \) be topological spaces, and let \( h : [0, 1] \times X \to Y \) be a continuous function, which we regard as a homotopy from \( f_0 = h|\{0\} \times X \) to \( f_1 = h|\{1\} \times X \). Then \( h \) determines a homotopy between the induced map of simplicial sets \( \text{Sing}_\bullet(f_0), \text{Sing}_\bullet(f_1) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y) \): this follows by applying Example \ref{3.1.4.5} to the composite map \( [0, 1] \times |\text{Sing}_\bullet(X)| \to [0, 1] \times X \xrightarrow{h} Y \).
Example 3.1.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and suppose we are given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$, which we identify with morphisms of simplicial sets $N_\bullet(F), N_\bullet(G) : N_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{D})$. By definition, a homotopy from $N_\bullet(F)$ to $N_\bullet(G)$ is a map of simplicial sets

$$h : \Delta^1 \times N_\bullet(\mathcal{C}) \simeq N_\bullet([1] \times \mathcal{C}) \to N_\bullet(\mathcal{D})$$

satisfying $h|_{\{0\} \times N_\bullet(\mathcal{C})} = N_\bullet(F)$ and $h|_{\{1\} \times N_\bullet(\mathcal{C})} = N_\bullet(G)$. By virtue of Proposition 1.2.2.1, this is equivalent to the datum of a functor $H : [1] \times \mathcal{C} \to \mathcal{D}$ satisfying $H|_{\{0\} \times \mathcal{C}} = F$ and $H|_{\{1\} \times \mathcal{C}} = G$. In other words, we have a canonical bijection

$$\{\text{Natural transformations from } F \text{ to } G\} \sim \{\text{Homotopies from } N_\bullet(F) \text{ to } N_\bullet(G)\}.$$

In particular, if there exists a natural transformation from $F$ to $G$, then $N_\bullet(F)$ and $N_\bullet(G)$ are homotopic.

Example 3.1.4.8. Let $X$ be a simplicial set, let $M_\ast$ be a chain complex of abelian groups, and let $K(M_\ast)$ denote the associated Eilenberg-MacLane space (Construction 2.5.6.3). Suppose we are given a pair of morphisms $f, g : X \to K(M_\ast)$ in the category of simplicial sets, which we can identify with morphisms $f', g' : N_\ast(X; \mathbb{Z}) \to M_\ast$ in the category of chain complexes (Corollary 2.5.6.13); here $N_\ast(X; \mathbb{Z})$ denotes the normalized Moore complex of $X$ (Construction 2.5.5.9). The following conditions are equivalent:

1. The morphisms $f$ and $g$ are homotopic, in the sense of Definition 3.1.4.1.
2. The chain maps $f'$ and $g'$ are chain homotopic, in the sense of Definition 2.5.0.5.

To prove this, we note that (1) is equivalent to the assertion that there is a homotopy from $f$ to $g$ (since $K(M_\ast)$ is a Kan complex; see Remark 2.5.6.4): that is, a map of simplicial sets $h : \Delta^1 \times X \to K(M_\ast)$ satisfying $h|_{\{0\} \times X} = f$ and $h|_{\{1\} \times X} = g$. By virtue of Corollary 2.5.6.13, this is equivalent to the existence of a chain map $h' : N_\ast(\Delta^1 \times X; \mathbb{Z}) \to M_\ast$ which is compatible with $f'$ and $g'$. For any such chain map $h'$, the composition

$$N_\ast(\Delta^1) \boxtimes N_\ast(X; \mathbb{Z}) \xrightarrow{EZ} N_\ast(\Delta^1 \times X) \xrightarrow{h'} M_\ast$$

determines a chain homotopy from $f'$ to $g'$ (where EZ denotes the Eilenberg-Zilber homomorphism of Example 2.5.7.12). More explicitly, this chain homotopy is given by the map of graded abelian groups

$$N_\ast(X; \mathbb{Z}) \to M_{\ast+1} \quad \sigma \mapsto h'(\tau \triangledown \sigma),$$

where $\tau$ is the generator of $N_1(\Delta^1) \simeq \mathbb{Z}$ and $\triangledown$ is the shuffle product of Construction 2.5.7.9.

This proves that (1) implies (2). Conversely, if (2) is satisfied, then there exists a chain map...
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\[ u : N_*(\Delta^1) \boxtimes N_*(X; Z) \to M_\ast \]

compatible with \( f' \) and \( g' \), and we can verify (1) by taking \( h' \)
to be the composite map

\[ N_*(\Delta^1 \times X; Z) \xrightarrow{\text{AW}} N_*(\Delta^1) \boxtimes N_*(X; Z) \xrightarrow{u} M_\ast \]

where \( \text{AW} \) is the Alexander-Whitney homomorphism of Construction 2.5.8.6.

**Notation 3.1.4.9.** Let \( f : X \to Y \) be a morphism of simplicial sets. We let \([f]\) denote the
homotopy class of \( f \): that is, the image of \( f \) in the set \( \pi_0 \text{Fun}(X, Y) \) of homotopy classes of
maps from \( X \) to \( Y \).

**Construction 3.1.4.10 (The Homotopy Category of Kan Complexes).** We define a category
\( \text{hKan} \) as follows:

- The objects of \( \text{hKan} \) are Kan complexes.
- If \( X \) and \( Y \) are Kan complexes, then \( \text{Hom}_{\text{hKan}}(X, Y) = [X, Y] = \pi_0(\text{Fun}(X, Y)) \) is the
set of homotopy classes of morphisms from \( X \) to \( Y \).
- If \( X, Y, \) and \( Z \) are Kan complexes, then the composition law

\[ \circ : \text{Hom}_{\text{hKan}}(Y, Z) \times \text{Hom}_{\text{hKan}}(X, Y) \to \text{Hom}_{\text{hKan}}(X, Z) \]

is characterized by the formula \([g] \circ [f] = [g \circ f] \).

We will refer to \( \text{hKan} \) as the homotopy category of Kan complexes.

**Remark 3.1.4.11.** Let \( \text{Kan} \) denote the full subcategory of \( \text{Set}_\Delta \) spanned by the Kan
complexes, and let \( \mathcal{C} \) be any category. Then precomposition with the quotient map \( \text{Kan} \to \text{hKan} \)
induces an isomorphism from the functor category \( \text{Fun}(\text{hKan}, \mathcal{C}) \) to the full subcategory
of \( \text{Fun}_(\text{Kan}, \mathcal{C}) \) spanned by those functors \( F : \mathcal{C} \to \text{Kan} \) which satisfy the following condition:

\((*)\) If \( X \) and \( Y \) are Kan complexes and \( u_0, u_1 : X \to Y \) are homotopic morphisms, then
\( F(u_0) = F(u_1) \) in \( \text{Hom}_\mathcal{C}(F(X), F(Y)) \).

For many purposes, it is useful to consider a more refined version of Construction 3.1.4.10.

**Construction 3.1.4.12 (The \( \infty \)-Category of Spaces).** Let \( \text{Set}_\Delta \) denote the category of
simplicial sets, which we view as a simplicial category (where the simplicial set of morphisms
from \( X \) to \( Y \) is given by \( \text{Fun}(X, Y) \)). Let \( \text{Set}_\Delta \) denote the full subcategory of \( \text{Set}_\Delta \)
spanned by the Kan complexes, so that \( \text{Set}_\Delta \) inherits the structure of a simplicial category. We
let \( \mathcal{S} \) denote the homotopy coherent nerve \( \text{N}^\text{hc}_*(\text{Set}_\Delta) \). Corollary 3.1.3.4 implies that the
simplicial category \( \text{Set}_\Delta \) is locally Kan, so the simplicial set \( \mathcal{S} = \text{N}^\text{hc}_*(\text{Set}_\Delta) \) is an \( \infty \)-category
(Theorem 2.4.5.1). We will refer to \( \mathcal{S} \) as the \( \infty \)-category of spaces.

**Remark 3.1.4.13.** The definition of the homotopy category \( \text{hKan} \) can be viewed as a special
case of Construction 2.4.6.1 applied to the simplicial category \( \text{Set}_\Delta \) of Construction 3.1.4.12.
Invoking Proposition 2.4.6.8 we see that the category \( \text{hKan} \) of Construction 3.1.4.10 can be
identified with the homotopy category of the \( \infty \)-category \( \mathcal{S} \) (as suggested by the notation).
3.1.5 Homotopy Equivalences and Weak Homotopy Equivalences

Let \( f : X \to Y \) be a morphism of Kan complexes. We will say that \( f \) is a homotopy equivalence if the homotopy class \([f]\) is an isomorphism in the homotopy category \( \text{hKan} \) of Construction 3.1.4.10. This definition can be extended to more general simplicial sets in multiple ways.

Definition 3.1.5.1. Let \( f : X \to Y \) be a morphism of simplicial sets. We will say that a morphism \( g : Y \to X \) is a homotopy inverse to \( f \) if the compositions \( g \circ f \) and \( f \circ g \) are homotopic to the identity morphisms \( \text{id}_X \) and \( \text{id}_Y \), respectively (in the sense of Definition 3.1.4.2). We say that \( f : X \to Y \) is a homotopy equivalence if it admits a homotopy inverse \( g \).

Example 3.1.5.2. Let \( f : X \to Y \) be a homotopy equivalence of topological spaces. Then the induced map of singular simplicial sets \( \text{Sing}_\bullet(f) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y) \) is a homotopy equivalence (see Example 3.1.4.6).

Remark 3.1.5.3. Let \( f : X \to Y \) be a morphism of simplicial sets. The condition that \( f \) is a homotopy equivalence depends only on the homotopy class \([f] \in \pi_0(\text{Fun}(X,Y))\). Moreover, if \( f \) is a homotopy equivalence, then its homotopy inverse \( g : Y \to X \) is determined uniquely up to homotopy.

Remark 3.1.5.4. Let \( f : X \to Y \) be a morphism of Kan complexes. If \( f \) is a homotopy equivalence, then the induced map of fundamental groupoids \( \pi_{\leq 1}(f) : \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y) \) is an equivalence of categories. In particular, \( f \) induces a bijection \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \).

Remark 3.1.5.5. Let \( f : X \to Y \) be a morphism of simplicial sets. The following conditions are equivalent:

- The morphism \( f \) is a homotopy equivalence.
- For every simplicial set \( Z \), composition with \( f \) induces a bijection \( \pi_0(\text{Fun}(Y,Z)) \to \pi_0(\text{Fun}(X,Z)) \).
- For every simplicial set \( W \), composition with \( f \) induces a bijection \( \pi_0(\text{Fun}(W,X)) \to \pi_0(\text{Fun}(W,Y)) \).

In particular (taking \( W = \Delta^0 \)), if \( f \) is a homotopy equivalence, then the induced map \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \) is a bijection.

Remark 3.1.5.6 (Two-out-of-Six). Let \( f : W \to X \), \( g : X \to Y \), and \( h : Y \to Z \) be morphisms of simplicial sets. If \( g \circ f \) and \( h \circ g \) are homotopy equivalences, then \( f \), \( g \), and \( h \) are all homotopy equivalences.
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**Remark 3.1.5.7** (Two-out-of-Three). Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of simplicial sets. If any two of the morphisms \( f \), \( g \), and \( g \circ f \) are homotopy equivalences, then so is the third.

**Remark 3.1.5.8.** Let \( \{ f_i : X_i \to Y_i \}_{i \in I} \) be a collection of homotopy equivalences of simplicial sets indexed by a set \( I \), and let \( f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \) be their product. Then:

- If \( I \) is finite, then \( f \) is a homotopy equivalence. This follows from Remark 3.1.5.5 and Corollary 1.1.6.26.

- If each of the simplicial sets \( X_i \) and \( Y_i \) is a Kan complex, then \( f \) is a homotopy equivalence. This follows from Remark 3.1.5.5 and Corollary 1.1.9.11.

- The morphism \( f \) need not be a homotopy equivalence in general (see Warning 1.1.6.27).

We now give some more examples of homotopy equivalences.

**Proposition 3.1.5.9.** Let \( F : C \to D \) be a functor between categories, and suppose that \( F \) admits either a left or a right adjoint. Then the induced map \( N_\bullet(F) : N_\bullet(C) \to N_\bullet(D) \) is a homotopy equivalence of simplicial sets.

**Proof.** Without loss of generality, we may assume that \( F \) admits a right adjoint \( G : D \to C \). Then there exist natural transformations \( u : \text{id}_C \to G \circ F \) and \( v : F \circ G \to \text{id}_D \) witnessing an adjunction between \( F \) and \( G \), so that the maps \( N_\bullet(F) \) and \( N_\bullet(G) \) are homotopy inverses by virtue of Example 3.1.4.7.

**Proposition 3.1.5.10.** Let \( f : X \to S \) be a trivial Kan fibration of simplicial sets. Then \( f \) is a homotopy equivalence.

**Proof.** Since \( f \) is a trivial Kan fibration, the lifting problem

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & X \\
\downarrow & & \downarrow f \\
S & \rightarrow & X
\end{array}
\]

admits a solution (Proposition 1.4.5.3). We can therefore choose a morphism of simplicial sets \( g : S \to X \) which is a section of \( f \): that is, \( f \circ g \) is the identity morphism from \( S \) to itself. We will complete the proof by showing that \( g \) is a homotopy inverse to \( f \). In fact, we claim that there exists a homotopy \( h \) from \( \text{id}_X \) to the composition \( g \circ f \). This follows from
the solubility of the lifting problem

\[
\begin{array}{c}
\{0,1\} \times X \\
\downarrow \quad \downarrow \quad \downarrow \\
X & h & \rightarrow & S.
\end{array}
\]

When working with simplicial sets which are not Kan complexes, it is usually better to work with a more liberal notion of homotopy equivalence.

**Definition 3.1.5.11.** Let \( f : X \rightarrow Y \) be a morphism of simplicial sets. We will say that \( f \) is a weak homotopy equivalence if, for every Kan complex \( Z \), precomposition with \( f \) induces a bijection \( \pi_0(\text{Fun}(Y,Z)) \rightarrow \pi_0(\text{Fun}(X,Z)) \).

**Proposition 3.1.5.12.** Let \( f : X \rightarrow Y \) be a morphism of simplicial sets. If \( f \) is a homotopy equivalence, then it is a weak homotopy equivalence. The converse holds if \( X \) and \( Y \) are Kan complexes.

**Proof.** The first assertion follows from Remark 3.1.5.5. For the second, assume that \( f \) is a weak homotopy equivalence. If \( X \) is a Kan complex, then precomposition with \( f \) induces a bijection \( \pi_0(\text{Fun}(Y,X)) \rightarrow \pi_0(\text{Fun}(X,X)) \). We can therefore choose a map of simplicial sets \( g : Y \rightarrow X \) such that \( g \circ f \) is homotopic to the identity on \( X \). It follows that \( f \circ g \circ f \) is homotopic to \( f = \text{id}_Y \circ f \). Invoking the injectivity of the map \( \pi_0(\text{Fun}(Y,Y)) \rightarrow \pi_0(\text{Fun}(X,Y)) \), we conclude that \( f \circ g \) is homotopic to \( \text{id}_Y \), so that \( g \) is a homotopy inverse to \( f \). \( \square \)

**Proposition 3.1.5.13.** Let \( f : A \hookrightarrow B \) be an anodyne morphism of simplicial sets. Then \( f \) is a weak homotopy equivalence.

**Remark 3.1.5.14.** We will later prove a (partial) converse to Proposition 3.1.5.13: if a monomorphism of simplicial sets \( f : A \hookrightarrow B \) is a weak homotopy equivalence, then \( f \) is anodyne (see Corollary 3.3.7.5).

**Proof of Proposition 3.1.5.13.** Let \( i : A \hookrightarrow B \) be an anodyne morphism of simplicial sets; we wish to show that \( i \) is a weak homotopy equivalence. Let \( X \) be any Kan complex. It follows from Corollary 3.1.3.6 that the restriction map \( \theta : \text{Fun}(B,X) \rightarrow \text{Fun}(A,X) \) is a trivial Kan fibration. In particular, \( \theta \) is a homotopy equivalence (Proposition 3.1.5.10), and therefore induced a bijection on connected components \( \pi_0(\text{Fun}(B,X)) \rightarrow \pi_0(\text{Fun}(A,X)) \) (Remark 3.1.5.5). \( \square \)

**Remark 3.1.5.15 (Two-out-of-Six).** Let \( f : W \rightarrow X \), \( g : X \rightarrow Y \), and \( h : Y \rightarrow Z \) be morphisms of simplicial sets. If \( g \circ f \) and \( h \circ g \) are weak homotopy equivalences, then \( f \), \( g \), and \( h \) are all weak homotopy equivalences.
Remark 3.1.5.16 (Two-out-of-Three). Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of simplicial sets. If any two of the morphisms \( f, g, \) and \( g \circ f \) are weak homotopy equivalences, then so is the third.

Proposition 3.1.5.17. Let \( f : X \to Y \) be a weak homotopy equivalence of simplicial sets. Then the induced map of normalized chain complexes \( N_*(X;Z) \to N_*(Y;Z) \) is a chain homotopy equivalence. In particular, \( f \) induces an isomorphism of homology groups \( H_*(X;Z) \to H_*(Y;Z) \).

Proof. Let \( M_* \) be a chain complex of abelian groups. We wish to show that precomposition with \( N_*(f;Z) \) induces a bijection

\[
\{ \text{Chain homotopy classes of maps } N_*(Y;Z) \to M_* \} \quad \theta \quad \{ \text{Chain homotopy classes of maps } N_*(X;Z) \to M_* \}.
\]

Let \( K(M_*) \) denote the Eilenberg-MacLane space associated to \( M_* \) (Construction 2.5.6.3). Using Example 3.1.4.8, we can identify \( \theta \) with the map \( \pi_0(Fun(Y,K(M_*))) \to \pi_0(Fun(X,K(M_*))) \) given by precomposition with \( f \). This map is bijective because \( f \) is a weak homotopy equivalence (by assumption) and \( K(M_*) \) is a Kan complex (Remark 2.5.6.4). □

Remark 3.1.5.18. There is a partial converse to Proposition 3.1.5.17. If \( f : X \to Y \) is a morphism between simply-connected simplicial sets and the induced map \( H_*(X;Z) \to H_*(Y;Z) \) is an isomorphism, one can show that \( f \) is a weak homotopy equivalence. Beware that this is not necessarily true if \( X \) and \( Y \) are not simply connected (see §[?] for further discussion).

Remark 3.1.5.19 (Coproducts of Weak Homotopy Equivalences). Let \( \{f(i) : X(i) \to Y(i)\}_{i \in I} \) be a collection of weak homotopy equivalences of simplicial sets indexed by a set \( I \). For every Kan complex \( Z \), we have a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Fun}(\prod_{i \in I} Y(i), Z) & \xrightarrow{\sim} & \text{Fun}(\prod_{i \in I} X(i), Z) \\
\downarrow & & \downarrow \\
\prod_{i \in I} \text{Fun}(Y(i), Z) & \xrightarrow{\sim} & \prod_{i \in I} \text{Fun}(X(i), Z),
\end{array}
\]

where the vertical maps are isomorphisms. Passing to the connected components (and using the fact that the functor \( Q \mapsto \pi_0(Q) \) preserves products when restricted to Kan complexes; see Corollary 1.1.9.13), we deduce that the map \( \pi_0(\text{Fun}(\prod_{i \in I} Y(i), Z)) \to \pi_0(\text{Fun}(\prod_{i \in I} X(i), Z)) \) is bijective. Allowing \( Z \) to vary, we conclude that the induced map \( \prod_{i \in I} X(i) \to \prod_{i \in I} Y(i) \) is also a weak homotopy equivalence.
Exercise 3.1.5.20. Let $G$ be the directed graph depicted in the diagram

$$
0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots
$$

and let $G$ denote the associated 1-dimensional simplicial set (see Warning 1.1.6.27). Show that the projection map $G \rightarrow \Delta^0$ is a weak homotopy equivalence, but not a homotopy equivalence.

Warning 3.1.5.21. Let $X$ and $Y$ be simplicial sets. The existence of a weak homotopy equivalence $f : X \rightarrow Y$ does not guarantee the existence of a weak homotopy equivalence $g : Y \rightarrow X$.

3.1.6 Fibrant Replacement

The formalism of Kan complexes is extremely useful as a combinatorial foundation for homotopy theory. However, when studying the homotopy theory of Kan complexes, it is often necessary to contemplate more general simplicial sets. For example, if $f_0, f_1 : S \rightarrow T$ are morphisms of Kan complexes, then a homotopy from $f_0$ to $f_1$ is defined as a morphism of simplicial sets $h : \Delta^1 \times S \rightarrow T$; here neither $\Delta^1$ nor the product $\Delta^1 \times S$ is a Kan complex (except in the trivial case $S = \emptyset$; see Exercise 1.1.9.2). When working with a simplicial set $X$ which is not a Kan complex, it is often convenient to replace $X$ by a Kan complex having the same weak homotopy type. This can always be achieved: more precisely, one can always find a weak homotopy equivalence $X \rightarrow Q$, where $Q$ is a Kan complex (Corollary 3.1.6.2).

Our goal in this section is to prove a “fiberwise” version of this result, which can be stated as follows:

**Proposition 3.1.6.1.** Let $f : X \rightarrow Y$ be a morphism of simplicial sets. Then $f$ can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where $f''$ is a Kan fibration and $f'$ is anodyne (hence a weak homotopy equivalence, by virtue of Proposition 3.1.5.13). Moreover, the simplicial set $Q(f)$ (and the morphisms $f'$ and $f''$) can be chosen to depend functorially on $f$, in such a way that the functor

$$\text{Fun}([1], \text{Set}_\Delta) \rightarrow \text{Set}_\Delta \quad (f : X \rightarrow Y) \rightarrow Q(f)$$

commutes with filtered colimits.

Before giving the proof of Proposition 3.1.6.1, let us note some of its consequences. Applying Proposition 3.1.6.1 in the special case $Y = \Delta^0$, we obtain the following:

**Corollary 3.1.6.2.** Let $X$ be a simplicial set. Then there exists an anodyne morphism $f : X \rightarrow Q$, where $Q$ is a Kan complex.
Remark 3.1.6.3. In the situation of Corollary 3.1.6.2, the Kan complex \( Q \) (and the anodyne morphism \( f \)) can be chosen to depend functorially on \( X \). This follows from the proof of Proposition 3.1.6.1 given below, but there are other (arguably more elegant) ways to achieve the same result. For example, we can take \( Q \) to be the simplicial set \( \text{Ex}^\infty(X) \) of Construction 3.3.6.1 (see Propositions 3.3.6.9 and 3.3.6.7), or the singular simplicial set \( \text{Sing}_\bullet(|X|) \) (see Proposition 1.1.9.8 and Theorem 3.5.4.1). These constructions also have non-aesthetic advantages: for example, the functors \( X \mapsto \text{Ex}^\infty(X) \) and \( X \mapsto \text{Sing}_\bullet(|X|) \) both preserve finite limits.

Corollary 3.1.6.4. Let \( f: X \to Y \) be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism \( f \) is anodyne.
2. The morphism \( f \) has the left lifting property with respect to Kan fibrations. That is, if \( g: Z \to S \) is a Kan fibration of simplicial sets, then every lifting problem

\[
\begin{array}{ccc}
X & \longrightarrow & Z \\
\downarrow f & \nearrow g & \\
Y & \longrightarrow & S
\end{array}
\]

admits a solution.

Proof. The implication (1) \( \Rightarrow \) (2) follows from Remark 3.1.2.6. To deduce the converse, we first apply Proposition 3.1.6.1 to write \( f \) as a composition \( X \xrightarrow{f'} Q \xrightarrow{f''} Y \), where \( f' \) is anodyne and \( f'' \) is a Kan fibration. If \( f \) satisfies condition (2), then the lifting problem

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Q \\
\downarrow f & \nearrow f'' & \\
Y & \xrightarrow{id} & Y
\end{array}
\]

admits a solution. It follows that \( f \) is a retract of \( f' \) (in the arrow category \( \text{Fun}([1], \text{Set}_\Delta) \)). Since the collection of anodyne morphisms is closed under retracts, it follows that \( f \) is anodyne.

Corollary 3.1.6.5. Let \( f: X \to Y \) be a morphism of simplicial sets, and let \( Z \) be a Kan complex. If \( f \) is a weak homotopy equivalence, then composition with \( f \) induces a homotopy equivalence \( \text{Fun}(Y, Z) \to \text{Fun}(X, Z) \).

Proof. Using Corollary 3.1.6.2, we can choose an anodyne morphism \( g: Y \to Y' \), where \( Y' \) is a Kan complex. Using Proposition 3.1.6.1 we can factor \( g \circ f \) as a composition
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$X \xrightarrow{g'} X' \xrightarrow{f'} Y'$, where $g'$ is anodyne and $f'$ is a Kan fibration. We then have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g'} & & \downarrow{g} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

where $f'$ is a Kan fibration between Kan complexes and the vertical maps are anodyne, and therefore weak homotopy equivalences. Using the two-out-of-three property, we deduce that $f'$ is also a weak homotopy equivalence (Remark 3.1.5.16). It follows that $f'$ is a homotopy equivalence (Proposition 3.1.5.12). We then obtain a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Fun}(X, Z) & \xleftarrow{\sim} & \text{Fun}(Y, Z) \\
\downarrow & & \downarrow \\
\text{Fun}(X', Z) & \xleftarrow{\sim} & \text{Fun}(Y', Z)
\end{array}
\]

where the lower horizontal map is a homotopy equivalence, and the vertical maps are trivial Kan fibrations (Corollary 3.1.3.6). In particular, the vertical maps are homotopy equivalences (Proposition 3.1.5.10), so the two-out-of-three property guarantees that the upper horizontal map is also a homotopy equivalence (Remark 3.1.5.7).

Recall that the homotopy category $h\text{Kan}$ of Construction 3.1.4.10 is defined as a quotient of the category of Kan complexes $\text{Kan}$ (by identifying morphisms which are homotopic). However, it can also be described as a localization of $\text{Kan}$, obtained by inverting the class of homotopy equivalences.

**Proposition 3.1.6.6.** Let $\mathcal{C}$ be a category and let $F : \text{Kan} \rightarrow \mathcal{C}$ be a functor. The following conditions are equivalent:

$(\ast)$ If $X$ and $Y$ are Kan complexes and $u_0, u_1 : X \rightarrow Y$ are homotopic morphisms, then $F(u_0) = F(u_1)$ in $\text{Hom}_\mathcal{C}(F(X), F(Y))$.

$(\ast')$ For every homotopy equivalence of Kan complexes $u : X \rightarrow Y$, the induced map $F(u) : F(X) \rightarrow F(Y)$ is an isomorphism in the category $\mathcal{C}$.

**Proof.** The implication $(\ast) \Rightarrow (\ast')$ is immediate (note that a morphism of Kan complexes $u : X \rightarrow Y$ is a homotopy equivalence if and only if its homotopy class $[u]$ is an isomorphism in the homotopy category $h\text{Kan}$). For the converse, assume that $(\ast')$ is satisfied, let $X$ and
Y be Kan complexes, and let $u_0, u_1 : X \to Y$ be a pair of homotopic morphisms. Let us regard $u_0$ and $u_1$ as vertices of the Kan complex $\text{Fun}(X, Y)$. Since $u_0$ and $u_1$ are homotopic, there exists an edge $e : \Delta^1 \to \text{Fun}(X, Y)$ satisfying $e(0) = u_0$ and $e(1) = u_1$. By virtue of Proposition 3.1.6.1, this morphism factors as a composition $\Delta^1 \overset{e} \to Q \overset{e''} \to \text{Fun}(X, Y)$, where $e'$ is anodyne and $e''$ is a Kan fibration. Since $\text{Fun}(X, Y)$ is a Kan complex (Corollary 3.1.3.4), it follows that $Q$ is also a Kan complex. Let us identify $e''$ with a morphism of Kan complexes $h : Q \times X \to Y$. Let $i_0 : X \hookrightarrow Q \times X$ be the product of the identity map $\text{id}_X$ with the inclusion $\{e'(0)\} \hookrightarrow Q$, and define $i_1 : X \hookrightarrow Q \times X$ similarly. Since $e'$ is anodyne, the restrictions $e'|_{\{0\}}$ and $e'|_{\{1\}}$ are anodyne. In particular, they are weak homotopy equivalences (Proposition 3.1.5.13) and therefore homotopy equivalences (Proposition 3.1.5.12), since $Q$ is a Kan complex. It follows that $i_0$ and $i_1$ are also homotopy equivalences, so that $F(i_0)$ and $F(i_1)$ are isomorphisms (by virtue of assumption (*')). Using the fact that $i_0$ and $i_1$ are left inverse to the projection map $\pi : Q \times X \to X$, we see that $F(\pi)$ is an isomorphism in $\mathcal{C}$ and that we have

$$F(u_0) = F(h) \circ F(i_0) = F(h) \circ F(\pi)^{-1} = F(h) \circ F(i_1) = F(u_1),$$

as desired.

**Corollary 3.1.6.7.** Let $\mathcal{C}$ be a category, let $\mathcal{E} \subseteq \text{Fun}(\text{Kan}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \text{Kan} \to \mathcal{C}$ which carry homotopy equivalences of Kan complexes to isomorphisms in the category $\mathcal{C}$. Then precomposition with the quotient map $\text{Kan} \to h\text{Kan}$ induces an isomorphism of categories $\text{Fun}(h\text{Kan}, \mathcal{C}) \to \mathcal{E}$.

**Proof.** Combine Remark 3.1.4.11 with Proposition 3.1.6.6.

**Variant 3.1.6.8.** Let $\mathcal{C}$ be a category, and let $\mathcal{E}' \subseteq \text{Fun}(\text{Set}_\Delta, \mathcal{C})$ be the full subcategory spanned by those functors $F : \text{Set}_\Delta \to \mathcal{C}$ which carry weak homotopy equivalences of simplicial sets to isomorphisms in the category $\mathcal{C}$. Then:

(a) For every functor $F \in \mathcal{E}'$, the restriction $F|_{\text{Kan}}$ factors (uniquely) as a composition $\text{Kan} \to h\text{Kan} \overset{F} \to \mathcal{C}$.

(b) The construction $F \mapsto \overline{F}$ induces an equivalence of categories $\mathcal{E}' \to \text{Fun}(h\text{Kan}, \mathcal{C})$.

**Remark 3.1.6.9.** Corollary 3.1.6.7 and Variant 3.1.6.8 can be stated more informally as follows:

- The homotopy category $h\text{Kan}$ can be obtained from the category $\text{Kan}$ of Kan complexes by formally adjoining inverses to all homotopy equivalences.

- The homotopy category $h\text{Kan}$ can be obtained from the category $\text{Set}_\Delta$ of simplicial sets by formally adjoining inverses to all weak homotopy equivalences.
Either of these assertions characterizes the homotopy category hKan up to equivalence (in fact, Corollary 3.1.6.7 even characterizes hKan up to isomorphism).

Proof of Variant 3.1.6.8. Let $\mathcal{E} \subseteq \text{Fun}(\text{Kan}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \text{Kan} \to \mathcal{C}$ which carry homotopy equivalences of Kan complexes to isomorphisms in $\mathcal{C}$. By virtue of Corollary 3.1.6.7 it will suffice to show that the restriction functor $F \mapsto F|_{\text{Kan}}$ induces an equivalence of categories $\mathcal{E}' \to \mathcal{E}$. Using Proposition 3.1.6.1, we can choose a functor $Q : \text{Set}_\Delta \to \text{Kan}$ and a natural transformation $u : \text{id}_{\text{Set}_\Delta} \to Q$ with the property that, for every Kan complex $X$, the induced map $u_X : X \to Q(X)$ is anodyne. For every morphism of simplicial sets $f : X \to Y$, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u_X} & & \downarrow{u_Y} \\
Q(X) & \xrightarrow{Q(f)} & Q(Y),
\end{array}
$$

where the vertical maps are weak homotopy equivalences (Proposition 3.1.5.13). It follows that if $f$ is a weak homotopy equivalence, then $Q(f)$ is also a weak homotopy equivalence (Remark 3.1.5.16) and therefore a homotopy equivalence (Proposition 3.1.5.12). In other words, the functor $Q$ carries weak homotopy equivalences of simplicial sets to homotopy equivalences of Kan complexes. It follows that precomposition with $Q$ induces a functor $\theta : \mathcal{E} \to \mathcal{E}'$. We claim that $\theta$ is homotopy inverse to the restriction functor $\mathcal{E}' \to \mathcal{E}$. This follows from the following pair of observations:

- For every functor $F : \text{Set}_\Delta \to \mathcal{C}$, $u$ induces a natural transformation $F \to F|_{\text{Kan}} \circ Q$, which depends functorially on $F$ and is an isomorphism for $F \in \mathcal{E}'$.

- For every functor $F_0 : \text{Kan} \to \mathcal{C}$, $u$ induces a natural transformation $F_0 \to (F_0 \circ Q)|_{\text{Kan}}$, which depends functorially on $F_0$ and is an isomorphism for $F_0 \in \mathcal{E}$.

We now turn to the proof of Proposition 3.1.6.1. We will use an easy version of Quillen’s “small object argument” (which we will revisit in greater generality in §[?]).

Proof of Proposition 3.1.6.1. Let $f : X \to Y$ be a morphism of simplicial sets. We construct a sequence of simplicial sets $\{X(m)\}_{m \geq 0}$ and morphisms $f(m) : X(m) \to Y$ by recursion. Set $X(0) = X$ and $f(0) = f$. Assuming that $f(m) : X(m) \to Y$ has been defined, let $S(m)$
denote the set of all commutative diagrams \( \sigma : \Lambda^n_i \to X(m) \to \Delta^n \to Y \), where \( 0 \leq i \leq n \), \( n > 0 \), and the left vertical map is the inclusion. For every such commutative diagram \( \sigma \), let \( C_\sigma = \Lambda^n_i \) denote the upper left hand corner of the diagram \( \sigma \), and \( D_\sigma = \Delta^n \) the lower left hand corner. Form a pushout diagram

\[
\begin{array}{ccc}
\Pi_{\sigma \in S(m)} C_\sigma & \rightarrow & X(m) \\
\downarrow & & \downarrow \\
\Pi_{\sigma \in S(m)} D_\sigma & \rightarrow & X(m + 1)
\end{array}
\]

and let \( f(m + 1) : X(m + 1) \to Y \) be the unique map whose restriction to \( X(m) \) is equal to \( f(m) \) and whose restriction to each \( D_\sigma \) is equal to \( u_\sigma \). By construction, we have a direct system of anodyne morphisms

\[
X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots
\]

Set \( Q(f) = \lim_{\to m} X(m) \). Then the natural map \( f' : X \to Q(f) \) is anodyne (since the collection of anodyne maps is closed under transfinite composition), and the system of morphisms \( \{ f(m) \}_{m \geq 0} \) can be amalgamated to a single map \( f'' : Q(f) \to Y \) satisfying \( f = f'' \circ f' \). It is clear from the definition that the construction \( f \mapsto Q(f) \) is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that \( f'' \) is a Kan fibration: that is, that every lifting problem \( \sigma : \Lambda^n_i \to \Delta^n \to Y \) admits a solution (provided that \( n > 0 \)). Let us abuse notation by identifying each \( X(m) \) with its image in \( Q(f) \). Since \( \Lambda^n_i \) is a finite simplicial set, its image under \( v \) is contained in
$X(m)$ for some $m \gg 0$. In this case, we can identify $\sigma$ with an element of the set $S(m)$, so that the lifting problem

$$
\begin{array}{ccc}
\Lambda^n & \xrightarrow{v} & X(m+1) \\
\downarrow & & \downarrow f(m+1) \\
\Delta^n & \xrightarrow{\text{\textbullet}} & Y
\end{array}
$$

admits a solution by construction.

**Example 3.1.6.10 (Path Fibrations).** If $f : X \to Y$ is a morphism of Kan complexes, then we can give a much more explicit proof of Proposition 3.1.6.1. Let $Q(f)$ denote the fiber product $X \times_{\text{Fun}(\{0\},Y)} \text{Fun}(\Delta^1,Y)$. Then $f$ factors as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where $f''$ is given by evaluation at the vertex $\{1\} \subseteq \Delta^1$ and $f'$ is obtained by amalgamating the identity morphism $\text{id}_X$ with the composition $X \xrightarrow{f} Y \xrightarrow{\delta} \text{Fun}(\Delta^1,Y)$. Moreover:

- The morphism $f'$ is a section of the projection map $Q(f) \to X$, which is a pullback of the evaluation map $\text{Fun}(\Delta^1,Y) \to \text{Fun}(\{0\},Y)$ and therefore a trivial Kan fibration (Corollary 3.1.3.6). It follows that $f'$ is a weak homotopy equivalence. Since it is also a monomorphism, it is anodyne (see Corollary 3.3.7.5).

- The morphism $f''$ factors as a composition

  $$
  Q(f) = X \times_{\text{Fun}(\{0\},Y)} \text{Fun}(\Delta^1,Y) \xrightarrow{u} X \times \text{Fun}(\{1\},Y) \xrightarrow{v} Y,
  $$

  where $u$ is a pullback of the restriction map $\text{Fun}(\Delta^1,Y) \to \text{Fun}(\partial \Delta^1,Y)$ (and therefore a Kan fibration by virtue of Corollary 3.1.3.3) and $v$ is a pullback of the projection map $X \to \Delta^0$ (and therefore a Kan fibration by virtue of our assumption that $X$ is a Kan complex). It follows that $f''$ is also a Kan fibration.

Then $q$ admits a section $f' : X \to Q(f)$ (whose second component is obtained by composing $f$ with the diagonal embedding $Y \hookrightarrow \text{Fun}(\Delta^1,Y)$).

The proof of Proposition 3.1.6.1 can be repurposed to obtain many analogous results.

**Exercise 3.1.6.11.** Let $f : X \to Y$ be a morphism of simplicial sets. Show that $f$ can be factored as a composition $X \xrightarrow{f'} P(f) \xrightarrow{f''} Y$, where $f'$ is a monomorphism and $f''$ is a trivial Kan fibration.
3.2 Homotopy Groups

Our goal in this section is to address the following:

**Question 3.2.0.1.** Let $f : X \to Y$ be a morphism of Kan complexes. Under what conditions does $f$ admit a homotopy inverse $g : Y \to X$?

Let us begin with a partial answer to Question 3.2.0.1. For every Kan complex $X$, let $\pi_{\leq 1}(X)$ denote the fundamental groupoid of $X$ (Definition 1.3.6.13). For each vertex $x \in X$, we let $\pi_1(X,x)$ denote the automorphism group $\text{Aut}_{\pi_{\leq 1}(X)}(x) = \text{Hom}_{\pi_{\leq 1}(X)}(x,x)$; we will refer to $\pi_1(X,x)$ as the fundamental group of $X$ (with respect to the base point $x$). Every morphism of Kan complexes $f : X \to Y$ induces a functor $\pi_{\leq 1}(f) : \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$. Moreover, if $f$ is a homotopy equivalence, then $\pi_{\leq 1}(f)$ is an equivalence of categories (Remark 3.1.5.4). In other words, every homotopy equivalence $f : X \to Y$ satisfies the following pair of conditions:

1. **(W0)** The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of sets: that is, $f$ induces a bijection from the set of connected components of $X$ to the set of connected components of $Y$.

2. **(W1)** For every choice of vertex $x \in X$ having image $y = f(x) \in Y$, the induced map of fundamental groups $\pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism.

However, these observations do not supply a complete answer to Question 3.2.0.1: conditions (W0) and (W1) are necessary for $f$ to be a homotopy equivalence, but they are not sufficient. In this section, we will remedy the situation by introducing a hierarchy of additional invariants. To each Kan complex $X$ and each vertex $x \in X$, we will associate a sequence of sets $\{\pi_n(X,x)\}_{n \geq 0}$, which enjoy the following features:

- For every nonnegative integer $n$, $\pi_n(X,x)$ is defined as the set of homotopy classes of pointed maps from the quotient $\Delta^n/\partial\Delta^n$ to $X$ (Construction 3.2.2.4). Here it is important to work in the homotopy theory of pointed simplicial sets, which we review in §3.2.1.

- When $n = 0$, we can identify $\pi_n(X,x)$ with the set $\pi_0(X)$ of connected components of $X$: in particular, it does not depend on the choice of base point $x$ (Example 3.2.2.6).

- For $n > 0$, the set $\pi_n(X,x)$ comes equipped with a natural group structure (Theorem 3.2.2.10), which we will construct in §3.2.3. For this reason, we will refer to $\pi_n(X,x)$ as the $n$th homotopy group of $X$ (with respect to the base point $x$). Moreover, the group $\pi_n(X,x)$ is abelian for $n \geq 2$. 


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• When $n = 1$, we can identify $\pi_1(X, x)$ with the fundamental group of $X$ as defined earlier: that is, with the automorphism group of $x$ as an object of the homotopy category $\pi_{\leq 1}(X)$ (Example 3.2.12).

• Let $f : X \to S$ be a Kan fibration between Kan complexes, let $x \in X$ be a vertex having image $s = f(x) \in S$, and let $X_s = \{s\} \times_S X$ denote the fiber of $f$ over the vertex $s$. Then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(S, s) \xrightarrow{\partial} \pi_n(X_s, x) \to \pi_n(X, x) \to \pi_n(S, s) \xrightarrow{\partial} \pi_{n-1}(X_s, x) \to \cdots$$

We construct this sequence in §3.2.4 and prove its exactness in §3.2.5 (Theorem 3.2.5.1).

• Let $f : X \to Y$ be a morphism of Kan complexes. In §3.2.6, we show that $f$ is a homotopy equivalence if and only if it induces a bijection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ and an isomorphism of homotopy groups $\pi_n(X, x) \to \pi_n(Y, f(x))$, for every choice of base point $x \in X$ and every positive integer $n$ (Theorem 3.2.6.1). This is a simplicial counterpart of a classical result of Whitehead ([39]). In §3.2.7, we apply this result to deduce some closure properties for the class of homotopy equivalences (Propositions 3.2.7.1 and 3.2.7.3).

3.2.1 Pointed Kan Complexes

In §3.1.4, we showed that the collection of Kan complexes can be organized into a category $h\text{Kan}$ whose morphisms are given by homotopy classes of maps (Construction 3.1.4.10). In this section, we describe a variant of this construction for Kan complexes which are equipped with a specified base point.

**Definition 3.2.1.1.** A pointed simplicial set is a pair $(X, x)$, where $X$ is a simplicial set and $x$ is a vertex of $X$. If $X$ is a Kan complex, then we refer to the pair $(X, x)$ as a pointed Kan complex.

**Remark 3.2.1.2.** We will often abuse terminology by identifying a pointed simplicial set $(X, x)$ with the underlying simplicial set $X$. In this case, we will refer to $x$ as the base point of $X$.

**Definition 3.2.1.3.** Let $(X, x)$ and $(Y, y)$ be simplicial sets, and suppose we are given a pair of pointed maps $f, g : X \to Y$, which we identify with vertices of the simplicial set $\text{Fun}(X, Y) \times_{\text{Fun}(\{x\}, Y)} \{y\}$. We will say that $f$ and $g$ are pointed homotopic if they belong to the same connected component of $\text{Fun}(X, Y) \times_{\text{Fun}(\{x\}, Y)} \{y\}$ (Definition 1.1.6.8).

**Definition 3.2.1.4.** Let $(X, x)$ and $(Y, y)$ be pointed simplicial sets, and suppose we are given a pair of pointed maps $f_0, f_1 : X \to Y$. A pointed homotopy from $f_0$ to $f_1$ is a
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morphism \( h : \Delta^1 \times X \to Y \) for which \( f_0 = h|_{\{0\} \times X} \), \( f_1 = h|_{\{1\} \times X} \), and \( h|_{\Delta^1 \times \{x\}} \) is the degenerate edge associated to the vertex \( y \in Y \).

**Proposition 3.2.1.5.** Let \((X,x)\) and \((Y,y)\) be pointed simplicial sets, and suppose we are given a pair of pointed morphisms \( f, g : X \to Y \). Then:

- The morphisms \( f \) and \( g \) are pointed homotopic if and only if there exists a sequence of pointed morphisms \( f = f_0, f_1, \ldots, f_n = g \) from \( X \) to \( Y \) having the property that, for each \( 1 \leq i \leq n \), either there exists a pointed homotopy from \( f_{i-1} \) to \( f_i \) or a pointed homotopy from \( f_i \) to \( f_{i-1} \).

- Suppose that \( Y \) is a Kan complex. Then \( f \) and \( g \) are pointed homotopic if and only if there exists a pointed homotopy from \( f \) to \( g \).

**Proof.** The first assertion follows by applying Remark 1.1.6.23 to the simplicial set \( \text{Fun}(X,Y) \times_{\text{Fun}(\{x\},Y)} \{y\} \).

If \( Y \) is a Kan complex, then the evaluation map \( \text{Fun}(X,Y) \to \text{Fun}(\{x\},Y) \) is a Kan fibration (Corollary 3.1.3.3), so the fiber \( \text{Fun}(X,Y) \times_{\text{Fun}(\{x\},Y)} \{y\} \) is a Kan complex (Remark 3.1.1.9). The second assertion now follows from Proposition 1.1.9.10.

**Example 3.2.1.6.** Let \((X,x)\) be a pointed simplicial set and let \((Y,y)\) be a pointed topological space. Suppose we are given a pair of continuous functions \( f_0, f_1 : |X| \to Y \) carrying \( x \) to \( y \), which we can identify with pointed morphisms \( f'_0, f'_1 : X \to \text{Sing}(Y) \). Let \( h : [0,1] \times |X| \to Y \) be a continuous function satisfying \( f_0 = h|_{\{0\} \times |X|} \), \( f_1 = h|_{\{1\} \times |X|} \), and \( h(t,x) = y \) for \( 0 \leq t \leq 1 \) (that is, \( h \) is a pointed homotopy from \( f_0 \) to \( f_1 \) in the category of topological spaces). Then the composite map

\[
|\Delta^1 \times X| \xrightarrow{\theta} |\Delta^1| \times |X| = [0,1] \times |X| \xrightarrow{h} Y
\]

classifies a morphism of simplicial sets \( h' : \Delta^1 \times X \to \text{Sing}(Y) \), which is a pointed homotopy from \( f'_0 \) to \( f'_1 \) (in the sense of Definition 3.2.1.4). By virtue of Corollary 3.5.2.2, the map \( \theta \) is a homeomorphism, so every pointed homotopy from \( f_0 \) to \( f_1 \) arises in this way. In other words, the construction \( h \mapsto h' \) induces a bijection

\[
\{(\text{Continuous pointed homotopies from } f_0 \text{ to } f_1)\} \xrightarrow{\sim} \{(\text{Simplicial pointed homotopies from } f'_0 \text{ to } f'_1)\}.
\]

**Example 3.2.1.7.** Let \((X,x)\) and \((Y,y)\) be pointed topological spaces, and let \( h : [0,1] \times X \to Y \) be a continuous function satisfying \( h(t,x) = y \) for \( 0 \leq t \leq 1 \), which we regard as
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a pointed homotopy from \( f_0 = h|_{\{0\}} \times X \) to \( f_1 = h|_{\{1\}} \times X \). Then \( h \) determines a homotopy between the induced map of simplicial sets \( \text{Sing}_\bullet(f_0), \text{Sing}_\bullet(f_1) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y) \): this follows by applying Example 3.2.1.6 to the composite map \([0, 1] \times |\text{Sing}_\bullet(X)| \to [0, 1] \times X \overset{h}{\to} Y\).

**Notation 3.2.1.8.** Let \((X, x)\) and \((Y, y)\) be pointed simplicial sets. We let \([X, Y]_\ast\) denote the set \(\pi_0(\text{Fun}(X, Y) \times_{\text{Fun}(\{x\}, Y)} \{y\})\) of pointed homotopy classes of morphisms from \((X, x)\) to \((Y, y)\). If \(f : X \to Y\) is a morphism of pointed simplicial sets, we denote its pointed homotopy class by \([f] \in [X, Y]_\ast\).

**Warning 3.2.1.9.** Notation 3.2.1.8 has the potential to create confusion. If \((X, x)\) and \((Y, y)\) are pointed simplicial sets and \(f : X \to Y\) is a morphism satisfying \(f(x) = y\), then we use the notation \([f]\) to represent both the homotopy class of \(f\) as a map of simplicial sets (that is, the image of \(f\) in the set \(\pi_0(\text{Fun}(X, Y))\)), and the pointed homotopy class of \(f\) as a map of pointed simplicial sets (that is, the image of \(f\) in the set \([X, Y]_\ast = \pi_0(\text{Fun}(X, Y) \times_{\text{Fun}(\{x\}, Y)} \{y\})\)). Beware that these usages are not the same: it is possible for a pair of pointed morphisms \(f, g : X \to Y\) to be homotopic without being pointed homotopic.

**Construction 3.2.1.10 (The Homotopy Category of Pointed Kan Complexes).** We define a category \(\text{hKan}_\ast\) as follows:

- The objects of \(\text{hKan}_\ast\) are pointed Kan complexes \((X, x)\).
- If \((X, x)\) and \((Y, y)\) are Kan complexes, then \(\text{Hom}_{\text{hKan}}((X, x), (Y, y)) = [X, Y]_\ast\) is the set of pointed homotopy classes of morphisms from \((X, x)\) to \((Y, y)\).
- If \((X, x)\), \((Y, y)\), and \((Z, z)\) are Kan complexes, then the composition law
  \[
  \circ : \text{Hom}_{\text{hKan}}((Y, y), (Z, z)) \times \text{Hom}_{\text{hKan}}((X, x), (Y, y)) \to \text{Hom}_{\text{hKan}}((X, x), (Z, z))
  \]
  is characterized by the formula \([g] \circ [f] = [g \circ f]\).

We will refer to \(\text{hKan}_\ast\) as the homotopy category of pointed Kan complexes.

3.2.2 The Homotopy Groups of a Kan Complex

Let \(X\) be a topological space and let \(x \in X\) be a point. For every positive integer \(n\), we let \(\pi_n(X, x)\) denote the set of homotopy classes of pointed maps \((S^n, x_0) \to (X, x)\), where \(S^n\) denotes a sphere of dimension \(n\) and \(x_0 \in S^n\) is a chosen base point. The set \(\pi_n(X, x)\) can be endowed with the structure of a group, which we refer to as the \(n\)th homotopy group of \(X\) (with respect to the base point \(x\)). Note that the sphere \(S^n\) can be realized as the quotient space \(|\Delta^n|/\partial \Delta^n|\), obtained from the topological simplex \(|\Delta^n|\) by collapsing its boundary to the point \(q\). We can therefore identify pointed maps \((S^n, x_0) \to (X, x)\) with maps of
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Let \( \sigma : \Delta^n \to \text{Sing}_\bullet(X) \) which carry the boundary \( \partial \Delta^n \) to the simplicial subset \( \{x\} \subseteq \text{Sing}_\bullet(X) \). In [23], Kan elaborated on this observation to give a direct construction of the homotopy group \( \pi_n(X,x) \) in terms of the simplicial set \( \text{Sing}_\bullet(X) \) (and the vertex \( x \)). Moreover, his construction can be applied directly to any Kan complex.

**Notation 3.2.2.1.** Let \( B \) be a simplicial set and let \( A \subseteq B \) be a simplicial subset. We let \( B/A \) denote the pushout \( B \coprod_A \{q_0\} \), formed in the category of simplicial sets. We regard \( B/A \) as a pointed simplicial set, with base point given by the vertex \( q_0 \).

**Remark 3.2.2.2.** Let \( B \) be a simplicial set and let \( A \) be a simplicial subset. Then the simplicial set \( B/A \) can be described more informally as follows: it is obtained from \( B \) by collapsing the simplicial subset \( A \subseteq B \) to a single vertex \( q_0 \). Beware that this informal description is a bit misleading when \( A = \emptyset \): in this case, the natural map \( B \to B/A \) is not surjective (instead, \( B/A \) can be described as the coproduct \( B_+ = B \coprod \{q_0\} \), obtained from \( B \) by adding a new base point).

**Example 3.2.2.3.** For \( n \geq 0 \), the geometric realization \( |\Delta^n/\partial \Delta^n| \) can be obtained from the topological \( n \)-simplex \( |\Delta^n| \) by collapsing the boundary \( |\partial \Delta^n| \) to a point (or by adding a new base point, in the degenerate case \( n = 0 \)). It follows that \( |\Delta^n/\partial \Delta^n| \) is homeomorphic to a sphere of dimension \( n \).

**Construction 3.2.2.4.** Let \( (X,x) \) be a pointed Kan complex and let \( n \) be a nonnegative integer. We let \( \pi_n(X,x) \) denote the set \([\Delta^n/\partial \Delta^n, X]_s\) of pointed homotopy classes of maps from \( \Delta^n/\partial \Delta^n \) to \( X \) (Notation 3.2.1.8). For \( n > 0 \), we will refer to \( \pi_n(X,x) \) as the \( n \)th homotopy group of \( X \) with respect to the base point \( x \) (see Theorem 3.2.2.10 below). In the special case \( n = 1 \), we refer to \( \pi_1(X,x) \) as the fundamental group of \( X \) with respect to the base point \( x \).

**Notation 3.2.2.5.** Let \( (X,x) \) be a pointed Kan complex and let \( n \) be a nonnegative integer. Then the set of pointed morphisms \( \Delta^n/\partial \Delta^n \to X \) can be identified with the set of \( n \)-simplices \( \sigma : \Delta^n \to X \) having the property that \( \sigma|_{\partial \Delta^n} \) is equal to the constant map \( \partial \Delta^n \to \{x\} \subseteq X \). In this case, we write \([\sigma]\) for the image of \( \sigma \) in the set \( \pi_n(X,x) \). Note that, if \( \tau \) is another \( n \)-simplex of \( X \) for which \( \tau|_{\partial \Delta^n} \) is the constant map \( \partial \Delta^n \to \{x\} \subseteq X \), then the equality \([\sigma] = [\tau]\) holds in \( \pi_n(X,x) \) if and only if there exists a homotopy \( h : \Delta^1 \times \Delta^n \to X \) satisfying \( \sigma = h|_{\{0\} \times \Delta^n} \), \( \sigma' = h|_{\{1\} \times \Delta^n} \), and \( h|_{\Delta^1 \times \partial \Delta^n} \) is the constant map taking the value \( x \).

**Example 3.2.2.6.** Let \( (X,x) \) be a pointed Kan complex. Then \( \pi_0(X,x) \) can be identified with the set \( \pi_0(X) \) of connected components of \( X \) (Definition 1.1.6.8). Beware that, unlike the higher homotopy groups \( \{\pi_n(X,x)\}_{n \geq 1} \), there is no naturally defined group structure on \( \pi_0(X,x) \).
Example 3.2.2.7. Let $X$ be a topological space and let $x \in X$ be a base point, which we identify with a vertex of the singular simplicial set $\text{Sing}_\bullet(X)$. For every positive integer $n$, we can identify $\pi_n(\text{Sing}_\bullet(X), x)$ with the set $\pi_n(X, x)$ of (pointed) homotopy classes of maps from the sphere $S^n \simeq |\Delta^n/\partial\Delta^n|$ into $X$.

Example 3.2.2.8. Let $X$ be a Kan complex, let $x$ be a vertex of $X$, and let $e, e' : x \to x$ be edges of $X$ which begin and end at the vertex $x$. Then the equality $[e] = [e']$ holds in the fundamental group $\pi_1(X, x)$ if and only if $e$ is homotopic to $e'$ as a morphism in the $\infty$-category $X$ (in the sense of Definition 1.3.3.1); see Corollary 1.3.3.7.

Remark 3.2.2.9. Let $n$ be a nonnegative integer. By virtue of Corollary 3.1.6.2, there exists an anodyne morphism $f : \Delta^n/\partial\Delta^n \to Q$, where $Q$ is a Kan complex. Let $q \in Q$ denote the image of the base point $q_0$ of $\Delta^n/\partial\Delta^n$. If $(X, x)$ is a pointed Kan complex, then precomposition with $f$ induces a trivial Kan fibration $\text{Fun}(Q, X) \to \text{Fun}(\Delta^n/\partial\Delta^n, X)$ (Theorem 3.1.3.5), hence also a trivial Kan fibration $\text{Fun}(Q, X) \times_{\text{Fun}(\{q\}, X)} \{x\} \to \text{Fun}(\Delta^n/\partial\Delta^n) \times_{\text{Fun}(\{q_0\}, X)} \{x\}$.

Passing to connected components, we see that $f$ induces a bijection $\text{Hom}_{\text{hKan}}(Q, X) \simeq \pi_n(X, x)$. In other words, the functor $(X, x) \mapsto \pi_n(X, x)$ is corepresentable (in the pointed homotopy category $\text{hKan}_*$) by the pointed Kan complex $(Q, q)$ (which can be regarded as a combinatorial incarnation of the $n$-sphere).

Theorem 3.2.2.10. Let $(X, x)$ be a pointed Kan complex and let $n$ be a positive integer. Then there is a unique group structure on the set $\pi_n(X, x)$ with the following properties:

(a) Let $e : \Delta^n \to \{x\} \to X$ be the constant map. Then the homotopy class $[e]$ is the identity element of $\pi_n(X, x)$.

(b) Let $f : \partial\Delta^{n+1} \to X$ be a morphism of simplicial sets, corresponding to a tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ of $n$-simplices of $X$ (see Exercise 1.1.2.8). Assume that each restriction $\sigma_i|_{\partial\Delta^n}$ is equal to the constant map $\partial\Delta^n \to \{x\} \subseteq X$. Then $f$ extends to a map $\Delta^{n+1} \to X$ if and only if the product $[\sigma_0]^{-1}[\sigma_1][\sigma_2]^{-1}[\sigma_3] \cdots [\sigma_{n+1}]^{-1}$ is equal to the identity element of $\pi_n(X, x)$.

Moreover, if $n \geq 2$, then the group $\pi_n(X, x)$ is abelian.

We will give the proof of Theorem 3.2.2.10 in §3.2.3.

Exercise 3.2.2.11. Show that when $n > 0$ is odd, condition (a) of Theorem 3.2.2.10 follows from condition (b) (beware that this is not true when $n$ is even).
Example 3.2.2.12. In the special case \( n = 1 \), we can rewrite condition (b) of Theorem 3.2.2.10 as follows:

- Let \( f, g, \) and \( h \) be edges of \( X \) which begin and end at the vertex \( x \). Then the equality \([h] = [g][f]\) holds (in the fundamental group \( \pi_1(X,x) \)) if and only if there exists a 2-simplex \( \sigma \) of \( X \) which witnesses \( h \) as a composition of \( f \) and \( g \) (in the sense of Definition 1.3.4.1), as indicated in the diagram

\[ x \xrightarrow{h} x \]

\[ x \xrightarrow{f} x \]

\[ x \xrightarrow{g} x \]

It follows that the fundamental group \( \pi_1(X,x) \) can be identified with the automorphism group of \( x \) as an object of the fundamental groupoid \( \pi_{\leq 1}(X) = hX \).

Warning 3.2.2.13. Let \((X,x)\) be a pointed Kan complex, so that \( \pi_1(X,x) \) can be identified with the set \( \text{Hom}_{\pi_{\leq 1}(X)}(x,x) \) of homotopy classes of paths from \( x \) to itself. We have adopted the convention that the multiplication on \( \pi_1(X,x) \) is given by composition in the homotopy category \( hX \). In other words, if \( f, g : x \to x \) are edges which begin and end at \( x \), then the product \([g][f] \in \pi_1(X,x)\) is the homotopy class of a path which can be described informally as traversing the path \( f \) first, followed by the path \( g \). Beware that the opposite convention is also common in the literature (note that his issue is irrelevant for the higher homotopy groups \( \{\pi_n(X,x)\}_{n \geq 2} \), since they are abelian).

Remark 3.2.2.14. Let \((X,x)\) be a pointed Kan complex. For \( n \geq 2 \), the homotopy group \( \pi_n(X,x) \) is abelian. We will generally emphasize this by using additive notation for the group structure on \( \pi_n(X,x) \): that is, we denote the group law by

\[ + : \pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x) \quad (\xi,\xi') \mapsto \xi + \xi'. \]

With this convention, we can restate property (b) of Theorem 3.2.2.10 as follows:

(b) Let \( f : \partial \Delta^{n+1} \to X \) be a morphism of simplicial sets, corresponding to a tuple \((\sigma_0, \sigma_1, \ldots, \sigma_{n+1})\) of \( n \)-simplices of \( X \). Then \( f \) extends to an \( (n+1) \)-simplex of \( X \) if and only if the sum \( \sum_{i=0}^{n+1} (-1)^i [\sigma_i] \) vanishes in \( \pi_n(X,x) \).

Remark 3.2.2.15 (Functoriality). Let \( f : X \to Y \) be a morphism of Kan complexes, let \( x \) be a vertex of \( X \), and set \( y = f(x) \). For each \( n \geq 1 \), the morphism \( f \) induces a homomorphism \( \pi_n(f) : \pi_n(X,x) \to \pi_n(Y,y) \), characterized by the formula \( \pi_n(f)([\sigma]) = [f(\sigma)] \) for each \( n \)-simplex \( \sigma \) of \( X \) for which \( \sigma|_{\partial \Delta^n} \) is the constant map \( \partial \Delta^n \to \{x\} \hookrightarrow X \).
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Remark 3.2.2.16. Let \( X \) be a Kan complex and let \( x \) be a vertex of \( X \). Then \( x \) can also be regarded as a vertex of the opposite simplicial set \( X^{\text{op}} \), which is also a Kan complex. For \( n \geq 1 \), we have an evident bijection \( \varphi : \pi_n(X, x) \simeq \pi_n(X^{\text{op}}, x) \). If \( n \geq 2 \), then this bijection is an isomorphism of abelian groups. Beware that, in the case \( n = 1 \), it is generally not an isomorphism of groups: instead, it is an anti-isomorphism (that is, it satisfies the identity \( \varphi(\xi \xi') = \varphi(\xi') \varphi(\xi) \) for \( \xi, \xi' \in \pi_1(X, x) \); see Warning 3.2.2.13 above).

Remark 3.2.2.17. Let \((X, x)\) be a pointed Kan complex and let \( n \) be a positive integer. Suppose that \( \sigma, \sigma' : \Delta^n \to X \) are \( n \)-simplices of \( X \) for which \( \sigma|_{\partial \Delta^n} \) and \( \sigma'|_{\partial \Delta^n} \) are equal to the constant map \( \partial \Delta^n \to \{x\} \subseteq X \). It follows from Theorem 3.2.2.10 that the equality \([\sigma] = [\sigma']\) holds (in the homotopy group \( \pi_n(X, x) \)) if and only if there exists an \((n+1)\)-simplex \( \tau \) of \( X \) such that \( d_0(\tau) = \sigma \), \( d_1(\tau) = \sigma' \), and \( d_i(\tau) \) is the constant map \( \Delta^n \to \{x\} \subseteq X \) for \( 2 \leq i \leq n+1 \).

Exercise 3.2.2.18 (Homotopy of Eilenberg-MacLane Spaces). Let \( M_* \) be a chain complex of abelian groups and let \( X = K(M_*) \) be the associated Eilenberg-MacLane space (Construction 2.5.6.3). Let \( x \in X \) be the vertex corresponding to the zero element, and let \( n \) be a positive integer. Note that a pointed map from \( \Delta^n/\partial \Delta^n \to \{x\} \subseteq X \) can be identified with a map of chain complexes \( N_*(\Delta^n, \partial \Delta^n ; \mathbb{Z}) \simeq \mathbb{Z}[n] \to M_* \): in other words, it can be identified with an \( n \)-cycle of the chain complex \( M_* \), which we will denote by \( \overline{\sigma} \).

1. Let \( \sigma, \sigma' : \Delta^n \to X \) be \( n \)-simplices whose restriction to \( \partial \Delta^n \) is equal to the constant map \( \partial \Delta^n \to \{x\} \subseteq X \). Show that \([\sigma] = [\sigma']\) in \( \pi_n(X, x) \) if and only if \( \overline{\sigma} \) and \( \overline{\sigma}' \) are homologous as \( n \)-cycles of \( M_* \) (use Remark 3.2.2.17).

2. Show that the \([\sigma] \mapsto [\overline{\sigma}]\) induces an isomorphism from \( \pi_n(X, x) \) to the homology group \( H_n(M) \).

In particular, if \( A \) is an abelian group and \( m \geq 0 \) is an integer, then the homotopy groups of the Eilenberg-MacLane space \( X = K(A, m) \) are given by

\[
\pi_n(X, x) = \begin{cases} A & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}
\]

3.2.3 The Group Structure on \( \pi_n(X, x) \)

Let \((X, x)\) be a pointed Kan complex and let \( n \geq 2 \) be an integer, which we regard as fixed throughout this section. Our goal is to give a proof of Theorem 3.2.2.10 which supplies a group structure on the set \( \pi_n(X, x) = [\Delta^n/\partial \Delta^n, X]_* \) (note that the case \( n = 1 \) of Theorem 3.2.2.10 is subsumed in our construction of the homotopy category \( \pi_{\leq 1}(X) = hX \), by virtue of Example 3.2.2.12).
Notation 3.2.3.1. Let $\Sigma$ denote the collection of all $n$-simplices $\sigma : \Delta^n \to X$ having the property that the restriction $\sigma|_{\partial \Delta^n}$ is equal to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. We let $e \in \Sigma$ denote the constant map $\Delta^n \to \{x\} \subseteq X$. Note that an $(n+2)$-tuple $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ of elements of $\Sigma$ can be identified with a map of simplicial sets $f : \partial \Delta^{n+1} \to X$, having the property that the restriction of $f$ to the $(n-1)$-skeleton of $\partial \Delta^{n+1}$ is equal to the constant map $sk_{n-1}(\partial \Delta^{n+1}) \to \{x\} \subseteq X$ (see Exercise 1.1.2.8). We will say that a tuple $\vec{\sigma}$ bounds if $f$ can be extended to an $(n+1)$-simplex of $X$: that is, if there exists an $(n+1)$-simplex $\tau$ of $X$ satisfying $\sigma_i = d_i(\tau)$ for $0 \leq i \leq n+1$.

The construction $\sigma \mapsto [\sigma]$ then determines a surjective map $\Sigma \to \pi_n(X,x)$. We will say that a pair of elements $\sigma, \sigma' \in \Sigma$ are homotopic if $[\sigma] = [\sigma']$ (that is, if there is a homotopy from $\sigma$ to $\sigma'$ which is constant along the boundary $\partial \Delta^n$).

Lemma 3.2.3.2. Let $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ be an $(n+2)$-tuple of elements of $\Sigma$. The condition that $\vec{\sigma}$ bounds depends only on the sequence of homotopy classes $\{[\sigma_i] \in \pi_n(X,x)\}_{0 \leq i \leq n+1}$. In other words, if $\vec{\sigma}' = (\sigma_0', \sigma_1', \ldots, \sigma_{n+1}')$ is another $(n+2)$-tuple of elements of $\Sigma$ satisfying $[\sigma_i'] = [\sigma_i]$ for $0 \leq i \leq n+1$ and $\vec{\sigma}'$ bounds, then $\vec{\sigma}'$ also bounds.

Proof. Let us identify $\vec{\sigma}$ and $\vec{\sigma}'$ with morphisms of simplicial sets $f,f' : \partial \Delta^{n+1} \to X$ (carrying the $(n-1)$-skeleton of $\partial \Delta^{n+1}$ to the vertex $x$). For $0 \leq i \leq n+1$, the equality $[\sigma_i] = [\sigma_i']$ allows us choose a homotopy $h_i : \Delta^1 \times \Delta^n \to X$ from $\sigma_i$ to $\sigma_i'$ which carries $\Delta^1 \times \partial \Delta^n$ to the vertex $\{x\} \subseteq X$. These maps can be amalgamated to a homotopy $h$ from $f$ to $f'$: that is, an edge joining $f$ to $f'$ in the simplicial set $Fun(\partial \Delta^{n+1}, X)$. If $\vec{\sigma}$ bounds, then $f$ can be extended to an $(n+1)$-simplex $\tau : \Delta^{n+1} \to X$. Since $X$ is a Kan complex, the restriction map $Fun(\Delta^{n+1}, X) \to Fun(\partial \Delta^{n+1}, X)$ is a Kan fibration (Corollary 3.1.3.3), so $h$ can be extended to a homotopy $\tilde{h}$ from $\tau$ to another map $\tau' : \Delta^{n+1} \to X$ satisfying $\tau'|_{\partial \Delta^{n+1}} = f'$. It follows that the tuple $\vec{\sigma}'$ also bounds.

Remark 3.2.3.3. Let $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ be an $(n+2)$-tuple of elements of $\pi_n(X,x)$, so that we can write $\eta_i = [\sigma_i]$ for some $n$-simplex $\sigma_i \in \Sigma$. We will say that the tuple of homotopy classes $\vec{\eta}$ bounds if the tuple of simplices $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ bounds, in the sense of Notation 3.2.3.1. By virtue of Lemma 3.2.3.2, this condition is independent of the choice of $\vec{\sigma}$.

With this terminology, Theorem 3.2.2.10 asserts (in the case $n \geq 2$) that there is a unique abelian group structure on the set $\pi_n(X,x)$ with the following pair of properties:

(a) The identity element of $\pi_n(X,x)$ is the homotopy class $[e]$.

(b) An $(n+2)$-tuple $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ bounds if and only if the sum $\sum_{i=0}^{n+1} (-1)^i \eta_i$ vanishes in $\pi_n(X,x)$. 

Lemma 3.2.3.4. Let \(0 \leq i \leq n + 1\), and suppose we are given a collection of homotopy classes \(\{\eta_j \in \pi_n(X,x)\}_{0 \leq j \leq n+1, j \neq i}\). Then there is a unique element \(\eta_i \in \pi_n(X,x)\) for which the tuple \(\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})\) bounds.

Proof. For \(j \neq i\), choose an element \(\sigma_j \in \Sigma\) satisfying \([\sigma_j] = \eta_j\). Then the tuple of \(n\)-simplices \((\sigma_0, \ldots, \sigma_i-1, \bullet, \sigma_{i+1}, \ldots, \sigma_{n+1})\) determines a map of simplicial sets \(f_0 : \Lambda_i^{n+1} \rightarrow X\) (see Exercise 1.1.2.14). Since \(X\) is a Kan complex, we can extend \(f_0\) to an \((n+1)\)-simplex \(\tau\) of \(X\). Then \(\eta_i = [d_i(\tau)]\) has the property that the tuple \(\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_n)\) bounds. This proves existence. To prove uniqueness, suppose we are given another element \(\eta_i' \in \pi_n(X,x)\) for which the tuple \((\eta_0, \ldots, \eta_i-1, \eta_i', \eta_{i+1}, \ldots, \eta_{n+1})\) bounds. Write \(\eta_i' = [\sigma_i']\) for some \(\sigma_i' \in \Sigma\), so that we can choose a simplex \(\tau' : \Delta^{n+1} \rightarrow X\) satisfying

\[
d_j(\tau') = \begin{cases} \sigma_i' & \text{if } j = i \\ \sigma_j & \text{otherwise.} \end{cases}
\]

Since the inclusion \(\Lambda_i^{n+1} \hookrightarrow \Delta^{n+1}\) is anodyne, so the restriction map \(\text{Fun}(\Delta^{n+1}, X) \rightarrow \text{Fun}(\Lambda_i^{n+1}, X)\) is a trivial Kan fibration (Corollary 3.1.3.6). It follows that there exists a homotopy from \(\tau\) to \(\tau'\) which is constant along the subset \(\Lambda_i^{n+1} \subseteq \Delta^{n+1}\), so that \(\eta_i = [d_i(\tau)] = [d_i(\tau')] = \eta_i'\).

As a special case of Lemma 3.2.3.4, we obtain several potential candidates for the composition law on \(\pi_n(X,x)\):

Lemma 3.2.3.5. Fix \(1 \leq i \leq n\). Then there is a unique function \(m_i : \pi_n(X,x) \times \pi_n(X,x) \rightarrow \pi_n(X,x)\) with the following property:

(*) Let \(\eta_{i-1}, \eta_i, \text{ and } \eta_{i+1}\) be elements of \(\pi_n(X,x)\). Then the \((n+2)\)-tuple

\[
([e], \ldots, [e], \eta_{i-1}, \eta_i, \eta_{i+1}, [e], \ldots, [e])
\]

bounds if and only if \(\eta_i = m_i(\eta_{i-1}, \eta_{i+1})\).

Example 3.2.3.6. Let \(\sigma\) be an element of \(\Sigma\), and let \(1 \leq i \leq n\). Then the degenerate \((n+1)\)-simplex \(\tau = s_i(\sigma)\) satisfies \(d_j(\tau) = \begin{cases} \sigma & \text{if } j \in \{i, i+1\} \\ e & \text{otherwise.} \end{cases}\) It follows that the multiplication map \(m_i : \pi_n(X,x) \times \pi_n(X,x) \rightarrow \pi_n(X,x)\) of Lemma 3.2.3.5 satisfies the identity \(m_i([e], [\sigma]) = [\sigma]\). A similar argument shows that \(m_i([\sigma], [e]) = [\sigma]\).

Lemma 3.2.3.7. Let \(\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_n)\) be an \((n+2)\)-tuple of elements of \(\pi_n(X,x)\), let \(1 \leq i \leq n\) be an integer, and let \(\alpha\) be another element of \(\pi_n(X,x)\). If \(\vec{\eta}\) bounds, then the tuple \((\eta_0, \ldots, \eta_{i-2}, m_i(\alpha, \eta_{i-1}), m_i(\alpha, \eta_i), \eta_{i+1}, \ldots, \eta_n)\) also bounds.
The definition of the multiplication $m_i$ then gives $m_i(\alpha, \eta_{i-1}) = [d_i(\rho)]$ and $m_i(\alpha, \eta_i) = [d_i(\rho')]$. The tuple $(s_i(\sigma_0), \ldots, s_i(\sigma_{i-2}), \rho, \rho', \bullet, \sigma, s_{i+1}(\sigma_{i+2}), \ldots, s_{i+1}(\sigma_{n+1}))$ then determines a map of simplicial sets $\Lambda^{n+2}_{i+1} \to X$ (Exercise 1.1.2.14). Since $X$ is a Kan complex, this map can be extended to an $(n+2)$-simplex of $X$. Let $\sigma'$ denote the $(i+1)$st face of this simplex. By construction, we have

$$d_j(\sigma') = \begin{cases} 
   d_i(\rho) & \text{if } j = i - 1 \\
   d_i(\rho') & \text{if } j = i \\
   \sigma_j & \text{otherwise},
\end{cases}$$

so that $\sigma'$ witnesses that the tuple $(\eta_0, \ldots, \eta_{i-2}, m_i(\alpha, \eta_{i-1}), m_i(\alpha, \eta_i), \eta_{i+1}, \ldots, \eta_{n+1})$ bounds.

**Lemma 3.2.3.8.** Let $\alpha$, $\beta$, and $\gamma$ be elements of $\pi_n(X, x)$. For $2 \leq i \leq n$, we have $m_i(\alpha, m_{i-1}(\beta, \gamma)) = m_{i-1}(\beta, m_i(\alpha, \gamma))$.

**Proof.** Applying Lemma 3.2.3.7 to the tuple $([e], \ldots, [e], \beta, m_{i-1}(\beta, \gamma), \gamma, [e], \ldots, [e])$, we deduce that the tuple $([e], \ldots, [e], \beta, m_i(\alpha, m_{i-1}(\beta, \gamma)), m_i(\alpha, \gamma), [e], \ldots, [e])$ bounds, which is equivalent to the asserted identity.

**Lemma 3.2.3.9.** Let $\alpha$ and $\beta$ be elements of $\pi_n(X, x)$. For $2 \leq i \leq n$, we have $m_i(\alpha, \beta) = m_{i-1}(\beta, \alpha)$.

**Proof.** Combining Lemma 3.2.3.8 with Example 3.2.3.6, we obtain

$$m_i(\alpha, \beta) = m_i(\alpha, m_{i-1}(\beta, [e])) = m_{i-1}(\beta, m_i(\alpha, [e])) = m_{i-1}(\beta, \alpha).$$

**Proof of Theorem 3.2.2.10.** For every pair of elements $\alpha, \beta \in \pi_n(X, x)$, let $\alpha \beta$ denote the homotopy class $m_1(\alpha, \beta)$, where $m_1 : \pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$ is the multiplication.
CHAPTER 3. KAN COMPLEXES

map of Lemma 3.2.3.5. We first note that this multiplication is associative: for every triple of elements \( \alpha, \beta, \gamma \in \pi_n(X, x) \), Lemmas 3.2.3.9 and 3.2.3.8 yield identities

\[
\alpha(\beta \gamma) = m_1(\alpha, m_1(\beta, \gamma)) = m_1(\alpha, m_2(\gamma, \beta)) = m_2(\gamma, m_1(\alpha, \beta)) = m_1(\alpha, m_1(\beta, \gamma)) = (\alpha \beta) \gamma.
\]

Example 3.2.3.6 shows that \([e]\) is a two-sided identity with respect to multiplication. For every element \( \alpha \in \pi_n(X, x) \), Lemma 3.2.3.4 implies that we can choose an element \( \beta \in \pi_n(X, x) \) for which the tuple \((\alpha, [e], \beta, [e], \ldots, [e])\) bounds, so that \( \alpha \beta = m_1(\alpha, \beta) = [e] \). This shows that \( \alpha \) has a right inverse, and a similar argument shows that \( \alpha \) has a left inverse. It follows that multiplication determines a group structure on the set \( \pi_n(X, x) \), having \([e]\) as the identity element.

We now verify that the multiplication on \( \pi_n(X, x) \) satisfies condition (b) of Theorem 3.2.2.10. Suppose we are given an \((n+1)\)-tuple \( \vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1}) \) of elements of \( \pi_n(X, x) \). We wish to show that \( \vec{\eta} \) bounds if and only if the product \( \eta_{n+1}^{(-1)} \eta_n^{(-1)} \cdots \eta_2^{(-1)} \eta_1^{(-1)} \) is equal to the identity element of \( \pi_n(X, x) \). If \( \vec{\eta} = ([e], [e], \ldots, [e]) \), there is nothing to prove. Otherwise, there exists some smallest positive integer \( i \) such that \( \eta_{i-1} \neq [e] \). We proceed by descending induction on \( i \). If \( i > n \), we must show that \( ([e], [e], \ldots, [e], \eta_n, \eta_{n+1}) \) bounds if and only if \( \eta_n = \eta_{n+1} \), which follows from Example 3.2.3.6. Let us therefore assume that \( 1 \leq i \leq n \).

Define \( \vec{\eta}' = (\eta'_0, \eta'_1, \ldots, \eta'_{n+1}) \) by the formula

\[
\eta'_j = \begin{cases} 
  m_i(\eta_{j-1}^{-1}, \eta_j) & \text{if } j = i - 1 \text{ or } j = i \\
  \eta_j & \text{otherwise.}
\end{cases}
\]

Invoking Lemma 3.2.3.9 repeatedly, we obtain

\[
\eta_{i-1}' = m_i(\eta_{i-1}^{-1}, \eta_{i-1}) = \begin{cases} 
  \eta_{i-1}^{-1} \eta_{i-1} & \text{if } i \text{ is odd} \\
  \eta_{i-1} \eta_{i-1}^{-1} & \text{if } i \text{ is even}
\end{cases} = [e]
\]

\[
\eta_i' = m_i(\eta_{i-1}^{-1}, \eta_i) = \begin{cases} 
  \eta_{i-1}^{-1} \eta_i & \text{if } i \text{ is odd} \\
  \eta_i \eta_{i-1}^{-1} & \text{if } i \text{ is even}
\end{cases}.
\]

We therefore have an equality

\[
\eta_0^{-1} \eta_1^{-1} \cdots \eta_n^{-1} = \eta'_0^{-1} \eta'_1^{-1} \cdots \eta'_{n+1}^{-1}.
\]

Invoking our inductive hypothesis, we conclude that this product vanishes if and only if the tuple \( \vec{\eta}' \) bounds. By virtue of Lemma 3.2.3.7, this is equivalent to the assertion that \( \vec{\eta}' \) bounds.
3.2. HOMOTOPY GROUPS

We now complete the proof of Theorem 3.2.2.10 by showing that the multiplication on \( \pi_n(X,x) \) is commutative. Fix a pair of elements \( \sigma, \sigma' \in \Sigma \). Then the tuples of \( n \)-simplices \((\sigma,e,\sigma',e,e,\ldots,e)\) and \((\sigma',e,\sigma,e,e,\ldots,e)\) determine maps of simplicial sets \( f,f' : \Lambda_{3+1}^n \to X \) (Exercise 1.1.2.14). Since \( X \) is a Kan complex, we can extend \( f \) and \( f' \) to \((n+1)\)-simplices of \( X \), which we will denote by \( \tau \) and \( \tau' \), respectively. Let \( \overline{\tau} : \Delta^{n+1} \to X \) denote the constant map taking the value \( x \). Then the tuple of \((n+1)\)-simplices \((\tau,s_0(\sigma),s_1(\sigma),\tau',e,e,e,\ldots,\overline{\tau})\) determines a map of simplicial sets \( g : \Lambda_{4}^{n+2} \to X \) (Exercise 1.1.2.14). Since \( X \) is a Kan complex, we can extend \( g \) to an \((n+2)\)-simplex of \( X \). Then the fourth face of this extension witnesses that the tuple of \( n \)-simplices \((d_3(\tau),e,e,d_3(\tau'),e,\ldots,e)\) bounds, so that we have an equality \([\sigma'][\sigma] = [d_3(\tau)] = [d_3(\tau')] = [\sigma][\sigma']\) in the homotopy group \( \pi_n(X,x) \).

\[ \square \]

3.2.4 The Connecting Homomorphism

\[ \text{OOWD} \]

Let \( S \) be a Kan complex, and let \( f : X \to S \) be a Kan fibration of simplicial sets (so that \( X \) is also a Kan complex). Fix a vertex \( x \in X \), let \( s = f(x) \) be its image in \( S \), and let \( X_s \) denote the fiber \( \{s\} \times_X X \) (so that \( X_s \) is also a Kan complex, and we can regard \( x \) as a vertex of \( X_s \)). In \[ 3.2.5 \], we will show that the homotopy groups of \( X, S, \) and \( X_s \) are related by a long exact sequence

\[ \cdots \to \pi_{n+1}(S,s) \overset{\partial}{\to} \pi_n(X_s,x) \to \pi_n(X,x) \to \pi_n(S,s) \overset{\partial}{\to} \pi_{n-1}(X_s,x) \to \cdots \]

(see Theorem 3.2.5.1 below). In this section, we set the stage by constructing the maps \( \partial : \pi_{n+1}(S,s) \to \pi_n(X_s,x) \) which appear in this sequence.

\[ \text{OOWE} \]

**Definition 3.2.4.1.** Let \( f : (X,x) \to (S,s) \) be a Kan fibration between pointed Kan complexes and let \( n \geq 0 \) be a nonnegative integer. Suppose we are given a pair of maps \( \sigma : \Delta^n \to X_s \) and \( \tau : \Delta^{n+1} \to S \), having the property that \( \sigma|_{\partial \Delta^n} \) and \( \tau|_{\partial \Delta^{n+1}} \) are the constant maps taking the values \( x \) and \( s \), respectively. We will say that \( \sigma \) is incident to \( \tau \) if there exists a simplex \( \overline{\tau} : \Delta^{n+1} \to X \) satisfying \( \tau = f(\overline{\tau}) \), \( \sigma = d_0(\overline{\tau}) \), and \( \overline{\tau}|_{\Lambda_{0}^{n+1}} : \Lambda_{0}^{n+1} \to X \) is the constant map taking the value \( x \).

\[ \text{OOWF} \]

**Proposition 3.2.4.2.** Let \( f : (X,x) \to (S,s) \) be a Kan fibration between pointed Kan complexes and let \( n \geq 0 \) be a nonnegative integer. Then there exists a unique function \( \partial : \pi_{n+1}(S,s) \to \pi_n(X_s,x) \) with the following property:

\[ (\ast) \text{ Let } \sigma : \Delta^n \to X_s \text{ and } \tau : \Delta^{n+1} \to S \text{ be simplices having the property that } \sigma|_{\partial \Delta^n} \text{ and } \tau|_{\partial \Delta^{n+1}} \text{ are the constant maps taking the values } x \text{ and } s, \text{ respectively. Then } \sigma \text{ is incident to } \tau \text{ (in the sense of Definition 3.2.4.1) if and only if } \partial([\tau]) = [\sigma]. \]
**Construction 3.2.4.3** (The Connecting Homomorphism). Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes. For each \( n \geq 0 \), we will refer to the map \( \partial : \pi_{n+1}(S, s) \to \pi_n(X, x) \) of Proposition 3.2.4.2 as the **connecting homomorphism** (for \( n \geq 1 \), it is a group homomorphism: see Proposition 3.2.4.4 below).

**Proof of Proposition 3.2.4.2** Fix a map \( \tau : \Delta^{n+1} \to S \) be an \((n+1)\)-simplex for which \( \tau|_{\partial \Delta^{n+1}} \) is the constant map taking the value \( s \). To prove Proposition 3.2.4.2, it will suffice to prove the following:

1. There exists an \( n \)-simplex \( \sigma : \Delta^n \to X \) such that \( \sigma|_{\partial \Delta^n} \) is the constant map taking the value \( x \) and \( \sigma \) is incident to \( \tau \).
2. Let \( \sigma' : \Delta^n \to X \) and \( \tau' : \Delta^{n+1} \to S \) have the property that \( \sigma'|_{\partial \Delta^n} \) and \( \tau'|_{\partial \Delta^{n+1}} \) are the constant maps taking the values \( x \) and \( s \), respectively, and suppose that \([\tau] = [\tau']\) in \( \pi_{n+1}(S, s) \). Then \( \sigma' \) is incident to \( \tau' \) if and only if \([\sigma] = [\sigma']\) in \( \pi_n(X, x) \).

Assertion (1) follows from the solvability of the lifting problem

\[
\begin{array}{ccc}
\Lambda_0^{n+1} & \xrightarrow{\sim} & X \\
\downarrow \tau & & \downarrow f \\
\Delta^{n+1} & \xrightarrow{\tau} & S,
\end{array}
\]

where the upper horizontal map is constant taking the value \( x \). Let \( \sigma' \) and \( \tau' \) be as in (2), and let \( \tau_0' : \partial \Delta^{n+1} \to X \) be the map given by the tuple of \( n \)-simplices \((\sigma', e, \ldots, e)\) (see Exercise 1.1.2.8) where \( e : \Delta^n \to X \) denotes the constant map taking the value \( x \). If \([\sigma] = [\sigma']\) in \( \pi_n(X, x) \), then we can choose a homotopy from \( \sigma \) to \( \sigma' \) (in the Kan complex \( X_s \)) which is constant along the boundary \( \partial \Delta^n \), and therefore a homotopy \( \tilde{h}_0 \) from \( \tau|_{\partial \Delta^{n+1}} \) to \( \tau_0' \) (also in the Kan complex \( X_s \)) which is constant along the simplicial subset \( \Lambda_0^{n+1} \subset \partial \Delta^{n+1} \).

Let \( h : \Delta^1 \times \Delta^{n+1} \to S \) be a homotopy from \( \tau \) to \( \tau' \) which is constant on \( \partial \Delta^{n+1} \). Since \( f \) is a Kan fibration, the homotopy extension lifting problem

\[
\begin{array}{ccc}
(\Delta^1 \times \partial \Delta^{n+1}) \coprod_{\{0\} \times \partial \Delta^{n+1}} (\{0\} \times \Delta^{n+1}) & \xrightarrow{\tilde{h}_0, \tau} & X \\
\downarrow \Delta^1 \times \Delta^{n+1} & & \downarrow f \\
\Delta^1 \times \Delta^{n+1} & \xrightarrow{h} & S
\end{array}
\]

admits a solution \( \tilde{h} : \Delta^1 \times \Delta^{n+1} \to X \) (Remark 3.1.4.3), which we can regard as a homotopy from \( \tilde{\tau} \) to another \((n + 1)\)-simplex \( \tilde{\tau}' : \Delta^{n+1} \to X \). By construction, this \((n + 1)\)-simplex witnesses that \( \sigma' \) is incident to \( \tau' \).
For the converse, suppose that \( \sigma' \) is incident to \( \tau' \), so that there exists an \((n + 1)\)-simplex \( \tilde{\tau}' : \Delta^{n+1} \to X \) satisfying \( d_0(\tilde{\tau}') = \sigma' \), \( f(\tilde{\tau}') = \tau' \), and \( \tilde{\tau}'|_{\Delta^0} \) is the constant map taking the value \( x \). Since \( f \) is a Kan fibration, the lifting problem

\[
(\Delta^1 \times \Lambda_0^{n+1}) \coprod_{\partial \Delta^1 \times \Lambda_0^{n+1}} (\partial \Delta^1 \times \Delta^{n+1}) \to X
\]

admits a solution, where \( \bar{\epsilon} : \Delta^1 \times \Lambda_0^{n+1} \to X \) is the constant map taking the value \( x \). Then \( \bar{h} \) is a homotopy from \( \bar{\epsilon} \) to \( \bar{\epsilon}' \) (in the Kan complex \( X \)) which is constant along the horn \( \Lambda_0^{n+1} \subseteq \Delta^{n+1} \), and it restricts to a homotopy from \( \sigma = d_0(\bar{\epsilon}) \) to \( \sigma' = d_0(\bar{\epsilon}') \) (in the Kan complex \( X_s \)) which is constant along the boundary \( \partial \Delta^n \). It follows that \([\sigma] = [\sigma']\) in \( \pi_n(X_s, x) \).

\[ \square \]

**Proposition 3.2.4.4.** Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes, and let \( n \geq 1 \) be a positive integer, and let \( \partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x) \) be as in Proposition 3.2.4.2. Then \( \partial \) is a group homomorphism.

**Proof.** To avoid confusion in the case \( n = 1 \), let us use multiplicative notation for the group structures on both \( \pi_{n+1}(S, s) \) and \( \pi_n(X_s, x) \). It is easy to see that the constant map \( \Delta^n \to \{x\} \subseteq X_s \) is incident to the constant map \( \Delta^{n+1} \to \{s\} \subseteq S \), so the map \( \partial \) carries the identity element of \( \pi_{n+1}(S, s) \) to the identity element of \( \pi_n(X_s, x) \). To complete the proof, it will suffice to show that if \( (\eta_0, \eta_1, \ldots, \eta_{n+1}) \) is an \((n + 2)\)-tuple of elements of \( \pi_{n+1}(S, s) \) for which the product \( \eta_0^{-1} \eta_1 \eta_2^{-1} \cdots \eta_{n+1}^{-1} \) vanishes in \( \pi_{n+1}(S, s) \), then the product \( \partial(\eta_0)^{-1} \partial(\eta_1) \partial(\eta_2)^{-1} \cdots \partial(\eta_{n+1})^{-1} \) vanishes in \( \pi_n(X_s, x) \). To prove this, choose simplices \( \tau_i : \Delta^{n+1} \to S \) for which each restriction \( \tau_i|_{\partial \Delta^{n+1}} \) is the constant map taking the value \( s \) and \([\tau_i] = \eta_i \). Using our assumption that \( f \) is a Kan fibration, we can lift each \( \tau_i \) to a simplex \( \tilde{\tau}_i : \Delta^{n+1} \to X \) carrying the horn \( \Lambda_0^{n+1} \) to the vertex \( x \in X \), so that \( \partial(\eta_i) = [d_0(\tilde{\tau}_i)] \). Since \( \pi_{n+1}(S, s) \) is abelian, the vanishing of the product \( \eta_0^{-1} \eta_1 \eta_2^{-1} \cdots \eta_{n+1}^{-1} \) guarantees that we can choose an \((n + 2)\)-simplex \( \rho : \Delta^{n+2} \to S \) such that \( d_0(\rho) = \tau_{i-1} \) for \( 1 \leq i \leq n + 2 \). Let \( \rho_0 : \Lambda_0^{n+2} \to X \) be the map given by the tuple of \((n + 1)\)-simplices \((\bullet, \tilde{\tau}_0, \tilde{\tau}_1, \ldots, \tilde{\tau}_{n+1})\) (see Exercise 1.1.2.14). Since \( f \) is a Kan
fibration, the lifting problem

\[
\begin{array}{ccc}
\Lambda^{n+2} & \xrightarrow{\bar{\rho}} & X \\
\downarrow & & \downarrow f \\
\Delta^{n+2} & \xrightarrow{\rho} & S
\end{array}
\]

admits a solution. Then \(\sigma = d_0(\rho)\) is an \((n+1)\)-simplex of \(X_s\) satisfying \(d_i(\sigma) = d_0(\tau_i)\) for \(0 \leq i \leq n + 1\), and therefore witnesses that the product

\[
[d_0(\sigma)]^{-1}[d_1(\sigma)][d_2(\sigma)]^{-1} \cdots [d_{n+1}(\sigma)]^{(-1)^n} = \partial(\eta_0)\partial(\eta_1)\partial(\eta_2)^{-1} \cdots \partial(\eta_{n+1})^{(-1)^n}
\]

vanishes in the homotopy group \(\pi_n(X_s, x)\).

In the special case \(n = 0\), we do not have a group structure on the set \(\pi_0(X_s, x)\), so we cannot assert that the connecting map \(\partial : \pi_1(S, s) \to \pi_0(X_s, x)\) is a group homomorphism. Nevertheless, the map \(\partial\) is compatible with the group structure on \(\pi_1(S, s)\) in the following sense:

**Variant 3.2.4.5.** Let \(f : X \to S\) be a Kan fibration between Kan complexes, let \(s\) be a vertex of \(S\), and set \(X_s = \{s\} \times_S X\). Then there is a unique left action \(a : \pi_1(S, s) \times \pi_0(X_s) \to \pi_0(X_s)\) of the fundamental group \(\pi_1(S, s)\) on \(\pi_0(X_s)\) with the following property;

(*) For each element \(\eta \in \pi_1(S, s)\) and each vertex \(x\) of \(X_s\), we have \(a(\eta, [x]) = \partial_x(\eta)\), where \(\partial_x : \pi_1(S, s) \to \pi_0(X_s, x) = \pi_0(X_s)\) is given by Proposition 3.2.4.2.

**Proof.** We first show that the function \(a\) is well-defined: that is, that the map \(\partial_x : \pi_1(S, s) \to \pi_0(X_s)\) depends only on the image of \(x\) in \(\pi_0(X_s)\). Fix an element \(\eta \in \pi_1(S, s)\), which we can write as the homotopy class of an edge \(v : s \to s\) in the Kan complex \(S\). Let \(x\) and \(x'\) be vertices belonging to the same connected component of \(X_s\), so that there exists an edge \(u : x' \to x\) of \(X\) satisfying \(f(u) = \text{id}_s\). We wish to show that \(\partial_x(\eta) = \partial_{x'}(\eta)\) in \(\pi_0(X_s)\). Since \(f\) is a Kan fibration, we can lift \(v\) to an edge \(\bar{v} : x \to y\) in \(X\). Using the fact that \(f\) is a Kan fibration, we can solve the lifting problem

\[
\begin{array}{ccc}
\Lambda^2 & \xrightarrow{(\bar{v} \bullet u)} & X \\
\downarrow & & \downarrow f \\
\Delta^2 & \xrightarrow{s_0(v)} & S
\end{array}
\]
to obtain a 2-simplex $\sigma$ of $X$ depicted in the diagram

$$
\begin{array}{c}
\text{x} \\
\downarrow \tilde{\nu} \\
\text{x}' \\
\downarrow \tilde{\nu}' \\
\text{y}.
\end{array}
$$

The edges $\tilde{\nu}$ and $\tilde{\nu}'$ then witness the identities $\partial_x(\eta) = [y] = \partial_{x'}(\eta)$ in $\pi_0(X_s)$.

We now complete the proof by showing that the function $a : \pi_1(S, s) \times \pi_0(X_s) \to \pi_0(X_s)$ determines a left action of $\pi_1(S, s)$ on $\pi_0(X_s)$. Note that the identity element of $\pi_1(S, s)$ is given by the homotopy class of the degenerate edge $\text{id}_s : s \to s$ of $S$. For each $x \in X_s$, we can lift $\text{id}_s$ to the edge $\text{id}_x : x \to x$ of $X$, which witnesses the identity $a([\text{id}_s], [x]) = \partial_x([\text{id}_s]) = [x]$ in $\pi_0(X_s)$. To complete the argument, it will suffice to show that for every pair of edges $g, g' : s \to s$ of $S$ and every vertex $x \in X_s$, we have an equality $a([g'][g], [x]) = a([g'], [y]) = a([g], [x])$ in $\pi_0(X_s)$. Since $f$ is a Kan fibration, we can lift $g$ to an edge $\tilde{g} : x \to y$ in $X$, and $g'$ to an edge $\tilde{g}' : y \to z$ in $X$. Since $X$ is a Kan complex, the map $(\tilde{g}', \bullet, \tilde{g}) : \Lambda^2_1 \to X$ can be completed to a 2-simplex $\sigma$ of $X$, as depicted in the diagram

$$
\begin{array}{c}
y \\
\downarrow \tilde{g} \\
x \\
\downarrow \tilde{g}' \\
z.
\end{array}
$$

The edges $\tilde{g}, \tilde{g}'$, and $\tilde{g}''$ then witness the identities $a([g], [x]) = [y], a([g'], [y]) = [z]$, and $a([g'][g], [x]) = [z]$ (respectively), so that we have an equality

$$a([g'][g], [x]) = [z] = a([g'], [y]) = a([g'], a([g], [x]))$$

as desired.

**Warning 3.2.4.6.** Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes. Then $x$ and $s$ can also be regarded as vertices of the opposite simplicial sets $X^\text{op}_s$ and $S^\text{op}$, respectively, and we have canonical bijections $\pi_{n+1}(S, s) \simeq \pi_{n+1}(S^\text{op}, s)$ and $\pi_n(X_s, x) \simeq \pi_n(X^\text{op}_s, x)$, respectively. However, these bijections are not necessarily compatible with the connecting homomorphisms Construction 3.2.4.3. The diagram

$$
\begin{array}{c}
\pi_{n+1}(S, s) \sim \pi_{n+1}(S^\text{op}, s) \\
\downarrow \partial \\
\pi_n(X_s, x) \sim \pi_n(X^\text{op}_s, x)
\end{array}
$$
commutes when \( n \) is odd, but *anticommutes* if \( n \geq 2 \) is even. This phenomenon is also visible in the case \( n = 0 \): in this case, the connecting maps \( \partial : \pi_1(S^{\text{op}}, s) \to \pi_0(X^{\text{op}}, x) \) determine a left action of the fundamental group \( \pi_1(S^{\text{op}}, s) \) on \( \pi_0(X^{\text{op}}, x) \simeq \pi_0(X, x) \), which can be interpreted as a right action of the group \( \pi_1(S, s) \) on \( \pi_0(X, x) \) (see Remark 3.2.2.16). To recover the left action of Variant 3.2.4.5, we must compose with the anti-homomorphism \( \pi_1(S, s) \to \pi_1(S, s) \) given by \( \eta \mapsto \eta^{-1} \).

### 3.2.5 The Long Exact Sequence of a Fibration

If \( (X, x) \) is a pointed Kan complex, then we regard each \( \pi_n(X, x) \) as a pointed set, with base point given by the homotopy class of the constant map \( \Delta^n \to \{x\} \subseteq X \) (if \( n \geq 1 \), then this is the identity element with respect to the group structure on \( \pi_n(X, x) \)). Recall that a diagram of pointed sets

\[
\cdots \to (G_{n+1}, e_{n+1}) \xrightarrow{f_n} (G_n, e_n) \xrightarrow{f_{n-1}} (G_{n-1}, e_{n-1}) \to \cdots
\]

is said to be *exact* if the image of each \( f_n \) is equal to the fiber \( f^{-1}_{n-1}\{e_{n-1}\} = \{g \in G_n : f_{n-1}(g) = e_{n-1}\} \). Our goal in this section is to prove the following:

**Theorem 3.2.5.1.** Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes. Then the sequence of pointed sets

\[
\cdots \to \pi_2(S, s) \xrightarrow{\partial} \pi_1(X, x) \to \pi_1(S, s) \xrightarrow{\partial} \pi_0(X, x) \to \pi_0(S, x) \to \pi_0(S, s)
\]

is exact; here \( \partial : \pi_{n+1}(S, s) \to \pi_n(X, x) \) denotes the connecting homomorphism of Construction 3.2.4.3.

Theorem 3.2.5.1 really amounts to three separate assertions, which we will formulate and prove individually (Propositions 3.2.5.2, 3.2.5.4, and 3.2.5.6).

**Proposition 3.2.5.2.** Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes and let \( n \geq 0 \) be an integer. Then the sequence of pointed sets

\[
\pi_n(X, x) \to \pi_n(X, x) \to \pi_n(S, s)
\]

is exact.

In the special case \( n = 0 \), the content of Proposition 3.2.5.2 can be formulated without reference to the base point \( x \in X \):

**Corollary 3.2.5.3.** Let \( f : X \to S \) be a Kan fibration between Kan complexes, let \( s \) be a vertex of \( S \), and set \( X_s = \{s\} \times_S X \). Then the image of the map \( \pi_0(X_s) \to \pi_0(X) \) is equal to the fiber of the map \( \pi_0(f) : \pi_0(X) \to \pi_0(S) \) over the connected component \( [s] \in \pi_0(S) \) determined by the vertex \( s \). In other words, a vertex \( x \in X \) satisfies \( |f(x)| = [s] \) in \( \pi_0(S) \) if and only if the connected component of \( x \) has nonempty intersection with the fiber \( X_s \).
3.2. HOMOTOPY GROUPS

Proof of Proposition 3.2.5.4. Fix an $n$-simplex $\sigma : \Delta^n \to X$ such that $\sigma|_{\partial \Delta^n}$ is the constant map carrying $\partial \Delta^n$ to the base point $x \in X$. We wish to show that the homotopy class $[\sigma]$ belongs to the image of the map $\pi_n(X_x, x) \to \pi_n(X, x)$ if and only if the image $[f(\sigma)]$ is equal to the base point of $\pi_n(S, s)$. The “only if” direction is clear, since the composite map $X_s \to X \xrightarrow{f} S$ is equal to the constant map taking the value $s$. For the converse, suppose that $[f(\sigma)]$ is the base point of $\pi_n(S, s)$. Then there exists a homotopy $h : \Delta^1 \times \Delta^n \to S$ from $f(\sigma)$ to the constant map $\sigma'_0 : \Delta^n \to \{s\} \subseteq S$, which is constant when restricted to the boundary $\partial \Delta^n$. Since $f$ is a Kan fibration, we can lift $h$ to a homotopy $\tilde{h} : \Delta^1 \times \Delta^n \to X$, where $\tilde{h}$ is constant along the boundary $\partial \Delta^n$ and $f(\sigma') = \sigma'_0$ (Remark 3.1.4.3). Then $\sigma'$ represents a homotopy class $[\sigma'] \in \pi_n(X_x, x)$, and the homotopy $\tilde{h}$ witnesses that $[\sigma]$ is equal to the image of $[\sigma']$ in $\pi_n(X, x)$. □

Proposition 3.2.5.4. Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be an integer. Then the sequence of pointed sets $\pi_{n+1}(S, s) \xrightarrow{\partial} \pi_n(X_x, x) \to \pi_n(X, x)$ is exact, where $\partial$ is the connecting homomorphism of Construction 3.2.4.3.

In the special case $n = 0$, Proposition 3.2.5.4 can also be formulated without reference to the base point $x \in X$.

Corollary 3.2.5.5. Let $f : X \to S$ be a Kan fibration between Kan complexes, let $s$ be a vertex of $S$, and set $X_s = \{s\} \times_S X$. Then two elements of $\pi_0(X_s)$ have the same image in $\pi_0(X)$ if and only if they belong to the same orbit of the action of the fundamental group $\pi_1(S, s)$ (see Variant 3.2.4.5). In other words, the inclusion of Kan complexes $X_s \hookrightarrow X$ induces a monomorphism of sets $(\pi_1(S, s) \backslash \pi_0(X_s)) \hookrightarrow \pi_0(X)$.

Proof. Combine Variant 3.2.4.5 with Proposition 3.2.5.4. □

Proof of Proposition 3.2.5.4. Fix an $n$-simplex $\sigma : \Delta^n \to X_s$ such that $\sigma|_{\partial \Delta^n}$ is the constant map carrying $\partial \Delta^n$ to the base point $x \in X_s$. By construction, the homotopy class $[\sigma] \in \pi_n(X_x, x)$ belongs to the image of the connecting homomorphism $\partial : \pi_{n+1}(S, s) \to \pi_n(X_x, x)$ if and only if there exists an $(n + 1)$-simplex $\tau : \Delta^{n+1} \to S$ such that $\tau|_{\partial \Delta^{n+1}}$ is the constant map taking the value $s$ and $\tau$ is incident to $\sigma$. This condition is equivalent to the existence of an $(n + 1)$-simplex $\tilde{\tau} : \Delta^{n+1} \to X$ satisfying $d_0(\tilde{\tau}) = \sigma$ and $d_i(\tilde{\tau})$ is equal to the constant map $e : \Delta^n \to \{x\} \subseteq X$ for $1 \leq i \leq n + 1$. In other words, it is equivalent to the assertion that the tuple of $n$-simplices of $X$ $(\sigma, e, e, \ldots, e)$ bounds, in the sense of Notation 3.2.3.1. For $n \geq 1$, this is equivalent to the vanishing of the image of $[\sigma]$ in the homotopy group $\pi_n(X, x)$ (Theorem 3.2.2.10). When $n = 0$, it is equivalent to the equality $[\sigma] = [x]$ in $\pi_0(X)$ by virtue of Remark 1.3.6.14. □
Proposition 3.2.5.6. Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes and let \( n \geq 0 \) be an integer. Then the sequence of pointed sets \( \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(f)} \pi_{n+1}(S, s) \xrightarrow{\partial} \pi_n(X, x) \) is exact, where \( \partial \) is the connecting homomorphism of Construction 3.2.4.3.

Corollary 3.2.5.7. Let \( f : (X, x) \to (S, s) \) be a Kan fibration between pointed Kan complexes. Then the image of the induced map \( \pi_1(f) : \pi_1(X, x) \to \pi_1(S, s) \) is equal to the stabilizer of \([x] \in \pi_0(X_s)\) (with respect to the action of \( \pi_1(S, s) \)) on \( \pi_0(X_s) \) supplied by Variant 3.2.4.5.

Proof. Combine Variant 3.2.4.5 with Proposition 3.2.5.6.

Proof of Proposition 3.2.5.6. Fix an \((n+1)\)-simplex \( \tau : \Delta^{n+1} \to S \) for which \( \tau|_{\partial\Delta^{n+1}} \) is the constant map taking the value \( s \). By construction, the connecting homomorphism \( \partial : \pi_{n+1}(S, s) \to \pi_n(X, x) \) carries \([\tau]\) to the base point of \( \pi_n(X, x) \) if and only if the constant map \( e : \Delta^n \to \{x\} \hookrightarrow X_s \) is incident to \( \tau \), in the sense of Definition 3.2.4.1. This is equivalent to the requirement that \( \tau \) can be lifted to a map \( \tilde{\tau} : \Delta^{n+1} \to X \) for which \( \tilde{\tau}|_{\partial\Delta^{n+1}} \) is the constant map taking the value \( x \), which clearly implies that that \([\tau] \) belongs to the image of the map \( \pi_{n+1}(f) : \pi_{n+1}(X, x) \to \pi_{n+1}(S, s) \). To prove the reverse implication, suppose that \([\tau] \) belongs to the image of \( \pi_{n+1}(f) \), so that we can write \([\tau] = [f(\tilde{\tau}')]\) for some map \( \tilde{\tau}' : \Delta^{n+1} \to X \) for which \( \tilde{\tau}'|_{\partial\Delta^{n+1}} \) is the constant map taking the value \( x \). It follows that there is a homotopy \( h : \Delta^1 \times \Delta^{n+1} \to S \) from \( f(\tilde{\tau}') \) to \( \tau \) which is constant along the boundary \( \partial\Delta^{n+1} \). Since \( f \) is a Kan fibration, we can lift \( h \) to a map \( \tilde{h} : \Delta^1 \times \Delta^{n+1} \to X \) such that \( h|\{0\} \times \Delta^{n+1} = \tilde{\tau}' \) and \( h|\Delta^1 \times \partial\Delta^{n+1} \) is the constant map taking the value \( x \) (Remark 3.1.4.3). The restriction \( \tilde{\tau} = h|\{1\} \times \Delta^{n+1} \) then satisfies \( f(\tilde{\tau}) = \tau \) and \( \tilde{\tau}|_{\partial\Delta^{n+1}} \) is the constant map taking the value \( x \).

3.2.6 Whitehead’s Theorem for Kan Complexes

Let \( f : X \to Y \) be a continuous function between nonempty topological spaces. If \( X \) and \( Y \) are CW complexes, then a classical theorem of Whitehead (see 39) asserts that \( f \) is a homotopy equivalence if and only if it induces a bijection \( \pi_0(X) \cong \pi_0(Y) \) and, for every base point \( x \in X \), the induced map of homotopy groups \( \pi_n(X, x) \to \pi_n(Y, f(x)) \) is an isomorphism for \( n > 0 \) (Corollary 3.5.3.10). Our goal in this section is to prove an analogous statement in the setting of Kan complexes.

Theorem 3.2.6.1. Let \( f : X \to Y \) be a morphism of Kan complexes. Then \( f \) is a homotopy equivalence if and only if it satisfies the following pair of conditions:

(a) The map of sets \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \) is a bijection.

(b) For every vertex \( x \in X \) having image \( y = f(x) \) in \( Y \) and every positive integer \( n \), the map of homotopy groups \( \pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y) \) is bijective.
Our first step will be to prove the easy direction of Theorem 3.2.6.1, which asserts that every homotopy equivalence \( f : X \to Y \) induces an isomorphism on homotopy groups. Here we encounter a slight annoyance: the hypothesis that \( f \) is a homotopy equivalence guarantees that it induces an isomorphism in the homotopy category \( h \text{Kan} \) of Kan complexes, but the homotopy groups of \( X \) and \( Y \) (with respect to base points \( x \in X \) and \( y \in Y \)) are computed by viewing \((X, x)\) and \((Y, y)\) as objects of the homotopy category \( h \text{Kan}_* \) of pointed Kan complexes. To address this point, we prove the following:

**Proposition 3.2.6.2.** Let \( f : (X, x) \to (Y, y) \) be a morphism of pointed Kan complexes. The following conditions are equivalent:

1. The underlying morphism of simplicial sets \( f : X \to Y \) is a homotopy equivalence (Definition 3.1.5.1): that is, there exists a morphism of simplicial sets \( g : Y \to X \) such that \( g \circ f \) and \( f \circ g \) are homotopic to the identity maps \( \text{id}_X \) and \( \text{id}_Y \), respectively.

2. The map \( f \) is a pointed homotopy equivalence: that is, there exists a morphism of pointed simplicial sets \( g : (Y, y) \to (X, x) \) such that \( g \circ f \) and \( f \circ g \) are pointed homotopic to the identity maps \( \text{id}_X \) and \( \text{id}_Y \), respectively.

*Proof.* The implication \( (2) \Rightarrow (1) \) is clear. For the converse, assume that \( f \) is a homotopy equivalence; we will prove that \( f \) induces an isomorphism in the pointed homotopy category \( h \text{Kan}_* \). Fix another pointed Kan complex \((Z, z)\), and consider the evaluation maps

\[
ev_x : \text{Fun}(X, Z) \to Z \quad \ev_y : \text{Fun}(Y, Z) \to Z.
\]

Since \( Z \) is a Kan complex, both \( \ev_x \) and \( \ev_y \) are Kan fibrations (Corollary 3.1.3.3). Let \( \text{Fun}(X, Z)_z = \{z\} \times_Z \text{Fun}(X, Z) \) and \( \text{Fun}(Y, Z)_z = \{z\} \times_Z \text{Fun}(Y, Z) \) denote the fibers of \( \ev_x \) and \( \ev_y \) over the vertex \( z \). We wish to show that precomposition with \( f \) induces a bijection

\[
\theta : \pi_0 \text{Fun}(Y, Z)_z = \text{Hom}_{h \text{Kan}_*}((Y, y), (Z, z)) \to \text{Hom}_{h \text{Kan}_*}((X, x), (Z, z)) = \pi_0 \text{Fun}(X, Z)_z.
\]

Since \( f \) is a homotopy equivalence, precomposition with \( f \) induces a homotopy equivalence of Kan complexes \( \text{Fun}(Y, Z) \to \text{Fun}(X, Z) \). It follows that the induced map of homotopy categories \( h \text{Fun}(Y, Z) \to h \text{Fun}(X, Z) \) is an equivalence (Remark 3.1.5.4). In particular, the induced map \( \pi_0(\text{Fun}(Y, Z)) \to \pi_0(\text{Fun}(X, Z)) \) is bijective. We have a commutative diagram of pointed sets

\[
\begin{array}{ccc}
\pi_0(\text{Fun}(Y, Z)_z) & \xrightarrow{\psi} & \pi_0(\text{Fun}(Y, Z)) \\
\downarrow{\theta} & & \downarrow{\text{id}} \\
\pi_0(\text{Fun}(X, Z)_z) & \xrightarrow{\psi} & \pi_0(\text{Fun}(X, Z))
\end{array}
\]

\[
\begin{array}{ccc}
\pi_0(\text{Fun}(Y, Z)) & \xrightarrow{\psi} & \pi_0(\text{Fun}(Y, Z)) \\
\downarrow{\theta} & & \downarrow{\text{id}} \\
\pi_0(\text{Fun}(X, Z)) & \xrightarrow{\psi} & \pi_0(\text{Fun}(X, Z))
\end{array}
\]
where each row is exact (Corollary 3.2.5.3), so that the induced map \(\text{im}(v) \to \text{im}(u)\) is a bijection. Using Variant 3.2.4.5, we see that the fundamental group \(\pi_1(Z, z)\) acts on both \(\pi_0(\text{Fun}(Y, Z)_z)\) and \(\pi_0(\text{Fun}(X, Z)_z)\), and we can identify the images of \(v\) and \(u\) with the quotient sets \(\pi_1(Z, z) \setminus \pi_0(\text{Fun}(Y, Z)_z)\) and \(\pi_1(Z, z) \setminus \pi_0(\text{Fun}(X, Z)_z)\), respectively (Corollary 3.2.5.5). Consequently, to show that \(\theta\) is bijective, it will suffice to show that \(\theta\) induces a bijection from each orbit of \(\pi_1(Z, z)\) in \(\pi_0(\text{Fun}(Y, Z)_z)\) to the corresponding orbit in \(\pi_0(\text{Fun}(X, Z)_z)\). Equivalently, we must show that for every pointed map \(g : (Y, y) \to (Z, z)\), the stabilizer of \([g] \in \pi_0(\text{Fun}(Y, Z)_z)\) is equal to the stabilizer of \([g \circ f] \in \pi_0(\text{Fun}(X, Z)_z)\).

By virtue of Corollary 3.2.5.7, this is equivalent to the assertion that the maps

\[
\pi_1(\text{Fun}(Y, Z), g) \to \pi_1(Z, z) \quad \pi_1(\text{Fun}(X, Z), g \circ f) \to \pi_1(Z, z)
\]

have the same image. This follows from the fact that \(f\) induces an isomorphism of fundamental groups \(\pi_1(\text{Fun}(Y, Z), g) \to \pi_1(\text{Fun}(X, Z), g \circ f)\) (because the functor of fundamental groupoids \(\tau_{\leq 1}(\text{Fun}(Y, Z)) \to \tau_{\leq 1}(\text{Fun}(X, Z))\) is an equivalence, by virtue of Remark 3.1.5.4).

\[\square\]

**Corollary 3.2.6.3.** Let \(f : (X, x) \to (Y, y)\) be a morphism of pointed Kan complexes, and suppose that the underlying morphism of simplicial sets \(X \to Y\) is a homotopy equivalence. Then, for every nonnegative integer \(n \geq 0\), the induced map \(\pi_n(X, x) \to \pi_n(Y, y)\) is a bijection.

**Definition 3.2.6.4.** Let \(X\) be a simplicial set. We will say that \(X\) is **contractible** if the projection map \(X \to \Delta^0\) is a homotopy equivalence (Definition 3.1.5.1). We say that \(X\) is **weakly contractible** if the projection map \(X \to \Delta^0\) is a weak homotopy equivalence (Definition 3.1.5.11).

**Remark 3.2.6.5.** Let \(X\) be a simplicial set. If \(X\) is contractible, then it is weakly contractible. The converse holds if \(X\) is a Kan complex (see Proposition 3.1.5.12).

**Example 3.2.6.6.** Let \(C\) be a category. If \(C\) has an initial object or a final object, then the simplicial set \(N_\bullet(C)\) is contractible (this is a special case of Proposition 3.1.5.9).

**Proposition 3.2.6.7.** Let \(X\) be a Kan complex. The following conditions are equivalent:

1. The projection map \(X \to \Delta^0\) is a trivial Kan fibration.
2. The Kan complex \(X\) is contractible: that is, the projection map \(X \to \Delta^0\) is a homotopy equivalence.
3. The Kan complex \(X\) is nonempty. Moreover, for each vertex \(x \in X\) and each \(n \geq 0\), the set \(\pi_n(X, x)\) has a single element.
(4) The Kan complex $X$ is connected. Moreover, there exists a vertex $x \in X$ such that the homotopy groups $\pi_n(X, x)$ are trivial for $n \geq 1$.

**Proof.** The implication $(1) \Rightarrow (2)$ is a special case of Proposition 3.1.5.10, the implication $(2) \Rightarrow (3)$ follows from Corollary 3.2.6.3, and the implication $(3) \Rightarrow (4)$ is immediate. We will complete the proof by showing that $(4)$ implies $(1)$. Assume that $X$ is connected, and fix a vertex $x \in X$ for which the homotopy groups $\pi_n(X, x)$ vanish for $n \geq 1$. We first prove the following:

$(\ast)$ Let $f : B \to X$ be a morphism of simplicial sets, let $A \subseteq B$ be a simplicial subset, and let $h : \Delta^1 \times A \to X$ be a homotopy from $f|_A$ to the constant map $A \to \{x\} \subseteq X$. Then $h$ can be extended to a homotopy $h' : \Delta^1 \times B \to X$ from $f$ to the constant map $B \to \{x\} \subseteq X$.

To prove $(\ast)$, we may assume without loss of generality that $B = \Delta^n$ and $A = \partial \Delta^n$ for some $n \geq 0$. Since $X$ is a Kan complex, we can extend $h$ to a homotopy $h' : \Delta^1 \times \Delta^n \to X$ from $f$ to some other map $f' : \Delta^n \to X$ for which $f'|_{\partial \Delta^n}$ is the constant map taking the value $x$ (Remark 3.1.4.3). Replacing $f$ by $f'$, we can reduce to the case where $f|_{\partial \Delta^n}$ and $h : \Delta^1 \times \partial \Delta^n \to X$ are the constant maps taking the value $x$. In this case, the existence of the desired extension follows from the vanishing of the homotopy group $\pi_n(X, x)$ (or from the connectedness of $X$, in the special case $n = 0$).

We now prove that the map $X \to \Delta^0$ is a trivial Kan fibration. Fix an integer $n \geq 0$ and a morphism of simplicial sets $f : \partial \Delta^n \to X$; we wish to show that $f$ can be extended to an $n$-simplex of $X$. Applying $(\ast)$ in the case $B = \partial \Delta^n$ and $A = \emptyset$, we conclude that there exists a homotopy $h : \Delta^1 \times \partial \Delta^n \to X$ from $f$ to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. Because $X$ is a Kan complex, we can extend $h$ to a map $\bar{h} : \Delta^1 \times \Delta^n \to X$ for which $\bar{h}|_{\{1\} \times \Delta^n}$ is the constant map taking the value $x$ (Remark 3.1.4.3). The restriction $\bar{h}|_{\{0\} \times \Delta^n}$ then provides the desired extension of $f$.

We establish a relative version of Proposition 3.2.6.7.

**Proposition 3.2.6.8.** Let $f : X \to S$ be a Kan fibration of simplicial sets. The following conditions are equivalent:

1. The morphism $f$ is a trivial Kan fibration.
2. For each vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a contractible Kan complex.
3. For each vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is connected. Moreover, for each vertex $x \in X_s$, the homotopy groups $\pi_n(X_s, x)$ vanish for $n > 0$. 

00X1
Proof. The implication (1) ⇒ (2) and the equivalence (2) ⇔ (3) follow by applying Proposition 3.2.6.7 to the fibers of \( f \). We will complete the proof by showing that (2) implies (1). Assume that (2) is satisfied; we wish to show that every lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{\sigma_0} & X \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\sigma} & S
\end{array}
\]

admits a solution. Let \( q : \Delta^1 \times \Delta^n \to \Delta^n \) be the map given on vertices by the formula

\[
q(i, j) = \begin{cases} 
  j, & \text{if } i = 0 \\
  n, & \text{if } i = 1
\end{cases}
\]

Then we can regard \( \sigma \circ q \) as a homotopy from \( \sigma : \Delta^n \to S \) to the constant map \( \Delta^n \to \{s\} \subseteq S \), where \( s \) denotes the vertex \( \sigma(n) \in S \). Since \( f \) is a Kan fibration, the restriction \( (\sigma \circ q)|_{\Delta^1 \times \partial \Delta^n} \) can be lifted to a homotopy \( h : \Delta^1 \times \partial \Delta^n \to X \) from \( \sigma_0 \) to some map \( \sigma'_0 : \partial \Delta^n \to X_s \) (Remark 3.1.4.3). It follows from assumption (2) that we can extend \( \sigma'_0 \) to an \( n \)-simplex \( \sigma' : \Delta^n \to X_s \). Invoking our assumption that \( f \) is a Kan fibration again, we see that \( h \) can be extended to a homotopy \( \tilde{h} : \Delta^1 \times \Delta^n \to X \) from \( \sigma \) to \( \sigma' \), where \( \sigma : \Delta^n \to X \) is an extension of \( \sigma_0 \) satisfying \( f(\sigma) = \sigma \).

Corollary 3.2.6.9. Let \( f : X \to S \) be a Kan fibration between Kan complexes. The following conditions are equivalent:

1. The morphism \( f \) is a trivial Kan fibration.
2. The morphism \( f \) is a homotopy equivalence.
3. The map \( f \) induces a bijection \( \pi_0(f) : \pi_0(X) \to \pi_0(S) \). Moreover, for each vertex \( x \in X \) having image \( s = f(x) \) in \( S \), the induced map \( \pi_n(f) : \pi_n(X, x) \to \pi_n(S, s) \) is an isomorphism for \( n > 0 \).
4. For each vertex \( s \in S \), the fiber \( X_s \) is connected. Moreover, the homotopy groups \( \pi_n(X_s, x) \) vanish for each vertex \( x \in X_s \) and each \( n > 0 \).

Proof. The implication (1) ⇒ (2) follows from Proposition 3.1.5.10, the implication (2) ⇒ (3) from Corollary 3.2.6.3, and the implication (4) ⇒ (1) from Proposition 3.2.6.8. The implication (3) ⇒ (4) follows from the long exact sequence of Theorem 3.2.5.1.

Remark 3.2.6.10. For the equivalence (1) ⇔ (2) of Corollary 3.2.6.9, the assumption that \( X \) and \( S \) are Kan complexes is not needed: these assertions hold more generally for any Kan fibration \( f : X \to S \) (Proposition 3.3.7.4).
Proof of Theorem 3.2.6.1. Let $f : X \to Y$ be a morphism of Kan complexes. Assume that the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and that, for every vertex $x \in X$ having image $y = f(x)$ in $Y$, the induced map $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism. We wish to prove that $f$ is a homotopy equivalence (the converse implication follows from Corollary 3.2.6.3). By virtue of Proposition 3.1.6.1, we can assume that $f$ factors as a composition $X \overset{f'}\to Q \overset{f''}\to Y$, where $f'$ is anodyne and $f''$ is a Kan fibration. Note that $Q$ is automatically a Kan complex (since $Y$ is a Kan complex and $f''$ is a Kan fibration). Moreover, the anodyne morphism $f'$ is a weak homotopy equivalence (Proposition 3.1.5.13) between Kan complexes, and is therefore a homotopy equivalence (Proposition 3.1.5.12). Consequently, to show that $f$ is a homotopy equivalence, it will suffice to show that $f''$ is a homotopy equivalence (Remark 3.1.5.7). In fact, we claim that $f''$ is a trivial Kan fibration. By virtue of Proposition 3.2.6.8, it will suffice to show that for every vertex $y \in Y$, the Kan complex $Q_y = \{y\} \times_Y Q$ is contractible. Since $\pi_0(f)$ is a bijection, there exists a vertex $x \in X$ such that $f(x)$ and $y$ belong to the same connected component of $Y$. Since $f''$ is a Kan fibration, the Kan complexes $Q_y$ and $Q_{f(x)}$ are homotopy equivalent (see Theorem 5.1.2.2). We may therefore assume without loss of generality that $y = f(x)$. Set $q = f'(x) \in Q$. Using the criterion of Proposition 3.2.6.7, we are reduced to proving that the set $\pi_n(Q_{f(x)}, q)$ is a singleton for each $n \geq 0$. Using the exact sequence

$$
\cdots \to \pi_2(Y, y) \overset{\partial}{\to} \pi_1(Q_y, q) \to \pi_1(Q, q) \to \pi_1(Y, y) \overset{\partial}{\to} \pi_0(Q_y, q) \to \pi_0(Q, q) \to \pi_0(Y, y)
$$

of Theorem 3.2.5.1, we are reduced to proving that each of the maps $\pi_n(f'') : \pi_n(Q, q) \to \pi_n(Y, y)$ is bijective. This follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\pi_n(X, x) & \overset{\partial}{\longrightarrow} & \pi_n(Q, q) \\
\downarrow & & \downarrow \\
\pi_n(Y, y) & & \\
\end{array}
$$

since the left vertical map is bijective by assumption and the upper horizontal map is bijective by virtue of Corollary 3.2.6.3.

3.2.7 Closure Properties of Homotopy Equivalences

We now apply Whitehead’s theorem (Theorem 3.2.6.1) to establish some stability properties for the collection of homotopy equivalences between Kan complexes (and weak homotopy equivalences between arbitrary simplicial sets).
Proposition 3.2.7.1. Suppose we are given a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow{f} & & \downarrow{f'} \\
S & \xrightarrow{h} & S',
\end{array}
\]

where \( f \) and \( f' \) are Kan fibrations and \( h \) is a homotopy equivalence. Then the following conditions are equivalent:

(1) The morphism \( g \) is a homotopy equivalence.

(2) For each vertex \( s \in S \) having image \( s' = h(s) \) in \( S' \), the map of fibers \( g_s : X_s \to X'_s \) is a homotopy equivalence.

Remark 3.2.7.2. In the situation of Proposition 3.2.7.1, the assumption that \( S \) and \( S' \) are Kan complexes can be eliminated at the cost of working with weak homotopy equivalences in place of homotopy equivalences: see Proposition 3.3.7.1.

Proof of Proposition 3.2.7.1. Assume first that (1) is satisfied. Let \( s \) be a vertex of \( S \) having image \( s' = h(s) \) in \( S' \); we wish to show that the induced map \( g_s : X_s \to X'_s \) is a homotopy equivalence. By virtue of Remark 3.1.5.5, it will suffice to show that for every simplicial set \( W \), the induced map \( \text{Fun}(W, X_s) \to \text{Fun}(W, X'_s) \) is bijective on connected components. Replacing \( X \) by \( \text{Fun}(W, X) \) (and making similar replacements for \( X', S, \) and \( S' \)), we may reduce to the problem of showing that \( g_s \) induces a bijection \( \pi_0(X_s) \to \pi_0(X'_s) \). Let us regard \( \pi_0(X_s) \) and \( \pi_0(X'_s) \) as endowed with actions of the fundamental groups \( \pi_1(S, s) \) and \( \pi_1(S', s') \), respectively (Variant 3.2.4.5). Using our assumption that \( g \) and \( h \) are homotopy equivalences, we conclude that the induced maps

\[
\pi_0(X) \to \pi_0(X') \quad \pi_0(S) \to \pi_0(S') \quad \pi_1(S, s) \to \pi_1(S', s')
\]

are bijective. Applying Corollaries 3.2.5.3 and 3.2.5.5, we conclude that \( g_s \) induces a bijection \( G \setminus \pi_0(X_s) \to G \setminus \pi_0(X'_s) \). It will therefore suffice to show that, for every vertex \( x \in X_s \), the stabilizer in \( G \) of the connected component \([x] \in \pi_0(X_s)\) is equal to the stabilizer of the connected component \([g(x)] \in \pi_0(X'_s)\). This follows from Corollary 3.2.5.7 since \( g \) induces an isomorphism \( \pi_1(X, x) \to \pi_1(X', g(x)) \).

We now show that (2) \( \Rightarrow \) (1). Assume that, for each vertex \( s \in S \) having image \( s' = h(s) \) in \( S' \), the induced map \( g_s : X_s \to X'_s \) is a homotopy equivalence. We wish to show that \( g \) is a homotopy equivalence. We first show that the map \( \pi_0(g) : \pi_0(X) \to \pi_0(X') \) is bijective. Our assumption that \( h \) is a homotopy equivalence guarantees that the map \( \pi_0(h) : \pi_0(S) \to \pi_0(S') \) is bijective. It will therefore suffice to show that, for each vertex \( s \in S \) having image \( s' = h(s) \), the induced map \( \pi_0(X) \times_{\pi_0(S)} \{s\} \to \pi_0(X') \times_{\pi_0(S')} \{s'\} \) is
bijective. Using Corollaries 3.2.5.3 and 3.2.5.5 we can identify this with the map of quotients \((\pi_1(S, s) \setminus \pi_0(X_s)) \to (\pi_1(S', s') \setminus \pi_0(X'_s))\). The desired result now follows from the bijectivity of the map \(\pi_0(g_s) : \pi_0(X_s) \to \pi_0(X'_s)\) and of the group homomorphism \(\pi_1(S, s) \to \pi_1(S', s')\).

To complete the proof that \(g\) is a homotopy equivalence, it will suffice (by virtue of Theorem 3.2.6.1) to show that for every vertex \(x \in X\) having image \(x' = g(x)\) and every positive integer \(n\), the group homomorphism \(\pi_n(X, x) \to \pi_n(X', x')\) is an isomorphism. Setting \(s = f(x)\) and \(s' = f(x')\), we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccc}
\pi_{n+1}(S, s) & \longrightarrow & \pi_n(X_s, x) & \longrightarrow & \pi_n(X, x) & \longrightarrow & \pi_n(S, s) & \longrightarrow & \pi_{n-1}(X_s, x) \\
\sim & & \sim & & \sim & & \sim & & \\
\pi_{n+1}(S', s') & \longrightarrow & \pi_n(X'_s, x') & \longrightarrow & \pi_n(X', x') & \longrightarrow & \pi_n(S', s') & \longrightarrow & \pi_{n-1}(X'_s, x').
\end{array}
\]

Our assumptions that \(g_s\) and \(h\) are homotopy equivalences guarantee that the outer vertical maps are bijective, and elementary diagram chase shows that that the middle vertical map is an isomorphism.

**Proposition 3.2.7.3.** Let \(W\) denote the full subcategory of \(\text{Fun}([1], \text{Set}_\Delta)\) spanned by those morphisms of simplicial sets \(f : X \to Y\) which are weak homotopy equivalences. Then \(W\) is closed under the formation of filtered colimits in \(\text{Fun}([1], \text{Set}_\Delta)\).

**Proof.** Suppose we are given a filtered diagram \(\{f_\alpha : X_\alpha \to Y_\alpha\}\) in \(W\), so that each \(f_\alpha\) is a weak homotopy equivalence of simplicial sets. We wish to show that the induced map \(f : (\lim \underset{\alpha}{\to} X_\alpha) \to (\lim \underset{\alpha}{\to} Y_\alpha)\) is also a weak homotopy equivalence. Using Proposition 3.1.6.1 we can choose a diagram of morphisms \(\{u_\alpha : Y_\alpha \hookrightarrow Y'_\alpha\}\) with the following properties:

- Each of the maps \(u_\alpha\) is anodyne, and the induced map \(u : (\lim \underset{\alpha}{\to} Y_\alpha) \to (\lim \underset{\alpha}{\to} Y'_\alpha)\) is anodyne.

- Each of the simplicial sets \(Y'_\alpha\) is a Kan complex, and (therefore) the colimit \(\lim \underset{\alpha}{\to} Y'_\alpha\) is also a Kan complex.

Since every anodyne morphism is a weak homotopy equivalence (Proposition 3.1.5.13), we can replace \(\{f_\alpha : X_\alpha \to Y_\alpha\}\) by the diagram of composite maps \(\{(u_\alpha \circ f_\alpha) : X_\alpha \to Y'_\alpha\}\), and therefore reduce to the case where each \(Y_\alpha\) is a Kan complex.

Let us regard the system of morphisms \(\{f_\alpha\}\) as a morphism from the filtered diagram of simplicial sets \(\{X_\alpha\}\) to the filtered diagram \(\{Y_\alpha\}\). Applying Proposition 3.1.6.1 again, we see that this diagram admits a factorization \(X_\alpha \xrightarrow{\{g_\alpha\}} X'_\alpha \xrightarrow{\{h_\alpha\}} Y_\alpha\) with the following properties:

- Each of the morphisms \(g_\alpha\) is anodyne, and the induced map \(g : (\lim \underset{\alpha}{\to} X_\alpha) \to (\lim \underset{\alpha}{\to} X'_\alpha)\) is anodyne.
Each of the morphisms $h_\alpha$ is a Kan fibration, and (therefore) the induced map 
$$\left( \lim_{\alpha \leftarrow} X'_\alpha \right) \to \left( \lim_{\alpha \leftarrow} Y_\alpha \right)$$ 
is also a Kan fibration.

Arguing as before, we can replace \( \{f_\alpha : X_\alpha \to Y_\alpha\} \) by the diagram of morphisms \( \{h_\alpha : X'_\alpha \to Y_\alpha\} \), and thereby reduce to the case where each \( f_\alpha \) is a Kan fibration. In this case, Proposition 3.2.6.8 guarantees that each \( f_\alpha \) is a trivial Kan fibration. It follows that the colimit map \( f : (\lim_{\alpha \leftarrow} X_\alpha) \to (\lim_{\alpha \leftarrow} Y_\alpha) \) is also a trivial Kan fibration, and therefore a (weak) homotopy equivalence by virtue of Proposition 3.1.5.10.

3.3 The $\text{Ex}^\infty$ Functor

Let \( f : X \to S \) be a Kan fibration of simplicial sets. If \( S \) is a Kan complex, then \( X \) is also a Kan complex. Moreover, for every vertex \( x \in X \) having image \( s = f(x) \in S \), Theorem 3.2.5.1 supplies an exact sequence of homotopy groups

$$\cdots \to \pi_2(S,s) \to \pi_1(X_s,x) \to \pi_1(X,x) \to \pi_1(S,s) \to \pi_0(X_s,x) \to \pi_0(X,x) \to \pi_0(S,s).$$

If \( S \) is not a Kan complex, then the results of §3.2.5 do not apply directly. However, one can obtain similar information by replacing \( f \) by a Kan fibration \( f' : X' \to S' \) between Kan complexes, using the following result:

**Theorem 3.3.0.1.** Let \( f : X \to S \) be a Kan fibration of simplicial sets. Then there exists a commutative diagram of simplicial sets

$$\begin{array}{ccc}
X & \xrightarrow{g'} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{g} & S'
\end{array}$$

with the following properties:

(a) The simplicial sets \( S' \) and \( X' \) are Kan complexes.

(b) The morphisms \( g \) and \( g' \) are weak homotopy equivalences.

(c) The morphism \( f' \) is a Kan fibration.

(d) For every vertex \( s \in S \), the induced map \( g'_s : X_s \to X'_{g(s)} \) is a homotopy equivalence of Kan complexes.

Note that we can almost deduce Theorem 3.3.0.1 formally from the results of §3.1.6. Given a Kan fibration \( f : X \to S \), we can always choose an anodyne map \( g : S \to S' \), where \( S' \) is a Kan complex (Corollary 3.1.6.2). Applying Proposition 3.1.6.1, we deduce that \( g \circ f \)
3.3. **THE $\text{Ex}^\infty$ FUNCTOR**

factors as a composition $X \xrightarrow{g'} X' \xrightarrow{f'} S'$, where $f'$ is a Kan fibration and $g'$ is anodyne. The resulting commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g'} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{g} & S'
\end{array}
$$

then satisfies conditions (a), (b), and (c) of Theorem 3.3.0.1. However, it is not so obvious that this diagram also satisfies condition (d). To guarantee this, it is convenient to adopt a different approach to the results of §3.1.6. Following Kan ([22]), we will introduce a functor $\text{Ex}^\infty : \text{Set}_\Delta \to \text{Set}_\Delta$ and a natural transformation of functors $\rho^\infty : \text{id}_{\text{Set}_\Delta} \to \text{Ex}^\infty$ with the following properties:

(a′) For every simplicial set $S$, the simplicial set $\text{Ex}^\infty(S)$ is a Kan complex (Proposition 3.3.6.9).

(b′) For every simplicial set $S$, the morphism $\rho^\infty_S : S \to \text{Ex}^\infty(S)$ is a weak homotopy equivalence (Proposition 3.3.6.7).

(c′) For every Kan fibration of simplicial sets $f : X \to S$, the induced map $\text{Ex}^\infty(f) : \text{Ex}^\infty(X) \to \text{Ex}^\infty(S)$ is a Kan fibration (Proposition 3.3.6.6).

(d′) The functor $\text{Ex}^\infty : \text{Set}_\Delta \to \text{Set}_\Delta$ commutes with finite limits (Proposition 3.3.6.4). In particular, for every morphism of simplicial sets $f : X \to S$ and every vertex $s \in S$, the canonical map $\text{Ex}^\infty(X_s) \to \{s\} \times_{\text{Ex}^\infty(S)} \text{Ex}^\infty(X)$ is an isomorphism (Corollary 3.3.6.5).

It follows from these assertions that for any Kan fibration $f : X \to S$, the diagram of simplicial sets

$$
\begin{array}{ccc}
X & \xrightarrow{\rho^\infty_X} & \text{Ex}^\infty(X) \\
\downarrow f & & \downarrow \text{Ex}^\infty(f) \\
S & \xrightarrow{\rho^\infty_S} & \text{Ex}^\infty(S)
\end{array}
$$

satisfies the requirements of Theorem 3.3.0.1.

Most of this section is devoted to the definition of the functor $\text{Ex}^\infty$ (and the natural transformation $\rho^\infty$) and the verification of assertions (a′) through (d′). The construction is rooted in classical geometric ideas. Let $n$ be a nonnegative integer, let

$$|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0, 1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}$$

denote the topological simplex of dimension $n$. This topological space admits a triangulation whose vertices are the barycenters of its faces. More precisely, there is a canonical
homeomorphism of topological spaces \(|\text{Sd}(\Delta^n)| \xrightarrow{\sim} |\Delta^n|\), where \(\text{Sd}(\Delta^n)\) denotes the nerve of the partially ordered set of faces of \(\Delta^n\) (Proposition 3.3.2.3). For every topological space \(Y\), composition with this homeomorphism induces a bijection 

\[ \varphi_n : \text{Sing}_n(Y) \xrightarrow{\sim} \text{Hom}_{\text{Set}}(\text{Sd}(\Delta^n), \text{Sing}_n(Y)). \]

Motivated by this observation, we define a functor \(X \mapsto \text{Ex}(X) = \text{Ex}_n(X)\) from the category of simplicial sets to itself by the formula \(\text{Ex}_n(X) = \text{Hom}_{\text{Set}}(\text{Sd}(\Delta^n), X)\) (Construction 3.3.2.5). The preceding discussion can then be summarized by noting that, when \(X = \text{Sing}_n(Y)\) is the singular simplicial set of a topological space \(Y\), the bijections \(\{\varphi_n\}_{n \geq 0}\) determine an isomorphism of semisimplicial sets \(\varphi : X \to \text{Ex}(X)\) (Example 3.3.2.9). Beware that \(\varphi\) is generally not an isomorphism of simplicial sets: that is, it need not be compatible with degeneracy operators.

In §3.3.3, we show that the functor \(\text{Ex} : \text{Set}_\Delta \to \text{Set}_\Delta\) admits a left adjoint (Corollary 3.3.3.4). We denote the value of this left adjoint on a simplicial set \(X\) by \(\text{Sd}(X)\), and refer to it as the subdivision of \(X\). It is essentially immediate from the definition that, in the special case where \(X = \Delta^n\) is a standard simplex, we recover the simplicial set \(\text{Sd}(\Delta^n)\) defined above. More generally, let say that a simplicial set \(X\) is braced if the collection of nondegenerate simplices of \(X\) is closed under face operators (Definition 3.3.1.1). If this condition is satisfied, then the subdivision \(\text{Sd}(X)\) can be identified with the nerve of the category \(\Delta^\text{nd}_X\) of nondegenerate simplices of \(X\) (Proposition 3.3.3.15). Moreover, we also have a canonical homeomorphism of topological spaces \(|\text{Sd}(X)| \to |X|\), which carries each vertex of \(N_*(\Delta^\text{nd}_X)\) to the barycenter of the corresponding simplex of \(|X|\) (Proposition 3.3.3.6).

In §3.3.4, we associate to every simplicial set \(X\) a pair of comparison maps 

\[ \lambda_X : \text{Sd}(X) \to X \quad \rho_X : X \to \text{Ex}(X); \]

we refer to \(\lambda_X\) as the last vertex map of \(X\) (Construction 3.3.4.3). In the special case \(X = \Delta^n\), the source and target of \(\lambda_X\) are both weakly contractible, so \(\lambda_X\) is automatically a weak homotopy equivalence. From this observation, it follows from a simple formal argument that \(\lambda_X\) is a weak homotopy equivalence for every simplicial set \(X\) (Proposition 3.3.4.8). In §3.3.5 we exploit this to show that the functor \(\text{Ex}\) carries Kan fibrations to Kan fibrations (Corollary 3.3.5.4), and that the comparison map \(\rho_X : X \to \text{Ex}(X)\) is a weak homotopy equivalence for every simplicial set \(X\) (Theorem 3.3.5.1). Consequently, the functor \(\text{Ex} : \text{Set}_\Delta \to \text{Set}_\Delta\) satisfies analogues of properties \((a')\), \((c')\), and \((d')\) above.

Unfortunately, the functor \(\text{Ex} : \text{Set}_\Delta \to \text{Set}_\Delta\) does not satisfy the analogue of condition \((a')\): in general, a simplicial set of the form \(\text{Ex}(X)\) need not satisfy the Kan extension condition. However, one can show that it satisfies a slightly weaker condition: for any morphism of simplicial sets \(f_0 : \Lambda^n_i \to \text{Ex}(X)\), the composite map \(\Lambda^n_i \xrightarrow{f_0} \text{Ex}(X) \xrightarrow{\rho_X} \text{Ex}^2(X)\) can be extended to an \(n\)-simplex of the simplicial set \(\text{Ex}^2(X) = \text{Ex}^2(\text{Ex}(X))\). We
apply this observation in §3.3.6 to deduce that the direct limit
\[ \text{Ex}^\infty(X) = \lim_{\longrightarrow} (X \xrightarrow{\rho_X} \text{Ex}(X) \xrightarrow{\rho_{\text{Ex}(X)}} \text{Ex}^2(X) \xrightarrow{\rho_{\text{Ex}^2(X)}} \text{Ex}^3(X) \to \cdots) \]
is a Kan complex (Proposition 3.3.6.9). Moreover, properties (b'), (c'), and (d') for the functor \( X \mapsto \text{Ex}^\infty(X) \) are immediate consequences of the analogous properties of the functor \( X \mapsto \text{Ex}(X) \).

We close this section by outlining some applications of the functor \( \text{Ex}^\infty \). In §3.3.7 we prove that, in the situation of Theorem 3.3.0.1, assertion (d) is a formal consequence of (b) and (c) (Proposition 3.3.7.1). Using this, we show that a Kan fibration of simplicial sets \( f : X \to S \) is a weak homotopy equivalence if and only if it is a trivial Kan fibration (Proposition 3.3.7.4), and that a monomorphism of simplicial sets \( g : X \to Y \) is a weak homotopy equivalence if and only if it is anodyne (Corollary 3.3.7.5). In §3.3.8 we prove a refinement of Theorem 3.3.0.1 which guarantees that every Kan fibration \( f : X \to S \) is actually isomorphic to the pullback of a Kan fibration \( f' : X' \to S' \) between Kan complexes (Theorem 3.3.8.1).

3.3.1 Digression: Braced Simplicial Sets

Let \( \Delta \) denote the simplex category (Definition 1.1.1.2), and let \( \Delta_{\text{inj}} \) denote the subcategory of \( \Delta \) spanned by the injective maps (Variant 1.1.1.6). Composition with the inclusion functor \( \Delta_{\text{inj}} \to \Delta^{\text{op}} \) determines a forgetful functor from the category \( \text{Set}_{\Delta} = \text{Fun}(\Delta^{\text{op}}, \text{Set}) \) of simplicial sets to the category \( \text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \) of semisimplicial sets (Remark 1.1.1.7). Our goal in this section is to show that this functor admits a faithful left adjoint, which we will denote by \( S_\bullet \mapsto S_\bullet^+ \). We begin by describing the essential image of this left adjoint.

**Definition 3.3.1.1.** Let \( X_\bullet \) be a simplicial set. We will say that \( X_\bullet \) is braced if, for every nondegenerate simplex \( \sigma \in X_n \) of dimension \( n > 0 \), the faces \( \{d_i(\sigma)\}_{0 \leq i \leq n} \) are also nondegenerate.

**Exercise 3.3.1.2.** Let \( C \) be a category. Show that the nerve \( N_\bullet(C) \) is braced if and only if \( C \) satisfies the following condition:

\[(\ast) \text{ For every pair of morphisms } f : X \to Y \text{ and } g : Y \to X \text{ in } C \text{ satisfying } g \circ f = \text{id}_X, \text{ we have } X = Y \text{ and } f = g = \text{id}_X.\]

In particular, for any partially ordered set \( Q \), the nerve \( N_\bullet(Q) \) is braced.

**Example 3.3.1.3.** Every simplicial set of dimension \( \leq 1 \) is braced.

**Notation 3.3.1.4.** Let \( X_\bullet \) be a simplicial set. For each nonnegative integer \( n \), we let \( X_\bullet^{\text{nd}} \subseteq X_\bullet \) denote the collection of nondegenerate \( n \)-simplices of \( X_\bullet \). If \( X_\bullet \) is braced...
(Definition 3.3.1.1), then the face maps \( \{ d_i : X_n \to X_{n-1} \}_{0 \leq i \leq n} \) carry \( X_n^{nd} \) into \( X_{n-1}^{nd} \). In this case, the construction \( [n] \mapsto X_n^{nd} \) determines a semisimplicial set, which we will denote by \( X_n^{nd} \).

The terminology of Definition 3.3.1.1 is motivated by the heuristic that a braced simplicial set \( X_\bullet \) is “supported” by the semisimplicial subset \( X_n^{nd} \subseteq X_\bullet \). This heuristic is supported by the following:

**Proposition 3.3.1.5.** Let \( X_\bullet \) and \( Y_\bullet \) be simplicial sets, and suppose that \( X_\bullet \) is braced. Then the restriction map

\[
\{\text{Morphisms of simplicial sets } f : X_\bullet \to Y_\bullet \} \to \{\text{Morphisms of semisimplicial sets } f_0 : X_n^{nd} \to Y_\bullet \}
\]

is a bijection.

**Proof.** Fix a morphism of semisimplicial sets \( f_0 : X_n^{nd} \to Y_\bullet \); we wish to show that \( f_0 \) extends uniquely to a morphism of simplicial sets from \( X_\bullet \) to \( Y_\bullet \). Let \( \sigma \) be an \( n \)-simplex of \( X_\bullet \). By virtue of Proposition 1.1.3.4, we can write \( \sigma \) uniquely as \( \alpha^*(\tau) \), where \( \alpha : [n] \to [m] \) is a nondecreasing surjection and \( \tau \) is a nondegenerate \( m \)-simplex of \( X_\bullet \). Define \( f(\sigma) = \alpha^* f_0(\tau) \in Y_n \). It is clear that any extension of \( f_0 \) to a morphism of simplicial sets \( X_\bullet \to Y_\bullet \) must be given by the construction \( \sigma \mapsto f(\sigma) \). It will therefore suffice to show that the construction \( \sigma \mapsto f(\sigma) \) is a morphism of simplicial sets.

Let \( \sigma, \tau, \) and \( \alpha \) be as above, and fix a nondecreasing map \( \beta : [n'] \to [n] \). We wish to prove that \( f(\beta^* \sigma) = \beta^* f(\sigma) \) in the set \( Y_{n'} \). Note that \( (\alpha \circ \beta) : [n'] \to [m] \) factors uniquely as a composition \( [n'] \overset{\alpha'}{\to} [m'] \overset{\beta'}{\to} [m] \), where \( \alpha' \) is surjective and \( \beta' \) is injective. Since \( X_\bullet \) is braced, \( \beta'^*(\tau) \) is a nondegenerate \( m' \)-simplex of \( X_\bullet \). We now compute

\[
f(\beta^* \sigma) = f(\beta^* \alpha^* \tau) = f(\alpha'^* \beta'^* \tau) = \alpha'^* f_0(\beta'^* \tau) = \alpha'^* \beta'^* f_0(\tau) = \beta^* \alpha^* f_0(\tau) = \beta^* f(\sigma).
\]

where the second and fifth equality follow from the identity \( \alpha \circ \beta = \beta' \circ \alpha' \), the third and sixth equality follow from the definition of \( f \), and the fourth equality from the fact that \( f_0 \) is a morphism of semisimplicial sets.
3.3. **THE Ex∞ FUNCTOR**

We now show that every semisimplicial set $S$ can be obtained from the procedure of Notation 3.3.1.4.

**Construction 3.3.1.6.** Let $S$ be a semisimplicial set. For each $n \geq 0$, we let $S_n^+$ denote the collection of pairs $(\alpha, \tau)$ where $\alpha : [n] \to [m]$ is a nondecreasing surjection of linearly ordered sets and $\tau$ is an element of $S_m$.

Let $\beta : [n'] \to [n]$ be a morphism in the category $\Delta$. For every element $(\alpha, \tau) \in S_n^+$, the composite map $\alpha \circ \beta : [n'] \to [m]$ factors uniquely as a composition $[n'] \xrightarrow{\alpha'} [m'] \xrightarrow{\beta'} [m]$, where $\alpha'$ is surjective and $\beta'$ is injective. We define a map $\beta^* : S_n^+ \to S_{n'}^+$ by the formula $\beta^*(\alpha, \tau) = (\alpha', \beta'^*(\tau)) \in S_{n'}^+$.

**Proposition 3.3.1.7.** Let $S$ be a semisimplicial set. Then:

1. The assignments
   
   $([n] \in \Delta) \mapsto S_n^+$  
   
   $([n'] \to [n]) \mapsto (\beta^* : S_n^+ \to S_{n'}^+)$
   
   of Construction 3.3.1.6 define a simplicial set $S^+$.

2. The construction $(\tau \in S_n) \mapsto ((\text{id}_{[n]}, \tau) \in S_n^+)$ determines a monomorphism of semisimplicial sets $\iota : S \hookrightarrow S^+$.

3. The simplicial set $S^+$ is braced, and $\iota$ induces an isomorphism from $S$ to the semisimplicial subset $(S^+)^{\text{nd}} \subseteq S^+$.

**Proof.** It follows immediately that for each $n \geq 0$, the function $\text{id}_{[n]}^* : S_n^+ \to S_n^+$ is the identity map. To prove (1), it will suffice to show that for every pair of composable morphisms $[n''] \xrightarrow{\gamma} [n'] \xrightarrow{\beta} [n]$ in $\Delta$, we have an equality $\gamma^* \circ \beta^* = (\beta \circ \gamma)^*$ of functions from $S_n^+$ to $S_{n''}^+$.

Fix an element $(\alpha, \tau) \in S_n^+$, where $\alpha : [n] \to [m]$ is a surjective nondecreasing function and $\tau$ is an element of $S_m$. There is a unique commutative diagram

\[
\begin{array}{ccc}
[n''] & \xrightarrow{\gamma} & [n'] \\
\downarrow{\alpha''} & & \downarrow{\alpha'} \\
[m''] & \xrightarrow{\gamma'} & [m'] \\
\downarrow{\beta'} & & \downarrow{\beta} \\
[m] & & [m]
\end{array}
\]

in the category $\Delta$, where the vertical maps are surjective and the lower horizontal maps are injective. We then compute

\[
(\gamma^* \circ \beta^*)(\alpha, \tau) = (\gamma^*(\alpha', \beta'^*\tau)) = (\alpha'', \gamma'^*(\beta'^*\tau)) = (\alpha'', (\beta' \circ \gamma')^*\tau) = (\beta \circ \gamma)^*(\alpha, \tau),
\]
which completes the proof of (1).

Assertion (2) is immediate from the definition. Note that if $\beta : [n'] \to [n]$ is a nondecreasing surjection, then the map $\beta^* : S^+_n \to S^+_{n'}$ is given by the formula $\beta^*(\alpha, \tau) = (\alpha \circ \beta, \tau)$. It follows that an $n$-simplex $\sigma = (\alpha, \tau)$ of $S^+_n$ is degenerate if and only if $\alpha : [n] \to [m]$ is not a bijection: that is, if and only if $\sigma$ belongs to the image of $\iota$. Since the image of $\iota$ is closed under face maps (by virtue of (2)), we conclude that $S^+_n$ is braced and that $\iota$ induces an isomorphism of semisimplicial sets $S_\bullet \simeq (S^+_\bullet)^{nd}$.

**Corollary 3.3.1.8.** Let $\text{Set}_{\Delta}^{br} \subseteq \text{Set}_{\Delta}$ denote the (non-full) subcategory whose objects are braced simplicial sets and whose morphisms are maps $f : X_\bullet \to Y_\bullet$ which carry nondegenerate simplices of $X_\bullet$ to nondegenerate simplices of $Y_\bullet$. Then the construction $X_\bullet \mapsto X_\bullet^{nd}$ induces an equivalence of categories $\text{Set}_{\Delta}^{br} \to \{\text{Semisimplicial sets}\}$, with homotopy inverse given by the construction $S_\bullet \mapsto S^+_\bullet$.

**Proof.** Let $X_\bullet$ and $Y_\bullet$ be braced simplicial sets. It follows from Proposition 3.3.1.5 that the restriction functor $\text{Hom}_{\text{Set}_{\Delta}}(X_\bullet, Y_\bullet) \to \text{Hom}_{\text{Fun}(\Delta^{op}_{\text{inj}}, \text{Set})}(X_{\bullet}^{nd}, Y_\bullet)$ is a bijection. Moreover, the image of $\text{Hom}_{\text{Set}_{\Delta}^{br}}(X_\bullet, Y_\bullet)$ under this bijection is the collection of morphisms of semisimplicial sets from $X_\bullet^{nd}$ to $Y_\bullet^{nd} \subseteq Y_\bullet$. This proves full-faithfulness, and the essential surjectivity follows from Proposition 3.3.1.7.

**Corollary 3.3.1.9.** Let $S_\bullet$ be a semisimplicial set. Then, for every simplicial set $Y_\bullet$, composition with the map $\iota : S_\bullet \mapsto S^+_\bullet$ induces a bijection

\[
\{\text{Morphisms of simplicial sets } f : S^+_\bullet \to Y_\bullet\}
\]

\[
\downarrow
\]

\[
\{\text{Morphisms of semisimplicial sets } f_0 : S_\bullet \to Y_\bullet\}\}.
\]

**Proof.** Combine Proposition 3.3.1.5 with Proposition 3.3.1.7.

**Corollary 3.3.1.10.** The forgetful functor

\[
\{\text{Simplicial sets}\} \to \{\text{Semisimplicial sets}\}
\]

has a left adjoint, given on objects by the construction $S_\bullet \mapsto S^+_\bullet$.

**Corollary 3.3.1.11.** Let $X_\bullet$ be a braced simplicial set. Then the inclusion of semisimplicial sets $\iota_0 : X_\bullet^{nd} \hookrightarrow X_\bullet$ extends uniquely to an isomorphism $g : (X_\bullet^{nd})^+ \simeq X_\bullet$.

**Proof.** It follows from Corollary 3.3.1.9 that $f_0$ extends uniquely to a map of simplicial sets $g : (X_\bullet^{nd})^+ \to X_\bullet$. To show that $f$ is an isomorphism, it will suffice to show that for every simplicial set $Y_\bullet$, composition with $g$ induces a bijection

\[
\text{Hom}_{\text{Set}_{\Delta}}(X_\bullet, Y_\bullet) \to \text{Hom}_{\text{Set}_{\Delta}}((X_\bullet^{nd})^+, Y_\bullet)
\]

\[
\simeq \text{Hom}_{\text{Fun}(\Delta^{op}_{\text{inj}}, \text{Set})}(X_\bullet^{nd}, Y_\bullet),
\]
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which is precisely the content of Proposition 3.3.1.5.

3.3.2 The Subdivision of a Simplex

Let \( n \geq 0 \) be a nonnegative integer and let 
\[
|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}
\]
be the topological \( n \)-simplex. For every nonempty subset \( S \subseteq [n] = \{0 < 1 < \cdots < n\} \), let 
\( |\Delta^S| \) denote the corresponding face of \( |\Delta^n| \), given by the collection of tuples 
\( (t_0, \ldots, t_n) \in |\Delta^n| \)
satisfying \( t_i = 0 \) for \( i \notin S \). Let \( b_S \) denote the barycenter of the simplex \( |\Delta^S| \): 
that is, the point 
\[(t_0, \ldots, t_n) \in |\Delta^S| \subseteq |\Delta^n| \text{ given by } t_i = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}
\]
The collection of barycenters \( \{b_S\}_{\emptyset \neq S \subseteq [n]} \) can be regarded as the vertices of a triangulation of \( |\Delta^n| \), which we indicate in the case \( n = 2 \) by the following diagram:

In this section, we show that this triangulation arises from the identification of \( |\Delta^n| \) with the geometric realization of another simplicial set (Proposition 3.3.2.3).

Notation 3.3.2.1. Let \( Q \) be a partially ordered set. We let \( \text{Chain}(Q) \) denote the collection of all nonempty, finite, linearly ordered subsets of \( Q \). We regard \( \text{Chain}(Q) \) as a partially ordered set, where the partial order is given by inclusion. In the special case where \( Q = [n] = \{0 < 1 < \ldots < n\} \) for some nonnegative integer \( n \), we denote the partially ordered set \( \text{Chain}(Q) \) by \( \text{Chain}[n] \).

Remark 3.3.2.2 (Functoriality). Let \( f : Q \to Q' \) be a nondecreasing map between partially ordered sets. Then \( f \) induces a map \( \text{Chain}(f) : \text{Chain}(Q) \to \text{Chain}(Q') \), which carries each nonempty linearly ordered subset \( S \subseteq Q \) to its image \( f(S) \subseteq Q' \). By means of this construction, we can regard \( Q \mapsto \text{Chain}(Q) \) as functor from the category of partially ordered sets to itself.
PROPOSITION 3.3.2.3. Let \( n \geq 0 \) be an integer. Then there is a unique homeomorphism of topological spaces

\[
f : |N_* (\text{Chain}[n])| \to |\Delta^n|
\]

with the following properties:

1. For every nonempty subset \( S \subseteq [n] \), the map \( f \) carries \( S \) (regarded as a vertex of \( N_* (\text{Chain}[n]) \)) to the barycenter \( b_S \in |\Delta^S| \subseteq |\Delta^n| \).

2. For every \( m \)-simplex \( \sigma : |\Delta^m| \to N_* (\text{Chain}[n]) \), the composite map

\[
|\Delta^m| \xrightarrow{\sigma} |N_* (\text{Chain}[n])| \xrightarrow{f} |\Delta^n|
\]

is affine: that is, it extends to an \( \mathbb{R} \)-linear map from \( \mathbb{R}^{m+1} \supseteq |\Delta^m| \) to \( \mathbb{R}^{n+1} \supseteq |\Delta^n| \).

Proof. Note that an affine map \( |\Delta^m| \to |\Delta^n| \) is uniquely determined by its values on the vertices of the topological \( m \)-simplex \( |\Delta^m| \). From this observation, it is easy to deduce that there is a unique continuous function \( f : |N_* (\text{Chain}[n])| \to |\Delta^n| \) which satisfies conditions (1) and (2) of Proposition 3.3.2.3. We will complete the proof by showing that \( f \) is a homeomorphism. Since the domain and codomain of \( f \) are compact Hausdorff spaces, it will suffice to show that \( f \) is a bijection. Unwinding the definitions, this can be restated as follows:

(*) For every point \( (t_0, t_1, \ldots, t_n) \in |\Delta^n| \), there exists a unique chain \( S_0 \subset S_1 \subset \cdots \subset S_m \) of subsets of \( [n] \) and positive real numbers \( (s_0, s_1, \ldots, s_m) \) satisfying the identities

\[
\sum s_i = 1 \quad \quad (t_0, t_1, \ldots, t_n) = \sum s_i b_{S_i}.
\]

We will deduce (*) from the following more general assertion:

(*) For every element \( (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \), there exists a unique (possibly empty) chain \( S_0 \subset S_1 \subset \cdots \subset S_m \) of subsets of \( [n] \) and positive real numbers \( (s_0, s_1, \ldots, s_m) \) satisfying \( (t_0, t_1, \ldots, t_n) = \sum s_i b_{S_i} \).

Note that, if \( (t_0, t_1, \ldots, t_n) \) and \( (s_0, s_1, \ldots, s_m) \) are as in (*), then we automatically have \( \sum_{i=0}^m s_i = \sum_{j=0}^n t_j \). It follows that assertion (*) is a special case of (*'). To prove (*'), let \( K \subseteq [n] \) be the collection of those integers \( j \) for which \( t_j \neq 0 \). We proceed by induction on the cardinality of \( k = |K| \). If \( k = 0 \) is empty, there is nothing to prove. Otherwise, set \( r = \min\{kt_i\}_{i \in K} \). We can then write

\[
(t_0, t_1, \ldots, t_n) = rb_K + (t'_0, t'_1, \ldots, t'_n)
\]

for a unique sequence of nonnegative real numbers \( (t'_0, \ldots, t'_n) \). Applying our inductive hypothesis to the sequence \( (t'_0, \ldots, t'_n) \), we deduce that there is a unique chain of subsets \( S_0 \subset S_1 \subset \cdots \subset S_{m-1} \) of \( [n] \) and positive real numbers \( (s'_0, s'_1, \ldots, s'_{m-1}) \) satisfying
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$(t'_0, t'_1, \ldots, t'_n) = \sum s_i b_{S_i}$. Note that each $S_i$ is contained in $K'$, and therefore properly contained in $K$. To complete the proof, we extend this sequence by setting $S_m = K$ and $s_m = r$.

\textbf{Remark 3.3.2.4 (Functoriality).} Let $\alpha : [m] \to [n]$ be a nondecreasing function between partially ordered sets, so that $\alpha$ induces a nondecreasing map $\text{Chain}[\alpha] : \text{Chain}[m] \to \text{Chain}[n]$ (Remark 3.3.2.2). If $\alpha$ is injective, then the diagram of topological spaces

\[
\begin{array}{ccc}
|N_\bullet(\text{Chain}[m])| & \xrightarrow{f_m} & |\Delta^m| \\
|N_\bullet(\text{Chain}[\alpha])| & \xrightarrow{|\alpha|} & |\Delta^n| \\
|N_\bullet(\text{Chain}[n])| & \xrightarrow{f_n} & |\Delta^n|
\end{array}
\]

is commutative, where the horizontal maps are the homeomorphisms supplied by Proposition 3.3.2.3. Beware that if $\alpha$ is not injective, this diagram does not necessarily commute. For example, the induced map $|\Delta^m| \to |\Delta^n|$ carries the barycenter of $|\Delta^m|$ to the point

\[
\left(\frac{|\alpha^{-1}(0)|}{m+1}, \frac{|\alpha^{-1}(1)|}{m+1}, \ldots, \frac{|\alpha^{-1}(n)|}{m+1}\right) \in |\Delta^n|,
\]

which need not be the barycenter of any face $|\Delta^n|$.

It will be convenient to repackage Proposition 3.3.2.3 (and Remark 3.3.2.4) as a statement about the singular simplicial set functor $\text{Sing}_\bullet : \text{Top} \to \text{Set}_\Delta$ of Construction 1.1.7.1. We first introduce a bit of notation (which will play an essential role throughout 3.3).

\textbf{Construction 3.3.2.5 (The Ex Functor).} Let $X$ be a simplicial set. For every non-negative integer $n$, we let $\text{Ex}_n(X)$ denote the collection of all morphisms of simplicial sets $N_\bullet(\text{Chain}[n]) \to X$. By virtue of Remark 3.3.2.2, the construction $[n] \in \Delta^{\text{op}} \mapsto (\text{Ex}_n(X) \in \text{Set})$ determines a simplicial set which we will denote by $\text{Ex}(X)$. The construction $X \mapsto \text{Ex}(X)$ determines a functor from the category of simplicial sets to itself, which we denote by $\text{Ex} : \text{Set}_\Delta \to \text{Set}_\Delta$.

\textbf{Remark 3.3.2.6.} The construction $X \mapsto \text{Ex}(X)$ can be regarded as a special case of Variant 1.1.7.6: it is the functor $\text{Sing}_T^\bullet : \text{Set}_\Delta \to \text{Set}_\Delta$ associated to the cosimplicial object $T$ of $\text{Set}_\Delta$ given by the construction $[n] \mapsto N_\bullet(\text{Chain}[n])$.

\textbf{Remark 3.3.2.7.} The functor $X \mapsto \text{Ex}(X)$ preserves filtered colimits of simplicial sets. To prove this, it suffices to observe that each of the simplicial sets $N_\bullet(\text{Chain}[n])$ has only finitely many nondegenerate simplices (since the partially ordered set $\text{Chain}[n]$ is finite).
Example 3.3.2.8. Let \( C \) be a category and let \( N_\bullet(C) \) denote the nerve of \( C \). Then \( n \)-simplices of the simplicial set \( \text{Ex}(N_\bullet(C)) \) can be identified with functors from the partially ordered set \( \text{Chain}[n] \) into \( C \) (see Proposition 1.2.2.1).

Example 3.3.2.9. Let \( X \) be a topological space and let \( \text{Sing}_\bullet(X) \) denote the singular simplicial set of \( X \). For each nonnegative integer \( n \), the \( n \)-simplices of \( \text{Sing}_\bullet(X) \) are given by continuous functions \( |\Delta^n| \to X \), and the \( n \)-simplices of \( \text{Ex}(\text{Sing}_\bullet(X)) \) are given by continuous functions \( |N_{n}(\text{Chain}[n])| \to X \). The homeomorphism \( |N_{n}(\text{Chain}[n])| \simeq |\Delta^n| \) of Proposition 3.3.2.3 determines a bijection \( \text{Sing}_n(X) \to \text{Ex}_n(\text{Sing}_\bullet(X)) \), and Remark 3.3.2.4 guarantees that these bijections are compatible with the face operators on the simplicial sets \( \text{Sing}_\bullet(X) \) and \( \text{Ex}(\text{Sing}_\bullet(X)) \). In other words, Proposition 3.3.2.3 supplies an isomorphism of semisimplicial sets \( \varphi : \text{Sing}_\bullet(X) \to \text{Ex}(\text{Sing}_\bullet(X)) \). Beware that \( \varphi \) is generally not an isomorphism of simplicial sets: that is, it usually does not commute with the degeneracy operators on \( \text{Sing}_\bullet(X) \) and \( \text{Ex}(\text{Sing}_\bullet(X)) \).

Variant 3.3.2.10 (Ex for Semisimplicial Sets). Note that, for every nonnegative integer \( n \), the simplicial set \( N_\bullet(\text{Chain}[n]) \) is braced (Exercise 3.3.1.2). If \( X \) is a semisimplicial set, we write \( \text{Ex}_n(X) \) for the collection of all morphisms of semisimplicial sets \( N_\bullet(\text{Chain}[n])^{nd} \to X \); here \( N_\bullet(\text{Chain}[n])^{nd} \) denotes the semisimplicial subset of \( N_\bullet(\text{Chain}[n]) \) spanned by the nondegenerate simplices. The construction \( [n] \mapsto \text{Ex}_n(X) \) determines a semisimplicial set, which we denote by \( \text{Ex}(X) \).

Note that, if \( X \) is the underlying semisimplicial set of a simplicial set \( Y \), then \( \text{Ex}(X) \) is the underlying semisimplicial set of the simplicial set \( \text{Ex}(Y) \) given by Construction 3.3.2.5 (this is a special case of Proposition 3.3.1.5). In other words, the construction \( X \mapsto \text{Ex}(X) \) determines a functor from the category of semisimplicial sets to itself which fits into a commutative diagram

\[
\begin{array}{ccc}
\{\text{Simplicial sets}\} & \xrightarrow{\text{Ex}} & \{\text{Simplicial sets}\} \\
\downarrow & & \downarrow \\
\{\text{Semisimplicial sets}\} & \xrightarrow{\text{Ex}} & \{\text{Semisimplicial sets}\} 
\end{array}
\]

3.3.3 The Subdivision of a Simplicial Set

Let \( n \geq 0 \) be a nonnegative integer. In §3.3.2, we showed that the topological \( n \)-simplex \( |\Delta^n| \) can be identified with the geometric realization of the set of its faces \( \text{Chain}[n] \), partially ordered by inclusion (Proposition 3.3.2.3). We now prove a generalization of this result, replacing the standard simplex \( \Delta^n \) by an arbitrary braced simplicial set \( X \) and the nerve \( N_\bullet(\text{Chain}[n]) \) by another simplicial set \( \text{Sd}(X) \), which we will refer to as the subdivision of \( X \).
Definition 3.3.3.1 (Subdivision). Let $X$ and $Y$ be simplicial sets. We will say that a morphism of simplicial sets $u : X \to \text{Ex}(Y)$ exhibits $Y$ as a subdivision of $X$ if, for every simplicial set $Z$, composition with $u$ induces a bijection $\text{Hom}_{\Delta}^\text{Set}(Y,Z) \to \text{Hom}_{\Delta}^\text{Set}(X,\text{Ex}(Z))$ (see Construction 3.3.2.5).

Notation 3.3.3.2. Let $X$ be a simplicial set. It follows immediately from the definitions that if there exists a simplicial set $Y$ and a morphism $u : X \to \text{Ex}(Y)$ which exhibits $Y$ as a subdivision of $X$, then the simplicial set $Y$ (and the morphism $u$) are uniquely determined up to isomorphism and depend functorially on $X$. To emphasize this dependence, we will denote $Y$ by $\text{Sd}(X)$ and refer to it as the subdivision of $X$.

Proposition 3.3.3.3. Let $X$ be a simplicial set. Then there exists another simplicial set $\text{Sd}(X)$ and a morphism $u : X \to \text{Ex}(\text{Sd}(X))$ which exhibits $\text{Sd}(X)$ as a subdivision of $X$, in the sense of Notation 3.3.3.2.

Proof. By virtue of Remark 3.3.2.6 this is a special case of Proposition 1.1.8.22. \qed

Corollary 3.3.3.4. The functor $\text{Ex} : \text{Set}_\Delta \to \text{Set}_\Delta$ admits a left adjoint, given by the construction $X \mapsto \text{Sd}(X)$.

Example 3.3.3.5. Let $n$ be a nonnegative integer. Then the identity map
\[ \text{id} : \text{N}_\bullet(\text{Chain}[n]) \to \text{N}_\bullet(\text{Chain}[n]) \]
determines a map of simplicial sets a map $u : \Delta^n \to \text{Ex}(\text{N}_\bullet(\text{Chain}[n]))$, which exhibits $\text{N}_\bullet(\text{Chain}[n])$ as the subdivision of $\Delta^n$. In particular, the subdivision $\text{Sd}(\Delta^2)$ is the 2-dimensional simplicial set indicated in the diagram

\[
\begin{align*}
\{1\} & \quad \downarrow & \quad \{1\} \\
\{0,1\} & \quad \downarrow & \quad \{1,2\} \\
\{0,1,2\} & \quad \downarrow & \quad & \{1\} \\
\{0\} & \quad \downarrow & \quad \{0,2\} & \quad \downarrow & \quad \{1\} \\
\end{align*}
\]

Proposition 3.3.3.6. Let $X$ be a braced simplicial set. Then there is a canonical homeomorphism of topological spaces $f_X : |\text{Sd}(X)| \to |X|$.
Proof. For every topological space $Y$, Example 3.3.2.9 supplies an isomorphism of semisimplicial sets $\text{Sing}_\bullet(Y) \to \text{Ex}(\text{Sing}_\bullet(Y))$. These isomorphisms depend functorially on $Y$, and can therefore be regarded as an isomorphism of functors $G \circ \text{Sing}_\bullet \cong G \circ \text{Ex} \circ \text{Sing}_\bullet$, where $G : \text{Set} \to \text{Fun}(\Delta^{\text{op}}_{\text{inj}}, \text{Set})$ denotes the forgetful functor from simplicial sets to semisimplicial sets. Passing to left adjoints, we conclude that for every semisimplicial set $S_\bullet$, we have a canonical homeomorphism $|\text{Sd}(S_\bullet)| \cong |S_\bullet|$, depending functorially on $S_\bullet$. Proposition 3.3.3.6 now follows from Corollary 3.3.1.11 (applied to the semisimplicial set $X^{\text{nd}}$).

Remark 3.3.3.7. The homeomorphisms $f_X : |\text{Sd}(X)| \cong |X|$ constructed in the proof of Proposition 3.3.3.6 are characterized by the following properties:

- In the special case where $X = \Delta^n$ is a standard simplex, $f_X$ is given by the composition

  $$|\text{Sd}(\Delta^n)| \cong |\text{N}_\bullet(\text{Chain}[n])| \overset{f}{\rightarrow} |\Delta^n|,$$

  where the first map is supplied by the identification $\text{Sd}(\Delta^n) \cong \text{N}_\bullet(\text{Chain}[n])$ of Example 3.3.3.5 and $f$ is the homeomorphism of Proposition 3.3.2.3.

- Let $u : X \to Y$ be a morphism of braced simplicial sets which carries nondegenerate simplices of $X$ to nondegenerate simplices of $Y$. Then the diagram of topological spaces

  $$\begin{array}{ccc}
  |\text{Sd}(X)| & \xrightarrow{f_X} & |X| \\
  \downarrow |\text{Sd}(u)| & & \downarrow |u| \\
  |\text{Sd}(Y)| & \xrightarrow{f_Y} & |Y|
  \end{array}$$

  commutes.

Warning 3.3.3.8. Let $u : X \to Y$ be a morphism of braced simplicial sets. If $u$ does not carry nondegenerate simplices of $X$ to nondegenerate simplices of $Y$, then the diagram of topological spaces

$$\begin{array}{ccc}
  |\text{Sd}(X)| & \xrightarrow{f_X} & |X| \\
  \sim |\text{Sd}(u)| & & \sim |u| \\
  |\text{Sd}(Y)| & \xrightarrow{f_Y} & |Y|
  \end{array}$$

does not necessarily commute (this phenomenon occurs already in the case where $X$ and $Y$ are simplices: see Remark 3.3.2.4).

In general, the subdivision $\text{Sd}(X)$ of a simplicial set $X$ can be computed as the colimit $\lim_{\Delta^n \to X} \text{Sd}(\Delta^n) = \lim_{\Delta^n \to X} \text{N}_\bullet(\text{Chain}[n])$, where the colimit is indexed by the category of simplices $\Delta_X$ introduced in Construction 1.1.8.19. When $X$ is braced, this colimit can be described more concretely.
Notation 3.3.3.9. Let $X$ be a simplicial set and let $\Delta_X$ be the category of simplices of $X$ (Construction 1.1.8.19). By definition, the objects of $\Delta_X$ are given by pairs $([n], \sigma)$, where $n$ is a nonnegative integer and $\sigma$ is an $n$-simplex of $X$. We let $\Delta_X^{nd}$ denote the full subcategory of $\Delta_X$ spanned by those pairs $([n], \sigma)$ where $\sigma$ is a nondegenerate $n$-simplex of $X$. We will refer to $\Delta_X^{nd}$ as the category of nondegenerate simplices of $X$.

Example 3.3.3.10. Let $S$ be a semisimplicial set, and let $S^+$ be the braced simplicial set given by Construction 3.3.1.6. Then the category of nondegenerate simplices $\Delta^{nd}_{S^+}$ can be described concretely as follows:

- The objects of $\Delta^{nd}_{S^+}$ are pairs $([n], \sigma)$, where $[n]$ is an object of $\Delta_{inj}$ and $\sigma$ is an element of $S_n$.
- A morphism from $([n], \sigma)$ to $([n'], \sigma')$ in $\Delta^{nd}_{S^+}$ is a strictly increasing function $\alpha : [n] \to [n']$ satisfying $\sigma = \alpha^*(\sigma')$ in the set $S_n$.

In other words, $\Delta^{nd}_{S^+}$ is the category obtained by applying the Grothendieck construction to the functor $S : \Delta^{op}_{inj} \to \text{Set}$ (see §3[?]).

Warning 3.3.3.11. Though the category $\Delta^{nd}_{X}$ is defined for any simplicial set $X$, it is primarily useful in the case where $X$ is braced (where we can use the description supplied by Example 3.3.3.10).

Exercise 3.3.3.12. Let $X$ be a simplicial set. Show that $X$ is braced if and only if the inclusion functor $\Delta^{nd}_X \hookrightarrow \Delta_X$ admits a left adjoint.

Example 3.3.3.13. Let $Q$ be a partially ordered set, and let $N_\bullet(Q)$ denote its nerve. By definition, the nondegenerate $n$-simplices of $N_\bullet(Q)$ can be identified with the strictly increasing functions $\sigma : \{0 < 1 < \cdots < n\} \to Q$. The construction $([n], \sigma) \mapsto \text{im}(\sigma)$ determines an isomorphism from the category of nondegenerate simplices $\Delta^{nd}_{N_\bullet(Q)}$ to the partially ordered set $\text{Chain}(Q)$ of Notation 3.3.2.1.

Construction 3.3.3.14. Let $X$ be a braced simplicial set. Every nondegenerate simplex $\sigma : \Delta^n \to X$ determines a functor

$$\text{Chain}[n] \simeq \Delta^{nd}_{\Delta^n} \to \Delta^{nd}_X,$$

which we can identify with an $n$-simplex $f_0(\sigma)$ of the simplicial set $\text{Ex}(N_\bullet(\Delta^{nd}_X))$ (Example 3.3.2.8). The construction $\sigma \mapsto f_0(\sigma)$ determines a map of semisimplicial sets $f_0 : X^{nd} \to \text{Ex}(N_\bullet(\Delta^{nd}_X))$, which extends uniquely to a map of simplicial sets $f : X \to \text{Ex}(N_\bullet(\Delta^{nd}_X))$ (Proposition 3.3.1.5).

Proposition 3.3.3.15. Let $X$ be a braced simplicial set. Then the morphism $f : X \to \text{Ex}(N_\bullet(\Delta^{nd}_X))$ of Construction 3.3.3.14 exhibits the nerve $N_\bullet(\Delta^{nd}_X)$ as a subdivision of $X$, in the sense of Definition 3.3.2.
Example 3.3.3.16. Let $Q$ be a partially ordered set. Combining Proposition 3.3.3.15 with Example 3.3.3.13 we obtain a canonical isomorphism $\text{Sd}(N_\bullet(Q)) \simeq N_\bullet(\text{Chain}(Q))$. In the special case $Q = [n]$, this recovers the isomorphism $\text{Sd}(\Delta^n) \simeq N_\bullet(\text{Chain}[n])$ of Example 3.3.3.5.

Remark 3.3.3.17 (Functoriality). Let $u : X \to Y$ be a morphism of braced simplicial sets. Then $u$ induces a morphism between their subdivisions

$$N_\bullet(\Delta_X^{nd}) \simeq \text{Sd}(X) \xrightarrow{\text{Sd}(u)} \text{Sd}(Y) \simeq N_\bullet(\Delta_Y^{nd}),$$

which can be identified with a functor $U : \Delta_X^{nd} \to \Delta_Y^{nd}$ (Proposition 1.2.2.1). If $u$ carries nondegenerate simplices of $X$ to nondegenerate simplices of $Y$, then the functor $U$ is easy to describe: it is given on objects by the formula $U([n], \sigma) = ([n], u(\sigma))$. More generally, $U$ carries an object $([n], \sigma) \in \Delta_X^{nd}$ to an object $([m], \tau) \in \Delta_Y^{nd}$, characterized by the requirement that $u(\sigma)$ factors as a composition $\Delta^n \to \Delta^m \to Y$ (see Proposition 1.1.3.4).

Warning 3.3.3.18. In the statement of Proposition 3.3.3.15, the hypothesis that $X$ is braced cannot be omitted. For example, let $X$ be the simplicial set $\Delta^2 \amalg \Delta^1 \amalg \Delta^0$ obtained from the standard 2-simplex by collapsing a single edge, which we depict informally by the diagram

![Diagram](image)

Then the subdivision of $X$ is the 2-dimensional simplicial set depicted in the diagram

![Diagram](image)
This simplicial set cannot arise as the nerve of a category, because it contains a nondegenerate 2-simplex \( \sigma \) for which \( d_2(\sigma) \) is degenerate.

The proof of Proposition 3.3.3.15 will make use of the following:

**Lemma 3.3.3.19.** The functor

\[
\{\text{Semisimplicial Sets}\} \to \{\text{Simplicial Sets}\} \quad S \mapsto N_*(\Delta^\text{nd}_{S^+})
\]

preserves colimits.

**Proof.** Let \( k \) be a nonnegative integer. For every semisimplicial set \( S \), Example 3.3.3.10 allows us to identify \( k \)-simplices of the nerve \( N_*(\Delta^\text{nd}_{S^+}) \) with the set of pairs \((\tau, \sigma)\), where \( \tau \) is a \( k \)-simplex of \( N_*(\Delta^\text{inj}_n) \) (given by a diagram of increasing functions \([n_0] \hookrightarrow [n_1] \hookrightarrow \cdots \hookrightarrow [n_k]\)) and \( \sigma \) is an element of the set \( S^{n_k} \). It follows that the functor \( S \mapsto N_*(\Delta^\text{nd}_{S^+}) \) preserves colimits. Allowing \( k \) to vary, we conclude that the functor \( S \mapsto N_*(\Delta^\text{nd}_{S^+}) \) preserves colimits.

**Proof of Proposition 3.3.3.15.** Let \( S \) be a semisimplicial set, and let \( S^+ \) denote the braced simplicial set given by Construction 3.3.1.6. Applying Construction 3.3.3.14, we obtain a comparison map of simplicial sets \( u_S : Sd(S^+) \to N_*(\Delta^\text{nd}_{S^+}) \). We wish to show that \( u_S \) is an isomorphism for every semisimplicial set \( S \). Note that the functor \( S \mapsto Sd(S^+) \) preserves colimits (since it is a left adjoint) and the functor \( S \mapsto N_*(\Delta^\text{nd}_{S^+}) \) also preserves colimits (by Lemma 3.3.3.19). Since every functor \( S : \Delta^{\text{op}}_{\text{inj}} \to \text{Set} \) can be written as a colimit of representable functors (see \([\?]\)), we may assume without loss of generality that \( S \simeq (\Delta^n)^\text{nd} \) is the semisimplicial set represented by an object \([n] \in \Delta_{\text{inj}} \). In this case, the desired comparison is immediate from the definition of subdivision (see Examples 3.3.3.5 and 3.3.3.16).

### 3.3.4 The Last Vertex Map

Let \( X \) be a simplicial set and let \( Sd(X) \) denote its subdivision (Notation 3.3.3.2). If \( X \) is braced, then Proposition 3.3.3.6 supplies a canonical homeomorphism of topological spaces \( |Sd(X)| \simeq |X| \). Beware that \( X \) and \( Sd(X) \) need not be isomorphic as simplicial sets: for example, the standard simplex \( X = \Delta^n \) has \( n + 1 \) vertices, while subdivision \( Sd(\Delta^n) \) has \( 2^{n+1} - 1 \) vertices. Nevertheless, we will prove in this section that \( X \) and \( Sd(X) \) are weakly homotopy equivalent. More precisely, for every simplicial set \( X \) there is a canonical weak homotopy equivalence \( \lambda_X : Sd(X) \to X \), which we refer to as the **last vertex map** (Construction 3.3.4.3).

**Notation 3.3.4.1.** Let \( Q \) be a partially ordered set. Every finite, nonempty, linearly ordered subset \( S \subseteq Q \) has a largest element, which we will denote by \( \text{Max}(S) \). The construction \( S \mapsto \text{Max}(S) \) determines a nondecreasing function \( \text{Max} : Q \to \text{Chain}(Q) \), where \( \text{Chain}(Q) \) is defined as in Notation 3.3.2.1.
Remark 3.3.4.2. Let \( f : P \to Q \) be a nondecreasing function between partially ordered sets. Then the diagram of partially ordered sets

\[
\begin{array}{ccc}
\text{Chain}(P) & \xrightarrow{\text{Max}} & P \\
\downarrow^{S \mapsto f(S)} & & \downarrow^{f} \\
\text{Chain}(Q) & \xrightarrow{\text{Max}} & Q
\end{array}
\]

is commutative.

Construction 3.3.4.3. Let \( X \) be a simplicial set. For every \( n \)-simplex \( \sigma : \Delta^n \to X \), we let \( \rho_X(\sigma) \) denote the composite map

\[
N_* (\text{Chain}[n]) \xrightarrow{\text{Max}} \Delta^n \xrightarrow{\sigma} X,
\]

which we regard as an \( n \)-simplex of the simplicial set \( \text{Ex}(X) \) of Construction 3.3.2.5. It follows from Remark 3.3.4.2 that the construction \( \sigma \mapsto \rho_X(\sigma) \) determines a map of simplicial sets \( \rho_X : X \to \text{Ex}(X) \).

Let \( u : X \to \text{Ex}(\text{Sd}(X)) \) be a map of simplicial sets which exhibits \( \text{Sd}(X) \) as a subdivision of \( X \) (Definition 3.3.3.1). Then there is a unique map of simplicial sets \( \lambda_X : \text{Sd}(X) \to X \) for which the composition \( X \to \text{Ex}(\text{Sd}(X)) \xrightarrow{\text{Ex}(\lambda_X)} \text{Ex}(X) \) is equal to \( \rho_X \). We will refer to \( \lambda_X \) as the last vertex map of \( X \).

Remark 3.3.4.4 (Functoriality). The morphisms \( \rho_X : X \to \text{Ex}(X) \) and \( \lambda_X : \text{Sd}(X) \to X \) depend functorially on the simplicial set \( X \). That is, for every map of simplicial sets \( f : X \to Y \), the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{\rho_X} & \text{Ex}(X) \\
\downarrow^{f} & & \downarrow^{\text{Ex}(f)} \\
Y & \xrightarrow{\rho_Y} & \text{Ex}(Y)
\end{array}
\quad
\begin{array}{ccc}
\text{Sd}(X) & \xrightarrow{\lambda_X} & X \\
\downarrow^{\text{Sd}(f)} & & \downarrow^{f} \\
\text{Sd}(Y) & \xrightarrow{\lambda_Y} & Y
\end{array}
\]

are commutative. We may therefore regard the constructions \( X \mapsto \rho_X \) and \( X \mapsto \lambda_X \) as natural transformations of functors

\[
\rho : \text{id}_{\text{Set}_\Delta} \to \text{Ex}, \quad \lambda : \text{Sd} \to \text{id}_{\text{Set}_\Delta}.
\]

Example 3.3.4.5. Let \( Q \) be a partially ordered set, so that we can identify the subdivision of \( N_*(Q) \) with the nerve of the partially ordered set \( \text{Chain}(Q) \) (Example 3.3.3.16). Under this identification, the last vertex map \( \lambda_{N_*(Q)} \) corresponds to the morphism \( N_*(\text{Chain}(Q)) \to N_*(Q) \) induced by \( \text{Max} : \text{Chain}(Q) \to Q \).
Example 3.3.4.6. Let $X$ be a discrete simplicial set (Definition 1.1.4.9). Then the maps

$$\rho_X : X \to \text{Ex}(X) \quad \lambda_X : \text{Sd}(X) \to X$$

are isomorphisms.

Example 3.3.4.7. Let $X$ be a braced simplicial set, so that the subdivision $\text{Sd}(X)$ can be identified with the nerve of the category of $\Delta^\text{nd}_X$ of nondegenerate simplices of $X$ (Proposition 3.3.3.15). Under this identification, the last vertex map $\lambda_X$ corresponds to a morphism of simplicial sets $N^\bullet(\Delta^\text{nd}_X) \to X$. Concretely, if $\tau$ is a $k$-simplex of $N^\bullet(\Delta^\text{nd}_X)$ corresponding to a diagram

$$\Delta^n_{0} \to \Delta^n_{1} \to \cdots \to \Delta^n_{k-1} \to \Delta^n_{k}$$

then $\lambda_X(\tau)$ is the $k$-simplex of $X$ given by the composition

$$\Delta^k \xrightarrow{f} \Delta^n_k \xrightarrow{\sigma_k} X,$$

where $f$ carries each vertex $\{i\} \subseteq \Delta^k$ to the image of the last vertex $\{n_i\} \subseteq \Delta^n$ under the map $\Delta^n_i \to \Delta^n_k$ given by horizontal composition in the diagram.

We can now state the main result of this section:

Proposition 3.3.4.8. Let $X$ be a simplicial set. Then the last vertex map $\lambda_X : \text{Sd}(X) \to X$ is a weak homotopy equivalence.

Remark 3.3.4.9. Proposition 3.3.4.8 has a counterpart for the comparison map $\rho_X : X \to \text{Ex}(X)$, which we will prove in §3.3.5 (see Theorem 3.3.5.1).

Proof of Proposition 3.3.4.8 For each integer $n \geq 0$, let $\text{sk}_n(X)$ denote the $n$-skeleton of the simplicial set $X$. Then the last vertex map $\lambda_X : \text{Sd}(X) \to X$ can be realized as a filtered colimit of the last vertex maps $\lambda_{\text{sk}_n(X)} : \text{Sd}(\text{sk}_n(X)) \to \text{sk}_n(X)$. Since the collection of weak homotopy equivalences is closed under the formation of filtered colimits (Proposition 3.2.7.3), it will suffice to show that each of the maps $\lambda_{\text{sk}_n(X)}$ is a weak homotopy equivalence. We may therefore replace $X$ by $\text{sk}_n(X)$, and thereby reduce to the case where $X$ is $n$-skeletal for some nonnegative integer $n \geq 0$. We proceed by induction on $n$. If $n = 0$, then the simplicial set $X$ is discrete and $\lambda_X$ is an isomorphism (Example 3.3.4.6). We will therefore assume that $n > 0$. 

CHAPTER 3. KAN COMPLEXES

Fix a Kan complex $Q$; we wish to show that composition with $\lambda_X : \text{Sd}(X) \to X$ induces a bijection $\pi_0(\text{Fun}(X, Q)) \to \pi_0(\text{Fun}(\text{Sd}(X), Q))$. In fact, we will show that the map $\text{Fun}(X, Q) \to \text{Fun}(\text{Sd}(X), Q)$ is a weak homotopy equivalence. Let $Y = \text{sk}_{n-1}(X)$ be the $(n-1)$-skeleton of $X$, so that we have a commutative diagram

$$
\begin{array}{ccc}
\text{Fun}(X, Q) & \xrightarrow{\theta} & \text{Fun}(\text{Sd}(X), Q) \\
\downarrow & & \downarrow \\
\text{Fun}(Y, Q) & \xrightarrow{\theta} & \text{Fun}(\text{Sd}(Y), Q),
\end{array}
$$

where the lower horizontal map is a homotopy equivalence by virtue of our inductive hypothesis (together with Corollary 3.1.6.5). It will therefore suffice to show that, for every morphism of simplicial sets $f : Y \to Q$, the induced map of fibers

$$\theta_f : \{f\} \times_{\text{Fun}(Y, Q)} \text{Fun}(X, Q) \to \{f\} \times_{\text{Fun}(\text{Sd}(Y'), Q)} \text{Fun}(\text{Sd}(X), Q)$$

is a homotopy equivalence (Proposition 3.2.7.1).

Let $S$ denote the collection of nondegenerate $n$-simplices of $X$, let $X' = \coprod_{\sigma \in S} \Delta^n$ denote their coproduct, and let $Y' = \coprod_{\sigma \in S} \partial \Delta^n$ denote the boundary of $X'$. Proposition 1.1.3.13 then supplies a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\coprod_{\sigma \in S} \partial \Delta^n & \to & \coprod_{\sigma \in S} \Delta^n \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\theta} & X,
\end{array}
$$

which we can use to identify $\theta_f$ with the induced map

$$\theta'_f : \{f\} \times_{\text{Fun}(Y', Q)} \text{Fun}(X', Q) \to \{f\} \times_{\text{Fun}(\text{Sd}(Y'), Q)} \text{Fun}(\text{Sd}(X'), Q).$$

Invoking Proposition 3.2.7.1 again, we are reduced to showing that the horizontal maps appearing in the diagram

$$
\begin{array}{ccc}
\text{Fun}(X', Q) & \xrightarrow{\theta} & \text{Fun}(\text{Sd}(X'), Q) \\
\downarrow & & \downarrow \\
\text{Fun}(Y', Q) & \xrightarrow{\theta} & \text{Fun}(\text{Sd}(Y'), Q)
\end{array}
$$

are homotopy equivalences. By virtue of Corollary 3.1.6.5 it will suffice to show that the last vertex maps $\lambda_{Y'} : \text{Sd}(Y') \to Y'$ and $\lambda_{X'} : \text{Sd}(X') \to X'$ are weak homotopy equivalences. In the first case, this follows from our inductive hypothesis (since $Y'$ has dimension $< n$). In the second, we can use Remark 3.1.5.19 to reduce to the problem of showing that the last vertex map $\lambda_{\Delta^n} : \text{Sd}(\Delta^n) \to \Delta^n$ is a weak homotopy equivalence. This is clear, since both $\text{Sd}(\Delta^n)$ and $\Delta^n$ are contractible by virtue of Example 3.2.6.6 (they can be realized as the nerves of partially ordered sets $\text{Chain}[n]$ and $[n]$, each of which has a largest element). \qed
3.3.5 Comparison of $X$ with $\text{Ex}(X)$

The goal of this section is to prove the following variant of Proposition 3.3.4.8:

**Theorem 3.3.5.1.** Let $X$ be a simplicial set. Then the comparison map $\rho_X : X \to \text{Ex}(X)$ of Construction 3.3.4.3 is a weak homotopy equivalence.

**Corollary 3.3.5.2.** Let $f : X \to Y$ be a morphism of simplicial sets. Then $f$ is a weak homotopy equivalence if and only $\text{Ex}(f)$ is a weak homotopy equivalence.

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{\rho_X} & & \downarrow^{\rho_Y} \\
\text{Ex}(X) & \xrightarrow{\text{Ex}(f)} & \text{Ex}(Y),
\end{array}
$$

where the vertical maps are weak homotopy equivalences (Theorem 3.3.5.1). The desired result now follows from the two-out-of-three property (Remark 3.1.5.16).

The proof of Theorem 3.3.5.1 will make use of the following fact, which we prove at the end of this section:

**Proposition 3.3.5.3.** Let $f : X \to Y$ be an anodyne morphism of simplicial sets. Then the induced map $\text{Sd}(f) : \text{Sd}(X) \to \text{Sd}(Y)$ is also anodyne.

**Corollary 3.3.5.4.** Let $f : X \to Y$ be a Kan fibration of simplicial sets. Then the induced map $\text{Ex}(f) : \text{Ex}(X) \to \text{Ex}(Y)$ is also a Kan fibration.

**Proof.** We must show that every lifting problem

$$
\begin{array}{ccc}
\Lambda^n_i & \rightarrow & \text{Ex}(X) \\
\downarrow & & \downarrow^{\text{Ex}(f)} \\
\Delta^n & \rightarrow & \text{Ex}(Y)
\end{array}
$$

admits a solution. This follows by applying Remark 3.1.2.6 to the associated lifting problem

$$
\begin{array}{ccc}
\text{Sd}(\Lambda^n_i) & \rightarrow & X \\
\downarrow & & \downarrow^f \\
\text{Sd}(\Delta^n) & \rightarrow & Y,
\end{array}
$$

since the left vertical map is anodyne by virtue of Proposition 3.3.5.3.
Corollary 3.3.5.5. Let $X$ be a Kan complex. Then the simplicial set $\text{Ex}(X)$ is also a Kan complex.

Proposition 3.3.5.6. Let $X$ and $Y$ be simplicial sets, where $Y$ is a Kan complex. Then the bijection
\[ \text{Hom}_{\text{Set}}(\text{Sd}(X), Y) \simeq \text{Hom}_{\text{Set}}(\text{Sd}(X), \text{Ex}(Y)) \]
respects homotopy. That is, for every pair of maps $f, g : \text{Sd}(X) \to Y$ having counterparts $f', g' : X \to \text{Ex}(Y)$, then $f$ is homotopic to $g$ if and only if $f'$ is homotopic to $g'$.

Proof. Assume first that $f$ and $g$ are homotopic, so that there exists a morphism of simplicial sets $h : \Delta^1 \times \text{Sd}(X) \to Y$ satisfying $h|_{\{0\} \times \text{Sd}(X)} = f$ and $h|_{\{1\} \times \text{Sd}(X)} = g$. The composite map
\[ \text{Sd}(\Delta^1 \times X) \to \text{Sd}(\Delta^1) \times \text{Sd}(X) \xrightarrow{\lambda_{\Delta^1} \times \text{id}} \Delta^1 \times \text{Sd}(X) \xrightarrow{h} Y \]
then determines a morphism of simplicial sets $h' : \Delta^1 \times X \to \text{Ex}(Y)$, which is immediately seen to be a homotopy from $f'$ to $g'$.

Conversely, suppose that $f'$ and $g'$ are homotopic. Since $\text{Ex}(Y)$ is a Kan complex (Corollary 3.3.5.5), we can choose a morphism of simplicial sets $h' : \Delta^1 \times X \to \text{Ex}(Y)$ satisfying $h'|_{\{0\} \times X} = f'$ and $h'|_{\{1\} \times X} = g'$, which we can identify with a map $u : \text{Sd}(\Delta^1 \times X) \to Y$. Let $v$ denote the composite map $\text{Sd}(\Delta^1 \times X) \to \text{Sd}(X) \xrightarrow{\text{i}} Y$, so that $u$ and $v$ have the same restriction to $\text{Sd}(\{0\} \times X)$. Note that the inclusion of simplicial sets $\{0\} \times X \hookrightarrow \Delta^1 \times X$ is anodyne (Proposition 3.1.2.7), so the subdivision $\text{Sd}(\{0\} \times X) \to \text{Sd}(\Delta^1 \times X)$ is also anodyne (Proposition 3.3.5.3). It follows that the restriction map $\text{Fun}(\text{Sd}(\Delta^1 \times X), Y) \to \text{Fun}(\text{Sd}(\{0\} \times X), Y)$ is a trivial Kan fibration, so that $u$ and $v$ belong to the same path component of $\text{Fun}(\text{Sd}(\Delta^1 \times X), Y)$ and are therefore homotopic. It follows that $f = v|_{\text{Sd}(\{1\} \times X)}$ and $g = u|_{\text{Sd}(\{1\} \times X)}$ are also homotopic.

We can now prove a special case of Theorem 3.3.5.1.

Proposition 3.3.5.7. Let $Y$ be a Kan complex. Then the comparison map $\rho_Y : Y \to \text{Ex}(Y)$ of Construction 3.3.4.3 is a homotopy equivalence.

Proof. Fix a simplicial set $X$. We wish to show that postcomposition with $\rho_Y$ induces a bijection
\[
\{\text{Maps of simplicial sets } X \to Y\}/\text{homotopy} \to \{\text{Maps of simplicial sets } X \to \text{Ex}(Y)\}/\text{homotopy}.
\]
By virtue of Proposition 3.3.5.6, this is equivalent to the assertion that precomposition with the last vertex map \( \lambda_X : Sd(X) \to X \) induces a bijection

\[
\{ \text{Maps of simplicial sets } X \to Y \}/\text{homotopy} \\
\{ \text{Maps of simplicial sets } Sd(X) \to Y \}/\text{homotopy},
\]

which follows from the fact that \( \lambda_X \) is a weak homotopy equivalence (Proposition 3.3.4.8). \( \square \)

To deduce Theorem 3.3.5.1 from Proposition 3.3.5.7, we will need the following:

**Proposition 3.3.5.8.** Let \( X \) be a simplicial set, and let \( \rho_X : X \to \text{Ex}(X) \) be the comparison map of Construction 3.3.4.3. Then the morphisms \( \rho_{\text{Ex}(X)}, \text{Ex}(\rho_X) : \text{Ex}(X) \to \text{Ex}(\text{Ex}(X)) \) are homotopic.

**Proof.** Let \( Q \) be a partially ordered set. Using Example 3.3.3.16, we can identify the subdivisions \( Sd(N_{\bullet}(Q)) \) and \( Sd(Sd(N_{\bullet}(Q))) \) with the nerves of partially ordered sets Chain(\( Q \)) and Chain(Chain(\( Q \))), respectively. Under this identification, the morphisms of simplicial sets

\[
Sd(\lambda_{N_{\bullet}(Q)}), \lambda_{Sd(N_{\bullet}(Q)))} : Sd(Sd(N_{\bullet}(Q))) \to Sd(N_{\bullet}(Q))
\]

can be correspond to with nondecreasing functions Chain(Chain(\( Q \)) \to Chain(\( Q \)), whose value on a linearly ordered subset \( \tilde{S} = (S_0 \subset S_1 \subset \cdots \subset S_n) \) of Chain(\( Q \)) are given by

\[
Sd(\lambda_{N_{\bullet}(Q)})(\tilde{S}) = \{ \text{Max}(S_0), \ldots, \text{Max}(S_n) \} \quad \lambda_{Sd(N_{\bullet}(Q)))}(\tilde{S}) = S_n.
\]

Note that we always have an inclusion \( \{ \text{Max}(S_0), \ldots, \text{Max}(S_n) \} \subseteq S_n \). It follows that there is a unique map of simplicial sets

\[
h_Q : \Delta^1 \times Sd(Sd(N_{\bullet}(Q))) \to Sd(N_{\bullet}(Q))
\]

satisfying \( h_Q|_{\{0\} \times Sd(N_{\bullet}(Q)))} = Sd(\lambda_{N_{\bullet}(Q)}(Q)) \) and \( h_Q|_{\{1\} \times Sd(N_{\bullet}(Q)))} = \lambda_{Sd(N_{\bullet}(Q))}, \) depending functorially on \( Q \).

Let \( \sigma \) be an \( n \)-simplex of the simplicial set \( \text{Ex}(X) \), which we identify with a map \( \sigma : Sd(\Delta^n) \to X \). We let \( f(\sigma) \) denote the composite map

\[
\Delta^1 \times Sd(\Delta^n) \xrightarrow{h_{[n]}} Sd(\Delta^n) \xrightarrow{\sigma} X,
\]

which we will identify with an \( n \)-simplex of the simplicial set \(
\text{Fun}(\Delta^1, \text{Ex}(\text{Ex}(X)))\). The construction \( \sigma \mapsto f(\sigma) \) then determines a morphism of simplicial sets \( f : \text{Ex}(X) \to \text{Fun}(\Delta^1, \text{Ex}(\text{Ex}(X))) \), which we can identify with a map \( \Delta^1 \times \text{Ex}(X) \to \text{Ex}(\text{Ex}(X)) \). By construction, this map is a homotopy from \( \rho_{\text{Ex}(X)} \) to \( \text{Ex}(\rho_X) \). \( \square \)
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Proof of Theorem 3.3.5.1. Let $X$ be a simplicial set. We wish to prove that the comparison map $\rho_X : X \to \text{Ex}(X)$ is a weak homotopy equivalence. Fix a Kan complex $Y$; we must show that composition with $\rho_X$ induces a bijection $\pi_0(\text{Fun}(\text{Ex}(X), Y)) \to \pi_0(\text{Fun}(X, Y))$. This map fits into a diagram

\[
\begin{array}{ccc}
\pi_0(\text{Fun}(\text{Ex}(X), Y)) & \xrightarrow{\circ \rho_X} & \pi_0(\text{Fun}(X, Y)) \\
\sim & f \mapsto \text{Ex}(f) & \sim \rho_Y \circ f \\
\pi_0(\text{Fun}(\text{Ex}(X), \text{Ex}(Y))) & \xrightarrow{\circ \rho_X} & \pi_0(\text{Fun}(X, \text{Ex}(Y))),
\end{array}
\]

where the vertical maps are bijective (Proposition 3.3.5.7) and the lower triangle commutes by the naturality of $\rho$. To show that the upper horizontal map is bijective, it will suffice to show that the upper triangle also commutes. Fix a map $f : \text{Ex}(X) \to Y$. We then compute

\[\text{Ex}(f \circ \rho_X) = \text{Ex}(f) \circ \text{Ex}(\rho_X) \sim \text{Ex}(f) \circ \rho_{\text{Ex}(X)} = \rho_Y \circ f\]

where the equality on the left follows from functoriality, the equality on the right from the naturality of $\rho$, and the homotopy in the middle is supplied by Proposition 3.3.5.8.

We close this section with the proof of Proposition 3.3.5.3.

Lemma 3.3.5.9. Let $J$ be a nonempty finite set, let $P(J)$ denote the collection of subsets of $J$ (partially ordered by inclusion), and set $P_-(J) = P(J) \setminus \{J\}$. Then the inclusion of simplicial sets

\[\theta : N_\bullet(P_-(J)) \hookrightarrow N_\bullet(P(J)) = \square^I\]

is anodyne.

Proof. Fix an element $j \in J$ and set $I = J \setminus \{j\}$, so that the simplicial cube $\square^I$ can be identified with the product $\Delta^1 \times \square^I \simeq \Delta^1 \times N_\bullet(P(I))$. Under this identification, $\theta$ corresponds to the inclusion map

\[\left(\Delta^1 \times N_\bullet(P_-(I))\right) \bigcup_{\{0\} \times N_\bullet(P_-(I))} \left(\{0\} \times N_\bullet(P(I))\right) \hookrightarrow \Delta^1 \times N_\bullet(P(I)),\]

which is anodyne by virtue of Proposition 3.1.2.7.

Proof of Proposition 3.3.5.3. Let $S$ be the collection of all morphisms of simplicial sets $f : X \to Y$ for which the induced map $\text{Sd}(f) : \text{Sd}(X) \to \text{Sd}(Y)$ is anodyne. Since the subdivision functor $\text{Sd}$ preserves colimits, the collection $S$ is weakly saturated (in the sense of Definition 1.4.4.15). To prove Proposition 3.3.5.3 it will suffice to show that $S$ contains every
horn inclusion. Fix a positive integer $n$ and another integer $0 \leq i \leq n$. We will complete the proof by showing that the inclusion $\Lambda^n_i \to \Delta^n$ induces an anodyne map $\text{Sd}(\Lambda^n_i) \to \text{Sd}(\Delta^n)$.

Let $J = [n] \setminus \{i\}$, let $P(J)$ denote the collection of all subsets of $J$, partially ordered by inclusion. Set $P_-(J) = P(J) \setminus \{\emptyset\}$, and $P_+(J) = P(J) \setminus \{\emptyset, J\}$. In what follows, we identify $\text{Sd}(\Delta^n)$ with the nerve of the partially ordered set $\text{Chain}[n]$, and $\text{Sd}(\Lambda^n_i)$ with the nerve of the partially ordered subset of $\text{Chain}[n]$ obtained by removing the elements $[n]$ and $J$ (Proposition 3.3.3.15). The construction $J \mapsto J \cup \{i\}$ determines an inclusion of partially ordered sets $P(J) \to \text{Chain}[n]$, hence a map of simplicial sets $g : [1, \times P_+(J)) \to \text{Sd}(\Delta^n)$.

Let $Z \subseteq \text{Sd}(\Delta^n)$ be the union of $\text{Sd}(\Lambda^n_i)$ with the image of $g$. An elementary calculation shows that the inverse image $g^{-1}(\text{Sd}(\Lambda^n_i))$ can be identified with the nerve of the subset $P_-(J) \subseteq P(J)$, so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
N_\bullet(P_-(J)) & \to & \text{Sd}(\Lambda^n_i) \\
\downarrow & & \downarrow \\
N_\bullet(P(J)) & \to & Z.
\end{array}
$$

The left vertical map is anodyne by virtue of Lemma 3.3.5.9 so the right vertical map is anodyne as well. Let $h : [1] \times P_+(J) \to \text{Chain}[n]$ be the map of partially ordered sets given $h(0, J_0) = J_0$ and $h(1, J_0) = J_0 \cup \{i\}$. Then $h$ determines a map of simplicial sets $\Delta^1 \times N_\bullet(P_+(J)) \to \text{Sd}(\Delta^n)$. An elementary calculation shows that this map of simplicial sets fits into a pushout diagram

$$
\begin{array}{ccc}
\{1\} \times N_\bullet(P_+(J)) & \coprod & \Delta^1 \times N_\bullet(P_+(J)) \\
\downarrow & & \downarrow \\
\Delta^1 \times N_\bullet(P_+(J)) & \to & \text{Sd}(\Delta^n).
\end{array}
$$

The left vertical map in this diagram is anodyne by virtue of Proposition 3.1.2.7 so the inclusion $Z \to \text{Sd}(\Delta^n)$ is also anodyne. It follows that the composite map $\text{Sd}(\Lambda^n_i) \to Z \to \text{Sd}(\Delta^n)$ is anodyne, as desired.

3.3.6 The $\text{Ex}^\infty$ Functor

Let $X$ be a simplicial set. In §3.1.6 we proved that one can always choose an embedding $j : X \to Q$, where $Q$ is a Kan complex and $j$ is a weak homotopy equivalence (Corollary 3.1.6.2). In [22], Kan gave an explicit construction of such an embedding, based on iteration of the construction $X \mapsto \text{Ex}(X)$.\[\]
CHAPTER 3. KAN COMPLEXES

Construction 3.3.6.1 (The Ex∞ Functor). For every nonnegative integer n, we let Ex^n denote the n-fold iteration of the functor Ex : Set_Δ → Set_Δ of Construction 3.3.2.5 given inductively by the formula

Ex^n(X) = \begin{cases} X & \text{if } n = 0 \\ Ex(Ex^{n-1}(X)) & \text{if } n > 0. \end{cases}

For every simplicial set X, we let Ex^∞(X) denote the colimit of the diagram

X \xrightarrow{\rho_X} Ex(X) \xrightarrow{\rho_{Ex(X)}} Ex^2(X) \xrightarrow{\rho_{Ex^2(X)}} Ex^3(X) \rightarrow \cdots,

where each ρ_{Ex^n(X)} denotes the comparison map of Construction 3.3.4.3, and we let ρ^∞_X : X → Ex^∞(X) denote the tautological map. The construction X ↦→ Ex^∞(X) determines a functor Ex^∞ from the category of simplicial sets to itself, and the construction X ↦→ ρ^∞_X determines a natural transformation of functors id_{Set_Δ} → Ex^∞.

Our goal in this section is to record the main properties of Construction 3.3.6.1. In particular, for every simplicial set X, we show that Ex^∞(X) is a Kan complex (Proposition 3.3.6.9) and that the comparison map ρ^∞_X : X → Ex^∞(X) is a weak homotopy equivalence (Proposition 3.3.6.7).

Proposition 3.3.6.2. Let X be a simplicial set. Then the comparison map ρ^∞_X : X → Ex^∞(X) is a monomorphism of simplicial sets. Moreover, it is bijective on vertices.

Proof. It will suffice to show that each of the comparison maps ρ_{Ex^n(X)} : Ex^n(X) → Ex^{n+1}(X) is a monomorphism which is bijective on vertices. Replacing X by Ex^n(X), we can reduce to the case n = 0. Fix m ≥ 0. On m-simplices, the comparison map ρ_X is given by the map of sets

Hom_{Set_Δ}(\Delta^m, X) → Hom_{Set_Δ}(Sd(\Delta^m), X)

induced by precomposition with the last vertex map \lambda_{\Delta^m} : Sd(\Delta^m) → \Delta^m. To complete the proof, it suffices to observe that this \lambda_{\Delta^m} is an epimorphism of simplicial sets (in fact, it admits a section \Delta^m → Sd(\Delta^m) ≃ N•(Chain[m]), given by the chain of subsets \{0\} ⊂ \{0, 1\} ⊂ \cdots ⊂ \{0, 1, \ldots, m\}, and an isomorphism in the special case m = 0. □

Example 3.3.6.3. Let X be a discrete simplicial set (Definition 1.1.4.9). Invoking Example 3.3.4.6 repeatedly, we deduce that the transition maps in the diagram

X \xrightarrow{\rho_X} Ex(X) \xrightarrow{\rho_{Ex(X)}} Ex^2(X) \xrightarrow{\rho_{Ex^2(X)}} Ex^3(X) \rightarrow \cdots,

are isomorphisms. It follows that the comparison map ρ^∞_X : X → Ex^∞(X) is also an isomorphism.
Proposition 3.3.6.4. The functor \( X \mapsto \operatorname{Ex}^\infty(X) \) preserves filtered colimits and finite limits.

Proof. It will suffice to show that, for every nonnegative integer \( n \), the functor \( X \mapsto \operatorname{Ex}^n(X) \) preserves filtered colimits and finite limits. Proceeding by induction on \( n \), we can reduce to the case \( n = 1 \). We now observe that the functor \( \operatorname{Ex} \) preserves all limits of simplicial sets (either by construction, or because it admits a left adjoint \( X \mapsto \operatorname{Sd}(X) \)), and preserves filtered colimits by virtue of Remark 3.3.2.7.

Corollary 3.3.6.5. Let \( f : X \to S \) be a morphism of simplicial sets. Let \( s \) be a vertex of \( S \), which we will identify (via Proposition 3.3.6.2) with its image in \( \operatorname{Ex}^\infty(S) \). Then the canonical map \( \operatorname{Ex}^\infty(X_s) \to \operatorname{Ex}^\infty(X)_s \) is an isomorphism of simplicial sets. Here \( X_s = \{s\} \times_S X \) denotes the fiber of \( f \) over the vertex \( s \), and \( \operatorname{Ex}^\infty(X)_s = \{s\} \times_{\operatorname{Ex}^\infty(S)} \operatorname{Ex}^\infty(X) \) is defined similarly.

Proof. Combine Proposition 3.3.6.4 with Example 3.3.6.3.

Proposition 3.3.6.6. Let \( f : X \to S \) be a morphism of simplicial sets. If \( f \) is a Kan fibration, then the induced map \( \operatorname{Ex}^\infty(f) : \operatorname{Ex}^\infty(X) \to \operatorname{Ex}^\infty(S) \) is also a Kan fibration.

Proof. Since the collection of Kan fibrations is stable under the formation of filtered colimits (Remark 3.1.1.8), it will suffice to show that each of the maps \( \operatorname{Ex}^n(f) : \operatorname{Ex}^n(X) \to \operatorname{Ex}^n(S) \) is a Kan fibration. Proceeding by induction on \( n \), we can reduce to the case \( n = 1 \), which follows from Corollary 3.3.5.4.

Proposition 3.3.6.7. Let \( X \) be a simplicial set. Then the comparison map \( \rho^\infty_X : X \to \operatorname{Ex}^\infty(X) \) is a weak homotopy equivalence.

Proof. By virtue of Proposition 3.2.7.3, it will suffice to show that for each \( n \geq 0 \), the composite map

\[
X \xrightarrow{\rho_X} \operatorname{Ex}(X) \xrightarrow{\rho_{\operatorname{Ex}(X)}} \cdots \xrightarrow{\rho_{\operatorname{Ex}^{n-1}(X)}} \operatorname{Ex}^n(X)
\]

is a weak homotopy equivalence. Proceeding by induction on \( n \), we are reduced to showing that each of the comparison maps \( \rho_{\operatorname{Ex}^{n-1}(X)} : \operatorname{Ex}^{n-1}(X) \to \operatorname{Ex}^n(X) \) is a weak homotopy equivalence, which is a special case of Theorem 3.3.5.1.

Corollary 3.3.6.8. Let \( f : X \to Y \) be a morphism of simplicial sets. Then \( f \) is a weak homotopy equivalence if and only \( \operatorname{Ex}^\infty(f) \) is a weak homotopy equivalence.

Proof. We have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\rho^\infty_X} & & \downarrow{\rho^\infty_Y} \\
\operatorname{Ex}^\infty(X) & \xrightarrow{\operatorname{Ex}^\infty(f)} & \operatorname{Ex}^\infty(Y),
\end{array}
\]
where the vertical maps are weak homotopy equivalences (Proposition 3.3.6.7). The desired result now follows from the two-out-of-three property (Remark 3.1.5.16).

**Proposition 3.3.6.9.** Let $X$ be a simplicial set. Then $\operatorname{Ex}^\infty(X)$ is a Kan complex.

**Proof.** Let $X$ be a simplicial set and suppose we are given a morphism of simplicial sets $f_0 : \Lambda^n_i \to \operatorname{Ex}^\infty(X)$, for some $n > 0$ and $0 \leq i \leq n$. We wish to show that $f_0$ can be extended to an $n$-simplex of $\operatorname{Ex}^\infty(X)$. Since the simplicial set $\Lambda^n_i$ has finitely many nondegenerate simplices, we can assume that $f_0$ factors as a composition $\Lambda^n_i \xrightarrow{f'_0} \operatorname{Ex}^m(X) \xrightarrow{p_{\operatorname{Ex}^m(X)}} \operatorname{Ex}^\infty(X)$, for some positive integer $m$. We will complete the proof by showing that $f'_0$ can be extended to an $n$-simplex of $\operatorname{Ex}^{m+1}(X)$: that is, that there exists a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{f'_0} & \operatorname{Ex}^m(X) \\
\downarrow & & \downarrow p_{\operatorname{Ex}^m(X)} \\
\Delta^n & \xrightarrow{f'} & \operatorname{Ex}^{m+1}(X).
\end{array}
$$

Replacing $X$ by $\operatorname{Ex}^{m-1}$, we can reduce to the case $m = 1$. In this case, $f'_0$ can be identified with a morphism of simplicial sets $g_0 : \operatorname{Sd}(\Lambda^n_i) \to X$. Unwinding the definitions, we see that the problem of finding a simplex $f' : \Delta^n \to \operatorname{Ex}^2(X)$, with the desired property is equivalent to the problem of finding a morphism $g : \operatorname{Sd}(\operatorname{Sd}(\Delta^n)) \to X$ whose restriction to $\operatorname{Sd}(\operatorname{Sd}(\Lambda^n_i))$ is equal to the composition

$$
\operatorname{Sd}(\operatorname{Sd}(\Lambda^n_i)) \xrightarrow{\operatorname{Sd}(\lambda_{\Lambda^n_i})} \operatorname{Sd}(\Lambda^n_i) \xrightarrow{g_0} X.
$$

Without loss of generality, we may assume that $X = \operatorname{Sd}(\Lambda^n_i)$ and that $g_0$ is the identity map. Let $\operatorname{Chain}[n]$ be the collection of all nonempty subsets of $[n]$ (Notation 3.3.2.1) and let $Q \subset \operatorname{Chain}[n]$ be the subset obtained by removing $[n]$ and $[n] \setminus \{i\}$. Using Proposition 3.3.3.15 we can identify $\operatorname{Sd}(\Lambda^n_i)$, $\operatorname{Sd}(\operatorname{Sd}(\Lambda^n_i))$, and $\operatorname{Sd}(\operatorname{Sd}(\Delta^n))$ with the nerves of the partially ordered sets $Q$, $\operatorname{Chain}(Q)$, and $\operatorname{Chain}(\operatorname{Chain}[n])$, respectively. To complete the proof, it will suffice to show that there exists a nondecreasing function of partially ordered sets $g : \operatorname{Chain}(\operatorname{Chain}[n]) \to Q$ having the property that, for every element $(S_0 \subset S_1 \subset \cdots \subset S_m)$ of $\operatorname{Chain}(Q)$, we have $g(S_0 \subset S_1 \subset \cdots \subset S_m) = \{\operatorname{Max}(S_0), \operatorname{Max}(S_1), \ldots, \operatorname{Max}(S_m)\} \in Q$. This requirement is satisfied if $g$ is defined by the formula

$$
g(S_0 \subset S_1 \subset \cdots \subset S_m) = \{\operatorname{Max}'(S_0), \operatorname{Max}'(S_1), \ldots, \operatorname{Max}'(S_m)\},
$$

where $\operatorname{Max}' : \operatorname{Chain}[n] \to [n]$ is the (non-monotone) map of sets given by

$$
\operatorname{Max}'(S) = \begin{cases} 
i & \text{if } S = [n] \text{ or } S = [n] \setminus \{i\} \\
\operatorname{Max}(S) & \text{otherwise}. \end{cases}
$$

\[\square\]
Corollary 3.3.6.10. Let \( X \) be a Kan complex. Then the comparison map \( \rho_X^\infty : X \to \text{Ex}^\infty(X) \) is a homotopy equivalence.

Proof. Since \( \text{Ex}^\infty(X) \) is also a Kan complex (Proposition 3.3.6.9), it will suffice to show that \( \rho_X^\infty \) is a weak homotopy equivalence (Proposition 3.1.5.12), which follows from Proposition 3.3.6.7.

3.3.7 Application: Characterizations of Weak Homotopy Equivalences

Let \( f : X \to S \) be a Kan fibration between Kan complexes. In \( \square 3.2.6 \) we proved that \( f \) is a homotopy equivalence if and only if it is a trivial Kan fibration (Corollary 3.2.6.9). We now apply the machinery of \( \square 3.3.6 \) to extend this result to the case where \( S \) is an arbitrary simplicial set. First, we need a slight generalization of Proposition 3.2.7.1.

Proposition 3.3.7.1. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{u} & X' \\
\downarrow & & \downarrow \\
S & \xrightarrow{v} & S',
\end{array}
\]

where the vertical maps are Kan fibrations and \( v \) is a weak homotopy equivalence. The following conditions are equivalent:

1. The morphism \( u \) is a weak homotopy equivalence.
2. For every vertex \( s \in S \), the induced map of fibers \( u_s : X_s \to X'_v(s) \) is a homotopy equivalence of Kan complexes.

Proof. Using Corollaries 3.3.6.8 and 3.3.6.5 we can replace (1) and (2) by the following assertions:

1'. The morphism \( \text{Ex}^\infty(u) : \text{Ex}^\infty(X) \to \text{Ex}^\infty(X') \) is a weak homotopy equivalence.

2'. For every vertex \( s \in S \), the induced map of fibers \( u_s : \text{Ex}^\infty(X)_s \to \text{Ex}^\infty(X')_v(s) \) is a homotopy equivalence of Kan complexes.

The equivalence of (1') and (2') follows by applying Proposition 3.2.7.1 to the diagram

\[
\begin{array}{ccc}
\text{Ex}^\infty(X) & \xrightarrow{\text{Ex}^\infty(u)} & \text{Ex}^\infty(X') \\
\downarrow & & \downarrow \\
\text{Ex}^\infty(S) & \xrightarrow{\text{Ex}^\infty(v)} & \text{Ex}^\infty(S').
\end{array}
\]
note that every simplicial set appearing in this diagram is a Kan complex (Proposition \[3.3.6.9\]), the vertical maps are Kan fibrations (Proposition \[3.3.6.6\]) and \(\text{Ex}^\infty(v)\) is a homotopy equivalence by virtue of Corollary \[3.3.6.8\].

**Corollary 3.3.7.2.** Let \(v: T \to S\) be a weak homotopy equivalence of simplicial sets. For every Kan fibration \(f: X \to S\), the projection map \(T \times_S X \to X\) is also a weak homotopy equivalence.

**Corollary 3.3.7.3.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \xrightarrow{u} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & S
\end{array}
\]

where the vertical maps are Kan fibrations. Then \(u\) is a weak homotopy equivalence if and only if, for each vertex \(s \in S\), the induced map \(u_s: Y_s \to X_s\) is a homotopy equivalence of Kan complexes.

**Proposition 3.3.7.4.** Let \(f: X \to S\) be a Kan fibration of simplicial sets. The following conditions are equivalent:

1. For every vertex \(s \in S\), the fiber \(X_s = \{s\} \times_S X\) is a contractible Kan complex.
2. The morphism \(f\) is a trivial Kan fibration.
3. The morphism \(f\) is a homotopy equivalence.
4. The morphism \(f\) is a weak homotopy equivalence.

**Proof.** The equivalence (1) \(\iff\) (2) follows from Proposition \[3.2.6.8\] and the equivalence (1) \(\iff\) (4) follows from Corollary \[3.3.7.3\]. The implication (2) \(\Rightarrow\) (3) is a special case of Corollary \[3.3.7.3\] and the implication (3) \(\Rightarrow\) (4) follows from from Proposition \[3.1.5.12\].

**Corollary 3.3.7.5.** Let \(f: X \to Y\) be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism \(f\) is anodyne.
2. The morphism \(f\) is both a monomorphism and a weak homotopy equivalence.

**Proof.** The implication (1) \(\Rightarrow\) (2) follows from Proposition \[3.1.5.13\] and Remark \[3.1.2.3\]. To prove the converse, assume that \(f\) is a weak homotopy equivalence and apply Proposition \[3.1.6.1\] to write \(f\) as a composition \(X \xrightarrow{f'} Q \xrightarrow{f''} Y\), where \(f'\) is anodyne and \(f''\) is a Kan fibration. Then \(f'\) is a weak homotopy equivalence (Proposition \[3.1.5.13\]), so \(f''\) is a weak
homotopy equivalence (Remark 3.1.5.16). Invoking Proposition 3.3.7.4, we conclude that $f''$ is a trivial Kan fibration. If $f$ is a monomorphism, then the lifting problem

\[
\begin{array}{ccc}
X & \xrightarrow{f'} & Q \\
\downarrow f & & \downarrow f'' \\
Y & \xrightarrow{\text{id}_Y} & Y
\end{array}
\]

admits a solution. It follows that $f$ is a retract of $f'$ (in the arrow category $\text{Fun}([1], \text{Set}_\Delta)$). Since the collection of anodyne morphisms is closed under retracts, we conclude that $f$ is anodyne.

### 3.3.8 Application: Extending Kan Fibrations

In the proof of Proposition 3.3.7.4 we made essential use of the fact that for any Kan fibration of simplicial sets $f : X \to S$ is (fiberwise) homotopy equivalent to a pullback $S \times_{S'} X' \to S$, where $f' : X' \to S'$ is a Kan fibration between Kan complexes. This can be achieved by taking $f'$ to be the Kan fibration $\text{Ex}^\infty(f) : \text{Ex}^\infty(X) \to \text{Ex}^\infty(S)$. Using a variant of this construction, one can obtain a more precise result.

**Theorem 3.3.8.1.** Let $f : X \to S$ be a Kan fibration between simplicial sets, and let $g : S \to S'$ be an anodyne map. Then there exists a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{g} & S'
\end{array}
\]

where $f'$ is a Kan fibration.

**Remark 3.3.8.2.** We refer the reader to [25] for a proof of Theorem 3.3.8.1 which is slightly different from the proof given below (it avoids the use of Kan’s $\text{Ex}^\infty$-functor by appealing instead to the theory of minimal Kan fibrations, which we will discuss in §[?]). See also [36] and [34].

**Remark 3.3.8.3.** If $f : X \to S$ is a Kan fibration of simplicial sets, then every vertex $s \in S$ determines a Kan complex $X_s = \{s\} \times_S X$. One can think of the construction $s \mapsto X_s$ as supplying a map from $S$ to the "space" of all Kan complexes. Roughly speaking, one can think of Theorem 3.3.8.1 as asserting that this "space" itself behaves like a Kan complex. We will return to this idea later (see §[?]).

The proof of Theorem 3.3.8.1 is based on the following observation:
Lemma 3.3.8.4. Let \( f : Y \rightarrow T \) be a Kan fibration of simplicial sets, and suppose we are given simplicial subsets \( X \subseteq Y \) and \( S \subseteq T \) satisfying the following conditions:

(a) The morphism \( f \) carries \( X \) to \( S \), and the restriction \( f_0 = f|_X \) is a Kan fibration from \( X \) to \( S \).

(b) For every vertex \( s \in S \), the inclusion of fibers \( X_s \hookrightarrow Y_s \) is a homotopy equivalence of Kan complexes.

Let \( Y' \subseteq Y \) denote the simplicial subset spanned by those simplices \( \sigma : \Delta^n \rightarrow Y \) having the property that the restriction \( \sigma|_{S \times T} \Delta^n \) factors through \( X \). Then the restriction \( f|_{Y'} : Y' \rightarrow T \) is a Kan fibration.

Proof. Set \( Y_S = S \times T Y \subseteq Y \). It follows from assumption (b) and Corollary 3.3.7.3 that the inclusion \( X \hookrightarrow Y_S \) is a weak homotopy equivalence, and is therefore anodyne (Corollary 3.3.7.5). Since \( f_0 \) is a Kan fibration, Theorem 3.1.3.5 guarantees that the map

\[
\text{Fun}(Y_S, Y_S) \rightarrow \text{Fun}(X, Y_S) \times_{\text{Fun}(X, S)} \text{Fun}(Y_S, S)
\]

is a trivial Kan fibration. We can therefore choose a homotopy \( H : \Delta^1 \times Y_S \rightarrow Y_S \) from \( \text{id}_Y = H|_{(0) \times Y_S} \) to \( r = H|_{(1) \times Y_S} \), such that \( f \circ H \) is the constant homotopy from \( f|_{Y_S} \) to itself.

Choose an anodyne map of simplicial sets \( i : A \hookrightarrow B \). We wish to show that every lifting problem of the form

\[
\begin{array}{ccc}
A & \xrightarrow{g_0} & Y' \\
\downarrow^i & & \downarrow^f|_{Y'} \\
B & \xrightarrow{g} & T
\end{array}
\]

admits a solution. Since \( f \) is a Kan fibration, we can choose a map \( g' : B \rightarrow Y \) satisfying \( g'|_A = g_0 \) and \( f \circ g = g \). Let \( B_S \subseteq B \) denote the simplicial subset given by the fiber product \( S \times T B \), and let \( g_1 : (A \cup B_S) \rightarrow Y \) be the map of simplicial sets characterized by \( g_1|_A = g_0 \) and \( g_1|_{B_S} = r \circ g'|_{B_S} \) (this map is well-defined, since \( r \circ g' \) and \( g_0 \) agree on the intersection \( A \cap B_S \)). Note that \( H \) induces a homotopy \( h_0 : \Delta^1 \times (A \cup B_S) \rightarrow Y \) from \( g'|_{A \cup B_S} \) to
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\[ g_1 \] compatible with the projection to \( S \). Since \( f \) is a Kan fibration, we can lift \( h_0 \) to a homotopy \( h : \Delta^1 \times B \to Y \) from \( g' \) to some map \( g : B \to Y \), compatible with the projection to \( S \) (Remark 3.1.4.3). It follows from the construction that \( g \) takes values in the simplicial subset \( Y' \subseteq Y \) and satisfies the requirements \( g|_A = g_0 \) and \( f \circ g = g_0 \).

**Proof of Theorem 3.3.8.1.** Let \( f : X \to S \) be a Kan fibration of simplicial sets. Let us abuse notation by identifying \( X \) and \( S \) with simplicial subsets of \( Y = \text{Ex}^\infty(X) \) and \( T = \text{Ex}^\infty(S) \), respectively (via the monomorphisms \( \rho_X^\infty : X \to \text{Ex}^\infty(X) \) and \( \rho_S^\infty : S \to \text{Ex}^\infty(S) \)), and let \( Y' \subseteq \text{Ex}^\infty(X) \) be the simplicial subset defined in the statement of Lemma 3.3.8.4. Let \( g : S \to S' \) be an anodyne morphism of simplicial sets. Since \( \text{Ex}^\infty(S) \) is a Kan complex (Proposition 3.3.6.9), the morphism \( \rho_S^\infty : S \to \text{Ex}^\infty(S) \) extends to a map of simplicial sets \( u : S' \to \text{Ex}^\infty(S) \). Set \( X' = S' \times_{\text{Ex}^\infty(S)} Y' \), so that we have a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow f & & \downarrow f' \\
S & \longrightarrow & S' \\
\downarrow g & & \downarrow u \\
& & \text{Ex}^\infty(S)
\end{array}
\]

where the right square and outer rectangle are pullback diagrams, so the left square is a pullback diagram as well. Since the projection map \( Y' \to \text{Ex}^\infty(S) \) is a Kan fibration (Lemma 3.3.8.4), it follows that \( f' \) is also a Kan fibration. \( \square \)

3.3.9 Towers of Kan Fibrations

We now study the behavior of Kan complexes with respect to inverse limits.

**Proposition 3.3.9.1.** Let \( X \) be the inverse limit of a tower of simplicial sets

\[
\cdots \xrightarrow{q_3} X(3) \xrightarrow{q_2} X(2) \xrightarrow{q_1} X(1) \xrightarrow{q_0} X(0).
\]

If each of the morphisms \( q_n \) is a Kan fibration, then the canonical map \( q : X \to X(0) \) is a Kan fibration.

**Proof.** This is a special case of Proposition 1.4.4.14 (which guarantees that the collection of all Kan fibrations is closed under transfinite composition in the opposite of the category of simplicial sets). \( \square \)

**Corollary 3.3.9.2.** Suppose we are given a tower of simplicial sets

\[
\cdots \xrightarrow{q_3} X(3) \xrightarrow{q_2} X(2) \xrightarrow{q_1} X(1) \xrightarrow{q_0} X(0).
\]

If \( X(0) \) is a Kan complex and each \( q_n \) is a Kan fibration, then the inverse limit \( X = \lim_{\leftarrow n \geq 0} X(n) \) is a Kan complex.
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Proof. Combine Proposition 3.3.9.1 with Remark 3.1.1.11.

Variant 3.3.9.3. Let $X$ be the inverse limit of a tower of simplicial sets

$$\cdots \xrightarrow{q_3} X(3) \xrightarrow{q_2} X(2) \xrightarrow{q_1} X(1) \xrightarrow{q_0} X(0).$$

If each of the morphisms $q_n$ is a trivial Kan fibration, then the canonical map $q : X \to X(0)$ is a trivial Kan fibration.

Proof. By virtue of Proposition 1.4.4.14, the collection of trivial Kan fibrations is closed under transfinite composition in the opposite of the category of simplicial sets.

Corollary 3.3.9.4. Let $X$ be the inverse limit of a tower of simplicial sets

$$\cdots \xrightarrow{q_3} X(3) \xrightarrow{q_2} X(2) \xrightarrow{q_1} X(1) \xrightarrow{q_0} X(0).$$

Assume that each of the simplicial sets $X(n)$ is weakly contractible and that each of the morphisms $q_n$ is a Kan fibration. Then the inverse limit $X = \varprojlim_{n \geq 0} X(n)$ is weakly contractible.

Proof. If each of these simplicial sets $X(n)$ is weakly contractible, then each of the morphisms $q_n : X(n+1) \to X(n)$ is a weak homotopy equivalence. Applying Proposition 3.3.7.4, we conclude that each $q_n$ is a trivial Kan fibration. Variant 3.3.9.3 then guarantees that the projection map $X \to X(0)$ is also a trivial Kan fibration. Since $X(0)$ is weakly contractible, it follows that $X$ is also weakly contractible.

Corollary 3.3.9.5. Let $X$ be the inverse limit of a tower of simplicial sets

$$\cdots \xrightarrow{q_3} X(3) \xrightarrow{q_2} X(2) \xrightarrow{q_1} X(1) \xrightarrow{q_0} X(0).$$

Assume that each of the simplicial sets $X(n)$ is a contractible Kan complex and that each of the morphisms $q_n$ is a Kan fibration. Then the inverse limit $X = \varprojlim_{n \geq 0} X(n)$ is a contractible Kan complex.

Proof. Combine Corollaries 3.3.9.2 and 3.3.9.4.

Proposition 3.3.9.6. Suppose we are given a commutative diagram of Kan complexes

$$\cdots \xrightarrow{p_3} X(3) \xrightarrow{p_2} X(2) \xrightarrow{p_1} X(1) \xrightarrow{p_0} X(0)$$

$$\xrightarrow{f_3} \cdots \xrightarrow{q_3} Y(3) \xrightarrow{q_2} Y(2) \xrightarrow{q_1} Y(1) \xrightarrow{q_0} Y(0).$$

Assume that for each $n \geq 0$, the morphisms $p_n$ and $q_n$ are Kan fibrations, and the morphism $f_n$ is a homotopy equivalence. Then the induced map $f : \varprojlim_{n \geq 0} X(n) \to \varprojlim_{n \geq 0} Y(n)$ is a homotopy equivalence of Kan complexes.
Proof. Set \( X = \lim_{\longleftarrow, n \geq 0} X(n) \) and \( Y = \lim_{\longleftarrow, n \geq 0} Y(n) \), so that \( X \) and \( Y \) are Kan complexes by virtue of Corollary 3.3.9.2. Let \( Q(f) = X \times_{\text{Fun}(\{0\}, Y)} \text{Fun}(\Delta^1, Y) \) be as in Example 3.1.6.10 so that \( f \) factors as a composition \( X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y \) where \( f' \) is a homotopy equivalence and \( f'' \) is a Kan fibration. We will complete the proof by showing that \( f'' \) is a trivial Kan fibration. By virtue of Proposition 3.3.7.4, it will suffice to show that for each vertex \( y \in Y \), the fiber \( Q(f)_y = \{ y \} \times_Y Q(f) \) is contractible.

For each \( n \geq 0 \), define \( Q(f)_n = X(n) \times_{\text{Fun}(\{0\}, Y(n))} \text{Fun}(\Delta^1, Y(n)) \) as in Example 3.1.6.10, so that \( f_n \) factors as a composition
\[
X(n) \xrightarrow{f'_n} Q(f)_n \xrightarrow{f''_n} Y(n),
\]
where \( f'_n \) is a homotopy equivalence and \( f''_n \) is a Kan fibration. Since \( f_n \) is also a homotopy equivalence, the morphism \( f''_n \) is a homotopy equivalence, and therefore a trivial Kan fibration (Proposition 3.3.7.4). Let \( Q(f)_n = \{ y \} \times_{Y(n)} \text{Fun}(\Delta^1, Y(n)) \) denote the fiber of \( f''_n \) over the image of \( y \), so that \( Q(f)_n \) is a contractible Kan complex. The simplicial set \( Q(f)_y \) can then be realized as the inverse limit of the tower
\[
\cdots \xrightarrow{r_3} Q(f)_3 \xrightarrow{r_2} Q(f)_2 \xrightarrow{r_1} Q(f)_1 \xrightarrow{r_0} Q(f)_0. 
\]
By virtue of Corollary 3.3.9.5, it will suffice to show that each of the morphisms \( r_n : Q(f_{n+1})_y \to Q(f_n)_y \) is a Kan fibration. To prove this, we observe that \( r_n \) factors as a composition
\[
Q(f_{n+1})_y = (X(n+1) \times \{ y \}) \times_{\text{Fun}(\partial \Delta^1, Y(n+1))} \text{Fun}(\Delta^1, Y(n+1))
\]
\[
\xrightarrow{r'} (X(n+1) \times \{ y \}) \times_{\text{Fun}(\partial \Delta^1, Y(n))} \text{Fun}(\Delta^1, Y(n))
\]
\[
\xrightarrow{r''} (X(n) \times \{ y \}) \times_{\text{Fun}(\partial \Delta^1, Y(n))} \text{Fun}(\Delta^1, Y(n))
\]
\[
= Q(f_n)_y.
\]
The morphism \( r'' \) is a pullback of \( p_n \), and therefore a Kan fibration. Similarly, the morphism \( r' \) is a pullback of the restriction map
\[
\text{Fun}(\Delta^1, Y(n+1)) \to \text{Fun}(\partial \Delta^1, Y(n+1)) \times_{\text{Fun}(\partial \Delta^1, Y(n))} \text{Fun}(\Delta^1, Y(n)),
\]
and is therefore a Kan fibration by virtue of Theorem 3.1.3.1. \( \square \)

### 3.4 Homotopy Pullback and Homotopy Pushout Squares

Recall that the category of simplicial sets admits arbitrary limits and colimits (Remark 1.1.1.13). In particular, given diagrams of simplicial sets
\[
T \to S \leftarrow X \quad \quad B \leftarrow A \to C,
\]
we can form the pullback $T \times_S X$ and the pushout $B \coprod_A C$. Beware that, in general, neither of these constructions is invariant under weak homotopy equivalence.

**Warning 3.4.0.1.** Suppose we are given a commutative diagram of of simplicial sets

$$
\begin{array}{ccc}
T & \rightarrow & S \\
\downarrow & & \downarrow \\
T' & \rightarrow & S'
\end{array}
\quad
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow & & \downarrow \\
\quad & & 
\end{array}
$$

for which the vertical maps are weak homotopy equivalences. Then the induced map $T \times_S X \rightarrow T' \times_S X'$ need not be a weak homotopy equivalence. For example, the pullback of the upper half of the diagram

$$
\begin{array}{ccc}
\{0\} & \rightarrow & \Delta^1 \\
\downarrow & & \downarrow \\
\{0\} & \sim & \Delta^0
\end{array}
\quad
\begin{array}{ccc}
\{1\} & \leftarrow & \{1\} \\
\downarrow & & \downarrow \\
\quad & & 
\end{array}
$$

is empty, while the pullback of the lower half is isomorphic to $\Delta^0$.

**Warning 3.4.0.2.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
C & \leftarrow & A \\
\downarrow & & \downarrow \\
C' & \leftarrow & A'
\end{array}
\quad
\begin{array}{ccc}
B & \rightarrow & B' \\
\downarrow & & \downarrow \\
\quad & & 
\end{array}
$$

in which the vertical maps are weak homotopy equivalences. Then the induced map $C \coprod_A B \rightarrow C' \coprod_A B'$ need not be a weak homotopy equivalence. For example, the pushout of the upper half of the diagram

$$
\begin{array}{ccc}
\Delta^1 & \leftarrow & \partial \Delta^1 \\
\downarrow & & \downarrow \\
\Delta^0 & \leftarrow & \partial \Delta^1
\end{array}
\quad
\begin{array}{ccc}
\Delta^1 & \rightarrow & \Delta^1 \\
\downarrow & & \downarrow \\
\Delta^0 & \rightarrow & \Delta^0
\end{array}
$$

is not weakly contractible (it has nontrivial homology in degree 1), but the pushout of the lower half is isomorphic to $\Delta^0$.

One can avoid the troublesome phenomena of Warnings 3.4.0.1 and 3.4.0.2 by restricting attention to diagrams which satisfy some additional conditions.
Proposition 3.4.0.3. Suppose we are given a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
T' & \xrightarrow{f'} & S', \\
\end{array}
\]
where the vertical maps are weak homotopy equivalences. If \( f \) and \( f' \) are Kan fibrations, then the induced map \( T \times_S X \to T' \times_{S'} X' \) is a weak homotopy equivalence.

Proof. We have a commutative diagram
\[
\begin{array}{ccc}
T \times_S X & \longrightarrow & T' \times_{S'} X' \\
\downarrow & & \downarrow \\
T & \longrightarrow & T', \\
\end{array}
\]
where the vertical maps are Kan fibrations (since they are pullbacks of \( f \) and \( f' \), respectively). By virtue of Proposition 3.3.7.1, it will suffice to show that for each vertex \( t \in T \) having image \( t' \in T' \), the induced map of fibers
\[
(T \times_S X)_t \simeq \{t\} \times_S X \to \{t'\} \times_{S'} X' = (T' \times_{S'} X')_{t'},
\]
is a homotopy equivalence of Kan complexes. This follows from the criterion of Proposition 3.3.7.1 applied to the diagram
\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow_{f} & & \downarrow_{f'} \\
S & \longrightarrow & S', \\
\end{array}
\]
\( \square \)

Proposition 3.4.0.4. Suppose we are given a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
C & \xleftarrow{i} & A \\
\downarrow & & \downarrow \\
C' & \xleftarrow{i'} & A', \\
\end{array}
\]
where the vertical maps are weak homotopy equivalences. If \( i \) and \( i' \) are monomorphisms, then the induced map \( \theta : C \coprod_A B \to C' \coprod_{A'} B' \) is also a weak homotopy equivalence.

Proof. Let \( X \) be a Kan complex; we will show that composition with \( \theta \) induces a homotopy equivalence of Kan complexes \( \text{Fun}(C' \coprod_{A'} B', X) \to \text{Fun}(C \coprod_A B, X) \) (and therefore a
bijection on connected components). This follows by applying Proposition 3.4.0.3 to the diagram

\[
\begin{array}{ccc}
\text{Fun}(C', X) & 
\longrightarrow & 
\text{Fun}(A', X) \\
\downarrow & & \downarrow \\
\text{Fun}(C, X) & 
\longrightarrow & 
\text{Fun}(A, X) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Fun}(B', X) & 
\longrightarrow & 
\text{Fun}(B, X) \\
\downarrow & & \downarrow \\
\text{Fun}(D', X) & 
\longrightarrow & 
\text{Fun}(D, X) \\
\end{array}
\]

note that the vertical maps in this diagram are weak homotopy equivalences by virtue of Corollary 3.1.6.5, and the right horizontal maps are Kan fibrations by virtue of Corollary 3.1.3.3

We can summarize Proposition 3.4.0.3 and Warning 3.4.0.1 more informally as asserting that the pullback construction

\[
(T \to S \xleftarrow{f} X) \mapsto T \times_{S} X
\]

has good behavior when \( f \) is a Kan fibration of simplicial sets, but not in general. Fortunately, the assumption that \( f \) is a Kan fibration is easy to arrange, using the fibrant replacement functor of Proposition 3.1.6.1.

**Construction 3.4.0.5 (The Homotopy Fiber Product).** Suppose we are given a diagram of simplicial sets \( T \to S \xleftarrow{f} X \). By virtue of Proposition 3.1.6.1, the morphism \( f \) factors as a composition

\[
X \xrightarrow{w} X' \xrightarrow{f'} S,
\]

where \( f' \) is a Kan fibration and \( w \) is a weak homotopy equivalence. We will refer to the fiber product \( T \times_{S} X' \) as a *homotopy fiber product* of \( T \) with \( X \) (relative to \( S \)), and denote it by \( T \times_{h}^{S} X \).

**Exercise 3.4.0.6** (Invariance of the Homotopy Fiber Product). Let \( T \to S \xleftarrow{f} X \) be a diagram of simplicial sets. Show that, up to weak homotopy equivalence, the homotopy fiber product \( T \times_{h}^{S} X \) does not depend on the factorization of \( f \) chosen in Construction 3.4.0.5. In other words, if \( f \) admits a pair of factorizations

\[
X \xrightarrow{w} X' \xrightarrow{f'} S \quad X \xrightarrow{w''} X'' \xrightarrow{f''} S,
\]

where both \( w' \) and \( w'' \) are weak homotopy equivalences and both \( f' \) and \( f'' \) are Kan fibrations, then the simplicial sets \( T \times_{S} X' \) and \( T \times_{S} X'' \) have the same weak homotopy type (hint: reduce to the case where \( w' \) is anodyne, and use Proposition 3.4.0.3). See Proposition 3.4.1.2 for a related statement.

**Exercise 3.4.0.7** (Symmetry). Let \( T \to S \xleftarrow{f} X \) be a diagram of simplicial sets. Show that the homotopy fiber products \( T \times_{h}^{S} X \) and \( X \times_{h}^{S} T \) have the same weak homotopy type (see Proposition 3.4.1.7 for a related statement).
Example 3.4.0.8 (Path Spaces). Let $S$ be a Kan complex containing a pair of vertices $s, t \in S$, and let

$$P_{s,t} = \{s\} \times_{\text{Fun}(\{0\}, S)} \text{Fun}(\Delta^1, S) \times_{\text{Fun}(\{1\}, S)} \{t\}$$

denote the Kan complex parametrizing paths in $S$ from $s$ to $t$ (so that the vertices of $P_{s,t}$ can be identified with edges of $S$ which originate at the vertex $s$ and terminate at the vertex $t$). Then $P_{s,t}$ can be identified with the fiber product $\{s\} \times_S \text{Fun}(\Delta^1, S)$, where $Q = \text{Fun}(\Delta^1, S) \times_{\text{Fun}(\{1\}, S)} \{t\}$ is a contractible Kan complex and the evaluation map $Q \to S$ is a Kan fibration (see Example 3.1.6.10). It follows that the Kan complex $P_{s,t}$ is a homotopy fiber product $\{s\} \times_{S} \{t\}$.

Remark 3.4.0.9 (Comparison with the Fiber Product). In the situation of Construction 3.4.0.5, there is always a map from the usual fiber product $T \times_S X$ to the homotopy fiber product $T \times_{h^S} X$. This map is a weak homotopy equivalence when $f : X \to S$ is a Kan fibration, but need not be a homotopy equivalence in general. For example, if $S$ is a Kan complex containing a pair of vertices $s, t \in S$, then the fiber product $\{s\} \times_S \{t\}$ is either empty (if $s \neq t$) or isomorphic to $\Delta^0$ (if $s = t$), but the homotopy fiber product $\{s\} \times_{h^S} \{t\}$ can be identified with the path space $P_{s,t}$ of Example 3.4.0.8 (which is potentially a much more useful invariant).

An awkward feature of Construction 3.4.0.5 is that, although the homotopy fiber product $T \times_{h^S} X$ is well-defined up to weak homotopy equivalence (Exercise 3.4.0.6), its isomorphism class depends on an auxiliary choice (namely, a factorization of the map $f : X \to S$ as a composition $j' \circ w$, where $j'$ is a Kan fibration and $w$ a weak homotopy equivalence). We will address this point in §3.4.1 by adopting a different perspective. Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
T & \longrightarrow & S \\
\end{array}
$$

We will say that (3.1) is a homotopy pullback square (or a homotopy Cartesian square) if the composite map

$$Y \to T \times_S X \to T \times_{h^S} X$$

is a weak homotopy equivalence (Definition 3.4.1.1). It is not hard to see that this condition does not depend on the factorization chosen in the construction of $T \times_{h^S} X$ (see Proposition 3.4.1.2 which is essentially a more precise version of Exercise 3.4.0.6).

Construction 3.4.0.5 has a counterpart for pushout squares. Given a diagram of simplicial sets $C \leftarrow A \overset{i}{\rightarrow} B$, we can always factor $i$ as a composition $A \overset{i'}{\rightarrow} B' \overset{w}{\rightarrow} B$, where $i'$ is a monomorphism and $w$ is a weak homotopy equivalence (Exercise 3.1.6.11). In this case, we denote the pushout $C \amalg_A B'$ by $C \amalg^h_A B$ and refer to it as the homotopy pushout of $C$. 
and $B$ along $A$. Using Proposition 3.4.0.4, it is not difficult to see that the weak homotopy type of $C \coprod_A B$ is independent of the choice of factorization $i = w \circ i'$. However, the isomorphism class of $C \coprod_A B$ does depend on this choice, which makes the construction somewhat cumbersome to work with. We remedy this point in §3.4.2 by introducing a collection of commutative diagrams

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
$$

which we will refer to as homotopy pushout squares or homotopy coCartesian diagrams (Definition 3.4.2.1), which are characterized by the requirement that the composite map $C \coprod_A B \rightarrow C \coprod_A B \rightarrow D$ is a weak homotopy equivalence (Remark 3.4.2.8).

The theory of homotopy pullback and homotopy pushout diagrams was introduced by Mather (in the setting of topological spaces, rather than simplicial sets). and have subsequently proven to be a very useful tool in algebraic topology. In [29], Mather established two fundamental results relating homotopy Cartesian and coCartesian diagrams, which are now known as the Mather cube theorems:

- Suppose we are given a homotopy pushout square of simplicial sets

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
$$

If $\overline{D} \rightarrow D$ is a Kan fibration, then the induced diagram

$$
\begin{array}{ccc}
A \times_D \overline{D} & \rightarrow & B \times_D \overline{D} \\
\downarrow & & \downarrow \\
C \times_D \overline{D} & \rightarrow & D
\end{array}
$$

is also a homotopy pushout square (Proposition 3.4.3.2). Stated more informally, the collection of homotopy pushout squares is stable under pullback by Kan fibrations. In §3.4.3 we establish a slightly more general (and homotopy invariant) version of this statement, which is known as Mather’s second cube theorem (Theorem 3.4.3.3).

- Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
C & \leftarrow & A & \rightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
C & \leftarrow & A & \rightarrow & B
\end{array}
$$
in which both squares are homotopy Cartesian. If \( i \) and \( \bar{i} \) are monomorphisms, then both squares in the induced diagram

\[
\begin{array}{ccc}
C & \longrightarrow & C \coprod_A B \\
\downarrow & & \downarrow \\
C & \longrightarrow & C \coprod_A B
\end{array}
\]

are also homotopy Cartesian (Proposition \ref{3.4.4.3}). In \( \S \) \ref{3.4.4} we establish a slightly more general (and homotopy invariant) version of this statement, which is known as Mather’s first cube theorem (Theorem \ref{3.4.4.4}).

The homotopy theory of topological spaces provides a rich supply of examples of homotopy pushout squares. Let \( X \) be a topological space which can be written as the union of two open subsets \( U, V \subseteq X \). In \( \S \) \ref{3.4.6} we show that the resulting diagram of singular simplicial sets

\[
\begin{array}{ccc}
\operatorname{Sing}_\bullet(U \cap V) & \longrightarrow & \operatorname{Sing}_\bullet(U) \\
\downarrow & & \downarrow \\
\operatorname{Sing}_\bullet(V) & \longrightarrow & \operatorname{Sing}_\bullet(X)
\end{array}
\]

is homotopy coCartesian (Theorem \ref{3.4.6.1}). To carry out the proof, we make use of the fact that the weak homotopy type of a simplicial set \( X \) can be recovered from its underlying semisimplicial set (see Proposition \ref{3.4.5.4} and Corollary \ref{3.4.5.5}), which we explain in \( \S \) \ref{3.4.5}). We conclude in \( \S \) \ref{3.4.7} by applying Theorem \ref{3.4.6.1} to deduce the classical Seifert-van Kampen theorem (Theorem \ref{3.4.7.1}) and the excision theorem for singular homology (Theorem \ref{3.4.7.3}).

**Remark 3.4.0.10.** The notions of homotopy pullback and homotopy pushout diagrams can be regarded as homotopy-invariant replacements for the usual notion of pullback and pushout diagrams, respectively. We will later make this heuristic precise by showing that a commutative diagram in the ordinary category of Kan complexes

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

is a homotopy pullback square (homotopy pushout square) if and only if it is a pullback square (pushout square) when regarded as a diagram in the \( \infty \)-category \( \mathcal{S} \) of Kan complexes (Construction \ref{3.1.4.12}); see Proposition [\?].
3.4.1 Homotopy Pullback Squares

We begin by reformulating Construction \(3.4.0.5\).

**Definition 3.4.1.1.** A diagram of simplicial sets

\[
\begin{align*}
Y & \longrightarrow X \\
\downarrow & \downarrow f \\
T & \longrightarrow S.
\end{align*}
\]

is homotopy Cartesian if there exists a factorization \(f = f' \circ w\) where \(f' : X' \to S\) is a Kan fibration, \(w : X \to X'\) is a weak homotopy equivalence, and the induced map \(Y \to T \times_S X'\) is a weak homotopy equivalence. In this case, we will also say that \(3.3\) is a homotopy pullback square.

Stated more informally, the diagram \(3.3\) is homotopy Cartesian if it exhibits \(Y\) as (weakly homotopy equivalent to) the homotopy fiber product \(T \times^h_S X\) of Construction \(3.4.0.5\).

This condition is actually independent of the choices involved in the construction of the homotopy fiber product, by virtue of the following:

**Proposition 3.4.1.2.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{align*}
Y & \longrightarrow X \\
\downarrow & \downarrow f \\
T & \longrightarrow S.
\end{align*}
\]

The following conditions are equivalent:

1. Diagram \(3.4\) is homotopy Cartesian, in the sense of Definition \(3.4.1.1\). That is, there exists a factorization \(X \xrightarrow{w} X' \xrightarrow{f'} X\) of the morphism \(f\) where \(f'\) is a Kan fibration, \(w\) is a weak homotopy equivalence, and the induced map \(Y \to T \times_S X'\) is a weak homotopy equivalence.

2. For every factorization \(X \xrightarrow{w} X' \xrightarrow{f'} X\) of the morphism \(f\), if \(f'\) is a Kan fibration and \(w\) is a weak homotopy equivalence, then the induced map \(Y \to T \times_S X'\) is also a weak homotopy equivalence.

**Proof.** The implication \(2 \Rightarrow 1\) follows from Proposition \(3.1.6.1\). To prove the converse, suppose that the morphism \(f\) admits two different factorizations

\[
X \xrightarrow{w'} X' \xrightarrow{f'} S \quad X \xrightarrow{w''} X'' \xrightarrow{f''} S,
\]
where both \( w' \) and \( w'' \) are weak homotopy equivalences and both \( f' \) and \( f'' \) are Kan fibrations. We wish to show that the induced map \( \rho' : Y \to T \times_S X' \) is a weak homotopy equivalence if and only if the induced map \( \rho'' : Y \to T \times_S X'' \) is a weak homotopy equivalence. To establish this equivalence, we may assume without loss of generality that \( w' \) is anodyne (since this can always be arranged using Proposition 3.1.6.1). In this case, the lifting problem

\[
\begin{array}{ccc}
X & \xrightarrow{w''} & X'' \\
\downarrow w' & & \downarrow f'' \\
X' & \xrightarrow{u} & S \\
\end{array}
\]

admits a solution \( u : X' \to X'' \) (Remark 3.1.2.6). Since \( w' \) and \( w'' \) are weak homotopy equivalences, the equality \( w'' = u \circ w' \) guarantees that \( u \) is also a weak homotopy equivalence (Remark 3.1.5.16), so that the map \( T \times_S X' \to T \times_S X'' \) is a weak homotopy equivalence by virtue of Proposition 3.4.0.3.

**Corollary 3.4.1.3.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \\
T & \xrightarrow{g} & S \\
\end{array}
\]  

where \( f \) is a weak homotopy equivalence. Then (3.5) is homotopy Cartesian if and only if \( g \) is a weak homotopy equivalence.

**Example 3.4.1.4.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \\
T & \xrightarrow{g} & S \\
\end{array}
\]  

where \( f \) and \( g \) are Kan fibrations. Then (3.6) is homotopy Cartesian if and only if, for each vertex \( t \in T \) having image \( s \in S \), the induced map \( Y_t \to X_s \) is a weak homotopy equivalence. This is essentially a reformulation of Proposition 3.3.7.1 (by virtue of Proposition 3.4.1.2).

**Example 3.4.1.5.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \\
T & \xrightarrow{g} & S \\
\end{array}
\]  

(3.7)
where \( f \) is a Kan fibration. Then (3.7) is homotopy Cartesian if and only if the induced map \( Y \to T \times_S X \) is a weak homotopy equivalence. In particular, if (3.7) is a pullback diagram, then it is also a homotopy pullback diagram. Beware that this conclusion is generally false if \( f \) is not a Kan fibration.

**Warning 3.4.1.6.** For a general pair of morphisms \( f : X \to S, \, u : T \to S \) in the category of simplicial sets, there need not exist a homotopy Cartesian diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow f \\
T & \longrightarrow & S.
\end{array}
\]

For example, if \( f : \{0\} \hookrightarrow \Delta^1 \) and \( u : \{1\} \hookrightarrow \Delta^1 \) are the inclusion maps, then the existence of a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & \{0\} \\
\downarrow & & \downarrow f \\
\{1\} & \longrightarrow & \Delta^1
\end{array}
\]  

(3.8) guarantees that the simplicial set \( Y \) is empty, in which case (3.8) is not a homotopy pullback square.

Note that Definition 3.4.1.1 is *a priori* asymmetric: it involves replacing the map \( f : X \to S \) by a Kan fibration, but leaving the map \( T \to S \) unchanged. However, this turns out to be irrelevant.

**Proposition 3.4.1.7** (Symmetry). A commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow f \\
T & \longrightarrow & S
\end{array}
\]

is homotopy Cartesian if and only if the transposed diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & T \\
\downarrow & & \downarrow g \\
X & \longrightarrow & S
\end{array}
\]

is homotopy Cartesian.

**Proof.** Using Proposition 3.1.6.1, we can choose factorizations

\[
\begin{align*}
X \xrightarrow{u_X} X' \xrightarrow{f'} S & \quad \quad T \xrightarrow{w_T} T' \xrightarrow{g'} S
\end{align*}
\]
of $f$ and $g$, where both $f'$ and $g'$ are Kan fibrations and both $w_X$ and $w_T$ are weak homotopy equivalences. We can then form a commutative diagram of simplicial sets

$$
\begin{array}{c}
Y \xrightarrow{u} T' \times_S X \xrightarrow{w_X} X \\
\downarrow v \quad \quad \downarrow v' \quad \quad \downarrow w_X \\
T \times_S X' \xrightarrow{w_T} T' \times_S X' \xrightarrow{g'} S \\
\downarrow \quad \downarrow \quad \downarrow \\
T \xrightarrow{w_T} T' \xrightarrow{g'} S.
\end{array}
$$

We wish to show that $u$ is a weak homotopy equivalence if and only if $v$ is a weak homotopy equivalence (see Proposition 3.4.1.2). This follows from the two-out-of-three property (Remark 3.1.5.16), since the morphisms $u'$ and $v'$ are weak homotopy equivalences by virtue of Corollary 3.3.7.2.

**Remark 3.4.1.8.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{c}
Y \xrightarrow{w_Y} Y' \xrightarrow{g'} T' \times_S X' \\
\downarrow \downarrow \downarrow \\
T \xrightarrow{w_T} T' \xrightarrow{g'} S,
\end{array}
$$

where $w_Y$ and $w_X$ are weak homotopy equivalences. Then the lower square is homotopy Cartesian if and only if the outer rectangle is homotopy Cartesian (see Corollary 3.4.1.10 for a slight generalization).

**Proposition 3.4.1.9 (Transitivity).** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{c}
Z \xrightarrow{Z} Y \xrightarrow{f} X \\
\downarrow h \quad \downarrow g \\
U \xrightarrow{h} T \xrightarrow{f} S
\end{array}
$$

where the square on the right is homotopy Cartesian. Then the left square is homotopy Cartesian if and only if the outer rectangle is a homotopy Cartesian.

**Proof.** By virtue of Proposition 3.1.6.1, the morphism $f$ factors as a composition $X \xrightarrow{w_X} X' \xrightarrow{f'} S$, where $f'$ is a Kan fibration and $w_X$ is a weak homotopy equivalence. Using Proposition 3.1.6.1 again, we can factor the induced map $Y \rightarrow T \times_S X'$ as a composition $Y \xrightarrow{w_Y} Y' \xrightarrow{\overline{g}} T \times_S X'$, where $\overline{g}$ is a Kan fibration and $w_Y$ is a weak homotopy equivalence.
Repeating this argument, we can factor the induced map $Z \to U \times_T Y'$ as a composition $Z \xrightarrow{w_Z} Z' \xrightarrow{\overline{h}} U \times_T Y'$, where $\overline{h}$ is a Kan fibration and $w_Z$ is a weak homotopy equivalence. We then obtain a commutative diagram

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow \scriptstyle{w_Z} & & \downarrow \scriptstyle{w_Y} \\
Z' & \rightarrow & Y'
\end{array}
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \scriptstyle{w_X} \\
& & Y' \\
\downarrow \scriptstyle{b'} & & \downarrow \scriptstyle{g'} \\
U & \rightarrow & T
\end{array}
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \scriptstyle{f'} \\
& & X' \\
\downarrow \scriptstyle{h'} & & \downarrow \scriptstyle{f} \\
Y' & \rightarrow & X'
\end{array}
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \scriptstyle{f} \\
& & X' \\
\downarrow \scriptstyle{S} & & \downarrow \scriptstyle{S} \\
T' & \rightarrow & S'
\end{array}
\]

where the upper vertical maps are weak homotopy equivalences and the lower vertical maps are Kan fibrations. It follows from our assumption (and Remark 3.4.1.8) that the lower right square in this diagram is homotopy Cartesian. To complete the proof, it will suffice (again using Remark 3.4.1.8) to show that the lower left square is homotopy Cartesian if and only if the lower rectangle is homotopy Cartesian. By virtue of the criterion of Example 3.4.1.4, we are reduced to showing that for each vertex $u \in U$ having images $t \in T$ and $s \in S$, respectively, the induced map $Z'_u \to Y'_t$ is a homotopy equivalence of Kan complexes if and only if the composite map $Z'_u \to Y'_t \to X'_s$ is a homotopy equivalence of Kan complexes. This follows from the two-out-of-three property (Remark 3.1.5.7), since the map of fibers $Y'_t \to X'_s$ is a weak homotopy equivalence (by the criterion of Example 3.4.1.4).

\[\square\]

**Corollary 3.4.1.10 (Homotopy Invariance).** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow \scriptstyle{w_Y} & & \downarrow \scriptstyle{w_X} \\
Y' & \rightarrow & X'
\end{array}
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \scriptstyle{w_T} \\
& & T' \\
\downarrow \scriptstyle{w_T} & & \downarrow \scriptstyle{w_S} \\
T & \rightarrow & S
\end{array}
\]

where the morphisms $w_X$, $w_T$, and $w_S$ are weak homotopy equivalences. Then any two of the following conditions imply the third:

1. The commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow \scriptstyle{w_Y} & & \downarrow \scriptstyle{w_X} \\
T & \rightarrow & S
\end{array}
\]

Corollary 3.4.1.10 (Homotopy Invariance). Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow \scriptstyle{w_Y} & & \downarrow \scriptstyle{w_X} \\
Y' & \rightarrow & X'
\end{array}
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \scriptstyle{w_T} \\
& & T' \\
\downarrow \scriptstyle{w_T} & & \downarrow \scriptstyle{w_S} \\
T & \rightarrow & S
\end{array}
\]

where the morphisms $w_X$, $w_T$, and $w_S$ are weak homotopy equivalences. Then any two of the following conditions imply the third:

1. The commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow \scriptstyle{w_Y} & & \downarrow \scriptstyle{w_X} \\
T & \rightarrow & S
\end{array}
\]
is homotopy Cartesian.

(2) The commutative diagram

\[
\begin{array}{ccc}
Y' & \rightarrow & X' \\
\downarrow & & \downarrow \\
T' & \rightarrow & S'
\end{array}
\]

is homotopy Cartesian.

(3) The morphism \(w_Y\) is a weak homotopy equivalence.

Proof. By virtue of Corollary 3.4.1.3, the bottom square in the commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
T & \rightarrow & S \\
\downarrow^{w_T} & & \downarrow^{w_S} \\
T' & \rightarrow & S'
\end{array}
\]

is homotopy Cartesian. Applying Propositions 3.4.1.9 and 3.4.1.7, we see that (1) is equivalent to the following:

\((1')\) The diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
T' & \rightarrow & S'
\end{array}
\]

is homotopy Cartesian.

If condition (3) is satisfied, then the equivalence \((1') \Leftrightarrow (2)\) is a special case of Remark 3.4.1.8. Conversely, if \((1')\) and \((2)\) are satisfied, then Propositions 3.4.1.9 and 3.4.1.7 guarantee that the upper square in the commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow^{w_Y} & & \downarrow^{w_X} \\
Y' & \rightarrow & X \\
\downarrow & & \downarrow \\
T' & \rightarrow & S'
\end{array}
\]

is homotopy Cartesian, so that \(w_Y\) is a weak homotopy equivalence by virtue of Corollary 3.4.1.3. \(\square\)
Proposition 3.4.1.11 (Summands). Suppose we are given a homotopy Cartesian diagram of simplicial sets

\[
\begin{array}{ccc}
Y & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
T & \xrightarrow{v} & S.
\end{array}
\]

Let \(X' \subseteq X\), \(T' \subseteq T\), and \(S' \subseteq S\) be summands satisfying \(f(X') \subseteq S' \supseteq v(T')\), and set \(Y' = g^{-1}(T') \cap u^{-1}(X') \subseteq Y\). Then the diagram of simplicial sets

\[
\begin{array}{ccc}
Y' & \xrightarrow{u} & X' \\
\downarrow{g} & & \downarrow{f} \\
T' & \xrightarrow{v} & S'
\end{array}
\]

is also homotopy Cartesian.

Proof. Consider the diagram of simplicial sets

\[
\begin{array}{ccc}
g^{-1}(T') & \rightarrow & Y \rightarrow X \\
\downarrow & & \downarrow \\
T' & \rightarrow & T \rightarrow S.
\end{array}
\]

The square on the left is a pullback diagram whose horizontal maps are Kan fibrations (Example 3.1.1.4), and is therefore homotopy Cartesian (Example 3.4.1.5). The square on the right is homotopy Cartesian by assumption. Applying Proposition 3.4.1.9, we deduce that bottom square of the commutative diagram

\[
\begin{array}{ccc}
Y'' & \rightarrow & X' \\
\downarrow & & \downarrow \\
g^{-1}(T') & \rightarrow & X \\
\downarrow & & \downarrow \\
T' & \rightarrow & S
\end{array}
\]

is homotopy Cartesian. In this diagram, the top square is a pullback diagram whose vertical maps are Kan fibrations (Example 3.1.1.4), and is therefore homotopy Cartesian (Example 3.4.1.5). Applying Proposition 3.4.1.9 again, we conclude that the outer rectangle in the diagram

\[
\begin{array}{ccc}
Y'' & \rightarrow & X' \rightarrow X' \\
\downarrow & & \downarrow \\
T' & \rightarrow & S' \rightarrow S
\end{array}
\]
is homotopy Cartesian. Here the square on the right is a pullback diagram of Kan fibrations (Example 3.1.1.4), and therefore homotopy Cartesian. Applying Proposition 3.4.1.9 again, we conclude that the left square is homotopy Cartesian, as desired.

### 3.4.2 Homotopy Pushout Squares

We now formulate a dual version of Definition 3.4.1.1.

**Definition 3.4.2.1.** A commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

is homotopy coCartesian if, for every Kan complex \(X\), the diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Fun}(A, X) & \leftarrow & \text{Fun}(B, X) \\
\uparrow & & \uparrow \\
\text{Fun}(C, X) & \leftarrow & \text{Fun}(D, X)
\end{array}
\]

is homotopy Cartesian (Definition 3.4.1.1).

A **homotopy pushout square** is a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

which is homotopy coCartesian.

We now summarize some of the formal properties enjoyed by Definition 3.4.2.1. For the most part, these are immediate consequences of their counterparts for homotopy Cartesian diagrams (proven in §3.4.1).

**Proposition 3.4.2.2 (Symmetry).** A commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

is homotopy coCartesian if and only if the transposed diagram

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & & \downarrow \\
B & \rightarrow & D
\end{array}
\]
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**Proposition 3.4.2.3** (Transitivity). Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & B' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
C & \rightarrow & C' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B & \rightarrow & C \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B' & \rightarrow & C' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
C & \rightarrow & C' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
D & \rightarrow & D' \\
\end{array}
\]

where the left square is homotopy coCartesian. Then the right square is homotopy coCartesian if and only if the outer rectangle is homotopy coCartesian.

**Proof.** Apply Proposition 3.4.1.9. □

**Proposition 3.4.2.4** (Homotopy Invariance). Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & B' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
C & \rightarrow & C' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B & \rightarrow & C \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B' & \rightarrow & C' \\
\end{array}
\quad
\begin{array}{ccc}
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
D & \rightarrow & D' \\
\end{array}
\]

where the morphisms \( w_A, w_B, \) and \( w_C \) are weak homotopy equivalences. Then any two of the following three conditions imply the third:

(1) The commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D \\
\end{array}
\]

is homotopy coCartesian.

(2) The commutative diagram

\[
\begin{array}{ccc}
A' & \rightarrow & B' \\
\downarrow & & \downarrow \\
C' & \rightarrow & D' \\
\end{array}
\]

is homotopy coCartesian.
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(3) The morphism \( w_D \) is a weak homotopy equivalence.

Proof. Combine Corollaries \[ \text{3.4.1.10 and 3.1.6.5} \]

Proposition 3.4.2.5. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{f'} & D
\end{array}
\]

where \( f \) is a weak homotopy equivalence. Then (3.9) is homotopy coCartesian if and only if \( f' \) is a weak homotopy equivalence.

Proof. For every Kan complex \( X \), we obtain a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Fun}(A, X) & \xleftarrow{u} & \text{Fun}(B, X) \\
\uparrow & & \uparrow \\
\text{Fun}(C, X) & \xleftarrow{u'} & \text{Fun}(D, X)
\end{array}
\]

where \( u \) is a homotopy equivalence of Kan complexes (Corollary \[ \text{3.1.6.5} \]). Applying Corollary \[ \text{3.4.1.3} \] we conclude that (3.10) is homotopy Cartesian if and only if \( u \) is a homotopy equivalence of Kan complexes. Consequently, (3.9) is homotopy coCartesian if and only if, for every Kan complex \( X \), the composition with \( f' \) induces a homotopy equivalence \( \text{Fun}(D, X) \to \text{Fun}(C, X) \). By virtue of Corollary \[ \text{3.1.6.5} \], this is equivalent to the requirement that \( f' \) is a weak homotopy equivalence.

Proposition 3.4.2.6. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \to & D
\end{array}
\]

where \( f \) is a monomorphism. Then (3.11) is homotopy coCartesian if and only if the induced map \( C \coprod_A B \to D \) is a weak homotopy equivalence.

Proof. For every Kan complex \( X \), we obtain a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Fun}(A, X) & \xleftarrow{u} & \text{Fun}(B, X) \\
\uparrow & & \uparrow \\
\text{Fun}(C, X) & \to & \text{Fun}(D, X)
\end{array}
\]
where $u$ is a Kan fibration (Corollary 3.1.3.3). It follows that the diagram (3.12) is homotopy Cartesian if and only if the induced map

$$\text{Fun}(D, X) \to \text{Fun}(C, X) \times_{\text{Fun}(A, X)} \text{Fun}(B, X) \simeq \text{Fun}(C \coprod_A B, X)$$

is a weak homotopy equivalence (Example 3.4.1.5). It follows that the diagram (3.11) is homotopy coCartesian if and only if, for every Kan complex $Q$, the induced map $\text{Fun}(D, X) \to \text{Fun}(C \coprod_A B, X)$ is a homotopy equivalence of Kan complexes. By virtue of Corollary 3.1.6.5, this is equivalent to the requirement that the morphism $C \coprod_A B \to D$ is a weak homotopy equivalence.

**Example 3.4.2.7.** Suppose we are given a pushout diagram of simplicial sets

$$\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D.
\end{array} \quad (3.13)$$

If $f$ is a monomorphism, then (3.13) is also a homotopy pushout diagram.

**Remark 3.4.2.8.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D.
\end{array} \quad (3.14)$$

Using Exercise 3.1.6.11, we can factor $f$ as a composition $A \rightarrow f' B' \rightarrow B$, where $f'$ is a monomorphism and $w$ is a weak homotopy equivalence (in fact, we can even arrange that $w$ is a trivial Kan fibration). Combining Propositions 3.4.2.6 and 3.4.2.4, we conclude that diagram (3.14) is homotopy coCartesian if and only if the induced map $u : C \coprod_A B' \to D$ is a weak homotopy equivalence. In particular, the condition that $u$ is a weak homotopy equivalence does not depend on the choice of factorization $f = w \circ f'$.

### 3.4.3 Mather’s Second Cube Theorem

Our goal in this section is to prove a theorem of Mather (Theorem 3.4.3.3), which asserts that the collection of homotopy pushout squares is stable under the formation of homotopy pullback. This is an analogue (and consequence) of a more elementary statement about sets:
Exercise 3.4.3.1. Suppose we are given a pushout square of sets

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D.
\end{array}
\]

Then, for every map of sets \( \overline{D} \rightarrow D \), the induced diagram

\[
\begin{array}{ccc}
A \times_D \overline{D} & \rightarrow & B \times_D \overline{D} \\
\downarrow & & \downarrow \\
C \times_D \overline{D} & \rightarrow & \overline{D}
\end{array}
\]

is also a pushout square.

Since limits and colimits in the category of simplicial sets are computed pointwise, Exercise 3.4.3.1 immediately implies that the collection of pushout squares in the category of simplicial sets is stable under the formation of pullback along any morphism of simplicial sets \( q : \overline{D} \rightarrow D \). This statement has an analogue for homotopy pushout diagrams of simplicial sets, provided that we assume that \( q \) is a Kan fibration.

Proposition 3.4.3.2. Suppose we are given a homotopy coCartesian diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D,
\end{array}
\]

and let \( q : \overline{D} \rightarrow D \) be a Kan fibration of simplicial sets. Then the induced diagram

\[
\begin{array}{ccc}
A \times_D \overline{D} & \rightarrow & B \times_D \overline{D} \\
\downarrow & & \downarrow \\
C \times_D \overline{D} & \rightarrow & \overline{D}
\end{array}
\]

is also homotopy coCartesian.

Proof. Choose a factorization of \( f \) as a composition \( A \xrightarrow{f'} B' \xrightarrow{w} B \), where \( f' \) is a monomorphism and \( w \) is a weak homotopy equivalence (Exercise 3.1.6.11). Set \( D' = B' \coprod_A C \). Our assumption that (3.15) is a homotopy pushout square guarantees that the induced map \( D' \rightarrow D \) is a weak homotopy equivalence (Remark 3.4.2.8). We have a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A \times_D \overline{D} & \rightarrow & B' \times_D \overline{D} & \rightarrow & B \times_D \overline{D} \\
\downarrow & & \downarrow & & \downarrow \\
C \times_D \overline{D} & \rightarrow & D' \times_D \overline{D} & \rightarrow & \overline{D}.
\end{array}
\]
The left square in this diagram is a pushout square (by virtue of Exercise \[3.4.3.1\]) and the map \( A \times_D D \to B' \times_D D \) is a monomorphism, so it is homotopy coCartesian (Example \[3.4.2.7\]). It follows from Corollary \[3.3.7.2\] that the horizontal maps on the right side of the diagram are weak homotopy equivalences, so the right square is also homotopy coCartesian (Proposition \[3.4.2.5\]). Applying Proposition \[3.4.2.3\], we deduce that the outer rectangle is also homotopy coCartesian, as desired. \( \square \)

We now formulate a homotopy-invariant version of Proposition \[3.4.3.2\].

**Theorem 3.4.3.3** (Mather’s Second Cube Theorem \[29\]). Suppose we are given a cubical diagram of simplicial sets

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D \\
\downarrow & & \downarrow \\
A & \to & B \\
& \downarrow & \\
& C & \to & D
\end{array}
\]

having the property that the faces

\[
\begin{array}{ccc}
\overline{A} & \to & \overline{B} \\
\downarrow & & \downarrow \\
\overline{A} & \to & \overline{C} \\
\downarrow & & \downarrow \\
A & \to & B \\
& \downarrow & \\
C & \to & D
\end{array}
\]

are homotopy Cartesian. If the bottom face

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D
\end{array}
\]

is homotopy coCartesian, then the top face

\[
\begin{array}{ccc}
\overline{A} & \to & \overline{B} \\
\downarrow & & \downarrow \\
\overline{C} & \to & \overline{D}
\end{array}
\]
is also homotopy coCartesian.

Proof. Using Proposition 3.1.6.1 we can factor $q$ as a composition $D \xrightarrow{w} D' \xrightarrow{q'} D$, where $w$ is a weak homotopy equivalence and $q$ is a Kan fibration. We then obtain another commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}

\begin{array}{ccc}
A \times_D D' & \xrightarrow{f'} & B \times_D D' \\
\downarrow & & \downarrow \\
C \times_D D' & \xrightarrow{g'} & D',
\end{array}

(3.17)

where the bottom face is homotopy coCartesian by virtue of Proposition 3.4.3.2. Since the diagrams (3.16) are homotopy Cartesian, the vertical arrows in (3.17) are weak homotopy equivalences. Applying Proposition 3.4.2.4 we conclude that the top face

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}

$$

is also homotopy coCartesian. \hfill \square

3.4.4 Mather’s First Cube Theorem

Our goal in this section is to prove a converse of Theorem 3.4.3.3 known as Mather’s first cube theorem. As before, we begin with an elementary statement about the category of sets.

Exercise 3.4.4.1. Suppose we are given a commutative diagram of sets

$$
\begin{array}{ccc}
C & \xleftarrow{i} & A \\
\downarrow & & \downarrow \\
C & \xleftarrow{i} & A \\
\downarrow & & \downarrow \\
C & \xleftarrow{i} & A \\
\downarrow & & \downarrow \\
C & \xleftarrow{i} & A
\end{array}

$$
where both squares are pullback diagrams, and $i$ is a monomorphism (so that $\bar{i}$ is also a monomorphism). Show that both squares in the resulting diagram

$$
\begin{array}{ccc}
C & \rightarrow & C \coprod_A B \\
\downarrow & & \downarrow \\
C & \rightarrow & C \coprod_A B
\end{array}
$$

are pullback squares.

**Warning 3.4.4.2.** The conclusion of Exercise 3.4.4.1 does not necessarily hold if the map $i$ is not injective. For example, let $G$ be a group with multiplication map $m : G \times G \rightarrow G$, and let $\pi, \pi' : G \times G \rightarrow G$ be the projection maps onto the two factors. Then the diagram of sets

$$
\begin{array}{ccc}
G & \leftarrow & G \times G \\
\downarrow & & \downarrow \\
* & \leftarrow & G
\end{array}
\quad
\begin{array}{ccc}
G \coprod_{G \times G} G & \rightarrow & G \\
\downarrow & & \downarrow \\
* & \rightarrow & *
\end{array}
$$

consists of pullback squares, but the induced diagram

$$
\begin{array}{ccc}
G & \rightarrow & G \coprod_{G \times G} G \\
\downarrow & & \downarrow \\
* & \rightarrow & *
\end{array}
\quad
\begin{array}{ccc}
G & \rightarrow & G \coprod_{G \times G} G \\
\downarrow & & \downarrow \\
* & \rightarrow & *
\end{array}
$$

does not (except in the case where $G$ is trivial).

Exercise 3.4.4.1 has an analogue for homotopy pullback diagrams of simplicial sets.

**Proposition 3.4.4.3.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
C & \leftarrow & A \rightarrow & B \\
\downarrow & & \downarrow & \downarrow \\
C & \leftarrow & A \rightarrow & B
\end{array}
$$

in which both squares are homotopy Cartesian. If $i$ and $\bar{i}$ are monomorphisms, then both squares in the induced diagram

$$
\begin{array}{ccc}
\bar{C} & \rightarrow & \bar{C} \coprod_A \bar{B} \\
\downarrow & & \downarrow \\
\bar{C} & \rightarrow & \bar{C} \coprod_A \bar{B}
\end{array}
\quad
\begin{array}{ccc}
C & \rightarrow & C \coprod_A B \\
\downarrow & & \downarrow \\
C & \rightarrow & C \coprod_A B
\end{array}
$$

are also homotopy Cartesian.
Proposition 3.4.4.3 is an immediate consequence of Example 3.4.2.7 together with the following homotopy-invariant statement:

**Theorem 3.4.4.4** (Mather’s First Cube Theorem). Suppose we are given a cubical diagram of simplicial sets

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & \downarrow & \downarrow \\
C & \to & D \\
\downarrow & \downarrow & \downarrow \\
A & \to & B & \to & C & \to & D \\
\end{array}
\]

having the property that the back and left faces

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & \downarrow & \downarrow \\
A & \to & B \\
\end{array}
\]

are homotopy Cartesian, and the top and bottom faces

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & \downarrow & \downarrow \\
C & \to & D \\
\end{array}
\]

are homotopy coCartesian. Then the front and right faces

\[
\begin{array}{ccc}
C & \to & D \\
\downarrow & \downarrow & \downarrow \\
B & \to & D \\
\end{array}
\]

are also homotopy Cartesian.

**Proof.** The proof will proceed in several steps, each of which involves replacing one or more of the terms in (3.18) by a weakly equivalent simplicial set (by virtue of Corollary 3.4.1.10 and Proposition 3.4.2.4, such replacements will not affect the truth of our hypotheses or of the desired conclusion). Let us denote each of the morphisms appearing in the diagram (3.18) by \(f_{XY}\), where \(X, Y \in \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, A, B, C, D\}\) are the source and target of \(f_{XY}\), respectively.
• By virtue of Proposition \[3.1.6.1\] the morphism \(f_{BB} : B \to B\) factors as a composition \(B \xrightarrow{w} B' \xrightarrow{f'_{BB}} B\), where \(w\) is anodyne and \(f'_{BB}\) is a Kan fibration. Replacing \(B\) by \(B'\) (and \(D\) by the pushout \(B' \coprod_{B} D\)), we can reduce to the case where \(f_{BB}\) is a Kan fibration. Similarly, we can arrange that the map \(f_{CC} : C \to C\) is a Kan fibration.

• By virtue of Exercise \[3.1.6.11\] the morphism \(f_{AC}\) factors as a composition \(A \xrightarrow{f'_{AC}} C \xrightarrow{w} C'\), where \(f'_{AC}\) is a monomorphism and \(w\) is a trivial Kan fibration. Replacing \(C\) by \(C'\), we can reduce to the case where \(f_{AC}\) is a monomorphism. Note that this replacement does not injure our hypothesis that \(f_{CC}\) is a Kan fibration.

• Since the left face

\[
\begin{array}{ccc}
A & \xrightarrow{f_{AC}} & C \\
\downarrow{f_{AA}} & & \downarrow{f_{CC}} \\
A & \longrightarrow & C
\end{array}
\]  \(\text{(3.19)}\)

is homotopy Cartesian and \(f_{CC}\) is a Kan fibration, the induced map \(\overline{A} \to A \times_C C\) is a weak homotopy equivalence (Example \[3.4.1.5\]). We may therefore replace \(\overline{A}\) by the fiber product \(A \times_C C\), and thereby reduce to the case where the diagram \(\text{(3.19)}\) is a pullback square (so that \(f_{AA}\) is also a Kan fibration). Note that this replacement does not injure our hypothesis that \(f_{AC}\) is a monomorphism.

• By virtue of Exercise \[3.1.6.11\] the morphism \(f_{AB}\) factors as a composition \(A \xrightarrow{f'_{AB}} B' \xrightarrow{w} B\), where \(f'_{AB}\) is a monomorphism and \(w\) is a trivial Kan fibration. Replacing \(B\) by \(B'\) (and \(B\) by the fiber product \(B' \times_B B'\)), we can reduce to the case where \(f_{AB}\) is a monomorphism.

• By virtue of Exercise \[3.1.6.11\] the morphism \(f_{AB}\) factors as a composition \(A \xrightarrow{f'_{AB}} B' \xrightarrow{w} B\), where \(f'_{AB}\) is a monomorphism and \(w\) is a trivial Kan fibration. Replacing \(B\) by \(B'\), we can reduce to the case where \(f_{AB}\) is a monomorphism.

• The back face

\[
\begin{array}{ccc}
A & \xrightarrow{f_{AB}} & B \\
\downarrow{f_{AA}} & & \downarrow{f_{BB}} \\
A & \longrightarrow & B
\end{array}
\]  \(\text{(3.20)}\)

is a homotopy pullback square in which the horizontal maps are monomorphisms and the vertical maps are Kan fibrations. It follows that, for every vertex \(a \in A\) having image \(b = f_{AB}(a) \in B\), the induced map of fibers \(\overline{A}_a \to \overline{B}_b\) is a homotopy equivalence. Let \(\overline{B}' \subseteq \overline{B}\) denote the simplicial subset spanned by those simplices \(\sigma : \Delta^n \to \overline{B}\) having


the property that the restriction $\sigma |_{A \times B \Delta^n}$ factors through $\overline{A}$. Applying Lemma 3.3.8.4 we deduce that the restriction $f_{\overline{B}} : \overline{B} \to B$ is also a Kan fibration. Moreover, the inclusion map $\overline{B} \hookrightarrow \overline{B}$ induces a homotopy equivalence of fibers $\overline{B}_b \hookrightarrow \overline{B}_b$, for each vertex $b \in B$. It follows that the inclusion $\overline{B} \hookrightarrow \overline{B}$ is a weak homotopy equivalence (Corollary 3.3.7.3). Replacing $\overline{B}$ by $\overline{B}'$, we can reduce to the case where the diagram (3.20) is a pullback square.

- By assumption, the top and bottom faces

\[
\begin{array}{ccc}
\overline{A} & \rightarrow & \overline{B} \\
\downarrow f_{\overline{AC}} & & \downarrow f_{AC} \\
\overline{C} & \rightarrow & \overline{D}
\end{array}
\quad
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f_{AC} & & \downarrow f_{AC} \\
C & \rightarrow & D
\end{array}
\quad
(3.21)
\]

are homotopy coCartesian. Since $f_{\overline{AC}}$ and $f_{AC}$ are monomorphisms, it follows that the induced maps

\[
\begin{array}{ccc}
\overline{C} \coprod_{\overline{A}} B & \rightarrow & \overline{D} \\
\downarrow f_{\overline{AC}} & & \downarrow f_{\overline{AC}} \\
C \coprod_A B & \rightarrow & D
\end{array}
\]

are weak homotopy equivalences (Proposition 3.4.2.6). We may therefore replace $D$ by the pushout $C \coprod_{\overline{A}} B$ and $\overline{D}$ by the pushout $\overline{C} \coprod_{\overline{A}} \overline{B}$, and thereby reduce to the case where the diagrams (3.21) are pushout squares.

- Applying Exercise 3.4.4.1 levelwise, we deduce that the front and right faces

\[
\begin{array}{ccc}
\overline{C} & \rightarrow & \overline{D} \\
\downarrow f_{\overline{AC}} & & \downarrow f_{\overline{AC}} \\
C & \rightarrow & D
\end{array}
\quad
\begin{array}{ccc}
B & \rightarrow & \overline{D} \\
\downarrow f_{\overline{BC}} & & \downarrow f_{\overline{BC}} \\
\overline{B} & \rightarrow & D
\end{array}
\quad
(3.22)
\]

are pullback squares in the category of simplicial sets. In particular, for every simplex $\sigma : \Delta^n \to D$, the projection map $\Delta^n \times_D \overline{D} \to \Delta^n$ is a pullback either of $f_{\overline{B}B}$ or of $f_{\overline{C}C}$, and is therefore a Kan fibration. Applying Remark 3.1.1.7 we conclude that $f_{\overline{D}D} : \overline{D} \to D$ is a Kan fibration. It follows that the diagrams (3.22) are also homotopy pullback squares, as desired.

\[\square\]

### 3.4.5 Digression: Weak Homotopy Equivalences of Semisimplicial Sets

Recall that a morphism of simplicial sets $f : X \to Y$ is a *weak homotopy equivalence* if, for every Kan complex $Z$, precomposition with $f$ induces a bijection $\pi_0(\text{Fun}(Y, Z)) \to \pi_0(\text{Fun}(X, Z))$ (Definition 3.1.5.11). Our goal in this section is to show that this condition
depends only on the underlying morphism of semisimplicial sets. To see this, we begin by recalling that the forgetful functor
\[
\{\text{Simplicial Sets}\} \to \{\text{Semisimplicial Sets}\}
\]
adopts a left adjoint, which we denote by \( X \mapsto X^+ \) (Corollary 3.3.1.10).

**Definition 3.4.5.1.** Let \( f : X \to Y \) be a morphism of semisimplicial sets. We will say that \( f \) is a **weak homotopy equivalence** if the induced map of simplicial sets \( f^+ : X^+ \to Y^+ \) is a weak homotopy equivalence, in the sense of Definition 3.1.5.11.

**Remark 3.4.5.2.** The collection of weak homotopy equivalences of semisimplicial sets is closed under the formation of filtered colimits. This follows immediately from the corresponding assertion for simplicial sets (Proposition 3.2.7.3), since the construction \( X \mapsto X^+ \) commutes with filtered colimits.

**Remark 3.4.5.3.** Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of semisimplicial sets. If any two of the morphisms \( f, g, \) and \( g \circ f \) are weak homotopy equivalences, then so is the third (see Remark 3.1.5.16).

When \( X \) is a simplicial set, we write \( v_X : X^+ \to X \) for the counit map (that is, the unique morphism of simplicial sets whose restriction to \((X^+)^{nd} \simeq X\) is the identity map).

To compare Definition 3.4.5.1 with Definition 3.1.5.11, we need the following:

**Proposition 3.4.5.4.** For every simplicial set \( X \), the counit map \( v_X : X^+ \to X \) is a weak homotopy equivalence.

**Corollary 3.4.5.5.** Let \( f : X \to Y \) be a morphism of simplicial sets. Then \( f \) is a weak homotopy equivalence (in the sense of Definition 3.1.5.11) if and only if the underlying morphism of semisimplicial sets is a weak homotopy equivalence (in the sense of Definition 3.4.5.1).

**Proof.** We have a commutative diagram of simplicial sets
\[
\begin{array}{ccc}
X^+ & \xrightarrow{f^+} & Y^+ \\
\downarrow v_X & & \downarrow v_Y \\
X & \xrightarrow{f} & Y,
\end{array}
\]
where the vertical maps are weak homotopy equivalences by virtue of Proposition 3.4.5.4. Invoking Remark 3.1.5.16, we deduce that \( f \) is a weak homotopy equivalence if and only if \( f^+ \) is a weak homotopy equivalence.
Corollary 3.4.5.6. For every semisimplicial set $X$, the inclusion map $\iota : X \hookrightarrow X^+$ is a weak homotopy equivalence of semisimplicial sets.

*Proof.* We wish to show that the map $\iota^+ : X^+ \to (X^+)^+$ is a weak homotopy equivalence of simplicial sets. This is clear, since $\iota^+$ is right inverse to the counit map $v_{X^+} : (X^+)^+ \to X^+$, which is a weak homotopy equivalence of simplicial sets by virtue of Proposition 3.4.5.4.

**Variant 3.4.5.7.** Let $X$ be a simplicial set, and let $\iota : X \hookrightarrow X^+$ be the inclusion map. Then the map $\text{Ex}(\iota) : \text{Ex}(X) \hookrightarrow \text{Ex}(X^+)$ is a weak homotopy equivalence of semisimplicial sets.

*Proof.* By virtue of Proposition 3.4.5.4 the counit map $v_X : X^+ \to X$ is a weak homotopy equivalence of simplicial sets. Applying Corollary 3.3.5.2 we deduce that the map $\text{Ex}(v_X) : \text{Ex}(X^+) \to \text{Ex}(X)$ is a weak homotopy equivalence of simplicial sets, hence also a weak homotopy equivalence of the underlying semisimplicial sets (Corollary 3.4.5.5). Since the composite map 

$$ \text{Ex}(X) \xrightarrow{\text{Ex}(\iota)} \text{Ex}(X^+) \xrightarrow{\text{Ex}(v_X)} \text{Ex}(X) $$

is the identity, it follows that $\text{Ex}(\iota)$ is also a weak homotopy equivalence of semisimplicial sets.

Corollary 3.4.5.8. Let $X$ and $Y$ be simplicial sets and let $f : X \to Y$ be a morphism of semisimplicial sets. Then $f$ is a weak homotopy equivalence of semisimplicial sets if and only if the induced map $\text{Ex}(f) : \text{Ex}(X) \to \text{Ex}(Y)$ is a weak homotopy equivalence of semisimplicial sets.

*Proof.* By definition, $f : X \to Y$ is a weak homotopy equivalence of semisimplicial sets if and only if the induced map $f^+ : X^+ \to Y^+$ is a weak homotopy equivalence of simplicial sets. By virtue of Corollary 3.3.5.2 this is equivalent to the assertion that $\text{Ex}(f^+) : \text{Ex}(X^+) \to \text{Ex}(Y^+)$ is a weak homotopy equivalence when viewed as a morphism of simplicial sets, or equivalently when viewed as a morphism of semisimplicial sets (Corollary 3.4.5.5). The desired result now follows by inspecting the commutative diagram of semisimplicial sets

$$ \begin{array}{ccc} \text{Ex}(X) & \xrightarrow{\text{Ex}(f)} & \text{Ex}(Y) \\ \downarrow & & \downarrow \\ \text{Ex}(X^+) & \xrightarrow{\text{Ex}(f^+)} & \text{Ex}(Y^+) \end{array} $$

since the vertical maps are weak homotopy equivalences by virtue of Variant 3.4.5.7.

We now turn to the proof of Proposition 3.4.5.4. The main ingredient we will need is the following is the following:
Lemma 3.4.5.9. Let \( \mathcal{C} \) be a category, and suppose that the collection of non-identity morphisms in \( \mathcal{C} \) is closed under composition. Then the counit map \( v_{N_\bullet(\mathcal{C})} : N_\bullet(\mathcal{C})^+ \to N_\bullet(\mathcal{C}) \) is a homotopy equivalence of simplicial sets.

Proof. Let \( \mathcal{C}^+ \) denote the category obtained from \( \mathcal{C} \) by formally adjoining a new identity morphism \( \text{id}_X^+ \) for each object \( X \in \mathcal{C} \). More precisely, the category \( \mathcal{C}^+ \) is defined as follows:

- The objects of \( \mathcal{C}^+ \) are the objects of \( \mathcal{C} \).
- For every pair of objects \( X, Y \in \mathcal{C}^+ \), we have
  \[
  \text{Hom}_{\mathcal{C}^+}(X,Y) = \begin{cases} 
  \text{Hom}_{\mathcal{C}}(X,Y) & \text{if } X \neq Y \\
  \text{Hom}_{\mathcal{C}}(X,Y) \amalg \{\text{id}_X^+\} & \text{if } X = Y.
  \end{cases}
  \]
- If \( f : X \to Y \) and \( g : Y \to Z \) are morphisms in \( \mathcal{C}^+ \), then the composition \( g \circ f \) is equal to \( g \) if \( f = \text{id}_Y^+ \), to the morphism \( f \) if \( g = \text{id}_Y^+ \), and is otherwise given by the composition law for morphisms in \( \mathcal{C} \).

Note that the collection of non-identity morphisms in \( \mathcal{C}^+ \) is closed under composition, so that the nerve \( N_\bullet(\mathcal{C}^+) \) is a braced simplicial set (Exercise 3.3.1.2). Unwinding the definitions, we see that the semisimplicial subset \( N_\bullet(\mathcal{C}^+) \) is identified with the \( N_\bullet(\mathcal{C}) \) (as a semisimplicial set). Using Corollary 3.3.1.11, we obtain a canonical isomorphism of simplicial sets \( N_\bullet(\mathcal{C}^+) \simeq N_\bullet(\mathcal{C}) \). Under this isomorphism, the counit map \( v_{N_\bullet(\mathcal{C})} \) is induced by the functor \( F : \mathcal{C}^+ \to \mathcal{C} \) which is the identity on objects, and carries each morphism \( f \in \text{Hom}_{\mathcal{C}}(X,Y) \subseteq \text{Hom}_{\mathcal{C}^+}(X,Y) \) to itself.

Let \( G : \mathcal{C} \to \mathcal{C}^+ \) be the functor which is the identity on objects, and which carries a morphism \( f \in \text{Hom}_{\mathcal{C}}(X,Y) \) to the morphism

\[
G(f) = \begin{cases} 
\text{id}_X^+ & \text{if } X = Y \text{ and } f = \text{id}_X \\
f & \text{otherwise}.
\end{cases}
\]

This functor is well-defined by virtue of our assumption that the collection of non-identity morphisms of \( \mathcal{C} \) is closed under composition. We will complete the proof by showing that the induced map \( N_\bullet(G) : N_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{C}^+) \) is a homotopy inverse of \( N_\bullet(F) = v_{N_\bullet(\mathcal{C})} \). One direction is clear: the composition \( \mathcal{C} \xrightarrow{G} \mathcal{C}^+ \xrightarrow{F} \mathcal{C} \) is the identity functor \( \text{id}_{\mathcal{C}} \), so \( N_\bullet(F) \circ N_\bullet(G) \) is equal to the identity. The composition \( \mathcal{C}^+ \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}^+ \) is not the identity functor on \( \mathcal{C}^+ \): for each object \( X \in \mathcal{C} \), it carries the morphism \( \text{id}_X \in \text{Hom}_{\mathcal{C}}(X,X) \subseteq \text{Hom}_{\mathcal{C}^+}(X,X) \) to the “new” identity morphism \( \text{id}_X^+ \). However, there is a natural transformation \( \alpha : G \circ F \to \text{id}_{\mathcal{C}^+} \), given by the construction \( (X \in \mathcal{C}^+) \mapsto \text{id}_X^+ \). It follows that the map of simplicial sets \( N_\bullet(G) \circ N_\bullet(F) \) is homotopic to the identity (Example 3.1.4.7).
**Proof of Proposition 3.4.5.4.** We proceed as in the proof of Proposition 3.3.4.8. For every simplicial set $X$, the counit map $v_X : X^+ \to X$ can be realized as a filtered colimit of counit maps $\{v_{\text{sk}_n(X)} : \text{sk}_n(X)^+ \to \text{sk}_n(X)\}_{n \geq 0}$. Since the collection of weak homotopy equivalences is closed under the formation of filtered colimits (Proposition 3.2.7.3), it will suffice to show that each of the maps $v_{\text{sk}_n(X)}$ is a weak homotopy equivalence. We may therefore replace $X$ by $\text{sk}_n(X)$, and thereby reduce to the case where $X$ is $n$-skeletal for some nonnegative integer $n \geq 0$. We now proceed by induction on $n$.

Let $Y = \text{sk}_{n-1}(X)$ be the $(n-1)$-skeleton of $X$. Let $S$ denote the collection of nondegenerate $n$-simplices of $X$, let $X' = \coprod_{\sigma \in S} \Delta^n$ denote their coproduct, and let $Y' = \coprod_{\sigma \in S} \partial \Delta^n$ denote the boundary of $X'$. Proposition 1.1.3.13 then supplies a pushout diagram of simplicial sets

$$
\begin{array}{cccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X.
\end{array}
$$

(3.23)

Note that both (3.23) and the induced diagram

$$
\begin{array}{cccc}
Y'^+ & \longrightarrow & X'^+ \\
\downarrow & & \downarrow \\
Y^+ & \longrightarrow & X^+
\end{array}
$$

are homotopy coCartesian (this is a special case of Example 3.4.2.7, since the maps $Y' \hookrightarrow X'$ and $Y'^+ \hookrightarrow X'^+$ are monomorphisms). Moreover, our inductive hypothesis guarantees that the maps $v_Y : Y^+ \to Y$ and $v_{Y'} : Y'^+ \to Y'$ are weak homotopy equivalences. Applying Proposition 3.4.2.4 to the commutative diagram

$$
\begin{array}{ccc}
Y'^+ & \longrightarrow & Y^+ \\
\downarrow & v_{Y'} & \downarrow v_Y \\
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X'^+ & \longrightarrow & X^+ \\
\downarrow & v_{X'} & \downarrow v_X \\
X' & \longrightarrow & X
\end{array}
$$

we are reduced to proving that $v_{X'}$ is a weak homotopy equivalence in the special case $X = \Delta^n$, which follows from Lemma 3.4.5.9. □
3.4.6 Excision

Let $X$ be a topological space which is a union of two open subsets $U, V \subseteq X$. Then the diagram

$$
\begin{array}{ccc}
U \cap V & \longrightarrow & U \\
\downarrow & & \downarrow \\
V & \longrightarrow & X
\end{array}
$$

is a pushout square in the category of topological spaces. Stated more informally, the topological space $X$ can be obtained by gluing $U$ and $V$ along their common open subset $U \cap V$. This observation has a homotopy-theoretic counterpart:

**Theorem 3.4.6.1** (Excision). Let $X$ be a topological space, and let $U, V \subseteq X$ be subsets whose interiors $\dot{U} \subseteq U$ and $\dot{V} \subseteq V$ comprise an open covering of $X$. Then the diagram of singular simplicial sets

$$
\begin{array}{ccc}
\text{Sing}_\bullet(U \cap V) & \longrightarrow & \text{Sing}_\bullet(U) \\
\downarrow & & \downarrow \\
\text{Sing}_\bullet(V) & \longrightarrow & \text{Sing}_\bullet(X)
\end{array}
$$

is homotopy coCartesian (Definition 3.4.2.1).

**Remark 3.4.6.2.** In the situation of Theorem 3.4.6.1, the canonical maps $\text{Sing}_\bullet(U) \leftrightarrow \text{Sing}_\bullet(U \cap V) \leftrightarrow \text{Sing}_\bullet(V)$ are monomorphisms. Consequently, the conclusion of Theorem 3.4.6.1 is equivalent to the assertion that the natural map

$$
\text{Sing}_\bullet(U) \coprod_{\text{Sing}_\bullet(U \cap V)} \text{Sing}_\bullet(V) \rightarrow \text{Sing}_\bullet(X)
$$

is a weak homotopy equivalence of simplicial sets (see Proposition 3.4.2.6).

**Warning 3.4.6.3.** In the situation of Theorem 3.4.6.1 it is generally not true that the diagram

$$
\begin{array}{ccc}
\text{Sing}_\bullet(U \cap V) & \longrightarrow & \text{Sing}_\bullet(U) \\
\downarrow & & \downarrow \\
\text{Sing}_\bullet(V) & \longrightarrow & \text{Sing}_\bullet(X)
\end{array}
$$

is a pushout square of simplicial sets. Concretely, this is because the image of a continuous function $f : |\Delta^n| \rightarrow X$ need not be contained in either $U$ or $V$.

Our goal in this section is to prove a stronger version Theorem 3.4.6.1, where we allow more general coverings of $X$. 
3.4. HOMOTOPY PULLBACK AND HOMOTOPY PUSHOUT SQUARES

Definition 3.4.6.4. Let $X$ be a topological space and let $\mathcal{U}$ be a collection of subsets of $X$. We say that a singular $n$-simplex $\sigma : |\Delta^n| \to X$ is $\mathcal{U}$-small if its image is contained in $U$, for some $U \in \mathcal{U}$. We let $\text{Sing}^U_n(X)$ denote the subset of $\text{Sing}_n(X)$ consisting of the $\mathcal{U}$-small simplices. Note that the subsets $\{\text{Sing}^U_n(X)\}_{n \geq 0}$ are stable under the face and degeneracy operators of the simplicial set $\text{Sing}_\bullet(X)$, and therefore determine a simplicial subset which we will denote by $\text{Sing}^U\bullet(X) \subseteq \text{Sing}_\bullet(X)$.

Remark 3.4.6.5. In the situation of Definition 3.4.6.4, the simplicial set $\text{Sing}^U\bullet(X)$ is given by the union $\bigcup_{U \in \mathcal{U}} \text{Sing}_\bullet(U)$, where we regard each $\text{Sing}_\bullet(U)$ as a simplicial subset of $\text{Sing}_\bullet(X)$.

Our main result can now be stated as follows:

Theorem 3.4.6.6. Let $X$ be a topological space and let $\mathcal{U}$ be a collection of subsets of $X$ satisfying $X = \bigcup_{U \in \mathcal{U}} \hat{U}$. Then the inclusion map $\text{Sing}^\mathcal{U}\bullet(X) \hookrightarrow \text{Sing}_\bullet(X)$ is a weak homotopy equivalence.

Proof of Theorem 3.4.6.1 from Theorem 3.4.6.6. Let $X$ be a topological space and let $\mathcal{U} = \{U, V\}$ be a pair of subsets of $X$. Then $\text{Sing}^\mathcal{U}\bullet(X)$ can be identified with the pushout

$$\text{Sing}_\bullet(U) \coprod_{\text{Sing}_\bullet(U \cap V)} \text{Sing}_\bullet(V),$$

formed in the category of simplicial sets. Theorem 3.4.6.6 then asserts that if $X = \hat{U} \cup \hat{V}$, then the inclusion

$$\text{Sing}_\bullet(U) \coprod_{\text{Sing}_\bullet(U \cap V)} \text{Sing}_\bullet(V) \hookrightarrow \text{Sing}_\bullet(X)$$

is a weak homotopy equivalence. By virtue of Remark 3.4.6.2, this is equivalent to Theorem 3.4.6.1. 

The proof of Theorem 3.4.6.6 is based on the observation that every singular $n$-simplex $\sigma : |\Delta^n| \to X$ can be “decomposed” into $\mathcal{U}$-small simplices by repeatedly applying the barycentric subdivision described in Proposition 3.3.2.3. To make this precise, we need the following geometric observation:

Lemma 3.4.6.7. Let $V$ be a normed vector space over the real numbers and let $K \subseteq V$ be the convex hull of a finite collection of points $v_0, v_1, \ldots, v_n \in V$, given by the image of a continuous function:

$$f : |\Delta^n| \to V \quad (t_0, t_1, \ldots, t_n) \mapsto t_0v_0 + t_1v_1 + \cdots + t_nv_n.$$

Let $\sigma$ be any $m$-simplex of the subdivision $\text{Sd}(\Delta^n)$, let $f_\sigma$ denote the composite map

$$|\Delta^m| \xrightarrow{|\sigma|} |\text{Sd}(\Delta^n)| \simeq |\Delta^n| \xrightarrow{f} V$$


(where the homeomorphism \( |\text{Sd}(\Delta^n)| \leq |\Delta^n| \) is supplied by Proposition 3.3.2.3, and let \( K_0 \subseteq K \) be the image of \( f_\sigma \). Then the diameters of \( K_0 \) and \( K \) satisfy the inequality \( \text{diam}(K_0) \leq \frac{n}{n+1} \text{diam}(K) \).

**Proof.** Let us denote the norm on the vector space \( V \) by \( |\cdot|_V \). Fix points \( x, y \in |\Delta^m| \); we wish to show that \( |f_\sigma(x) - f_\sigma(y)|_V \leq \frac{n}{n+1} \text{diam}(K) \). Note that, if we regard the point \( x \) as fixed, then the function \( y \mapsto |f_\sigma(x) - f_\sigma(y)|_V \) is convex, and therefore achieves its supremum at some vertex of \( |\Delta^m| \). We may therefore assume without loss of generality that \( y \) is a vertex of \( |\Delta^m| \). Similarly, we may assume that \( x \) is a vertex of \( |\Delta^m| \). We may also assume that \( x \neq y \) (otherwise there is nothing to prove). Exchanging \( x \) and \( y \) if necessary, it follows that there exist disjoint nonempty subsets \( A, B \subseteq \{0, 1, \ldots, n\} \) of cardinality \( a = |A| \) and \( b = |B| \) satisfying

\[
    f_\sigma(x) = \sum_{i \in A} v_i^a, \quad f_\sigma(y) = \sum_{i \in A \cup B} v_i^a + b.
\]

We then compute

\[
    |f_\sigma(x) - f_\sigma(y)|_V = \left\| \sum_{(i,j) \in A \times B} v_i v_j - v_j v_i \right\|_V \\
    \leq \sum_{(i,j) \in A \times B} \left( |v_i - v_j|_V \right) a(a + b) \\
    \leq \sum_{(i,j) \in A \times B} \left( \text{diam}(K) \right) a(a + b) \\
    = \frac{b}{a + b} \text{diam}(K) \\
    \leq \frac{n}{n+1} \text{diam}(K).
\]

\( \square \)

**Proof of Theorem 3.4.6.6.** Let \( X \) be a topological space and let \( \mathcal{U} \) be a collection of subsets of \( X \) satisfying \( X = \bigcup_{U \in \mathcal{U}} U \). For each \( k \geq 0 \), let \( Y(k) \subseteq \text{Sing}_\bullet(X) \) denote the semisimplicial subset spanned by those singular \( n \)-simplices \( f : |\Delta^n| \to X \) having the property that, for every \( m \)-simplex \( \sigma \) of the iterated subdivision \( \text{Sd}^k(\Delta^n) \), the composite map

\[
    |\Delta^m| \xrightarrow{|\sigma|} |\text{Sd}^k(\Delta^n)| \simeq |\Delta^n| \xrightarrow{f} X
\]

is \( \mathcal{U} \)-small; here the identification \( |\text{Sd}^k(\Delta^n)| \simeq |\Delta^n| \) is given by iteratively applying the barycentric subdivision of Proposition 3.3.2.3. By construction, we have inclusions of semisimplicial sets

\[
    \text{Sing}_\bullet^{\mathcal{U}}(X) = Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq \cdots \subseteq \text{Sing}_\bullet(X).
\]
We first claim that \( \text{Sing}_\bullet(X) = \bigcup_{k \geq 0} Y(k) \). Fix a continuous function \( f : |\Delta^n| \to X \), regarded as an \( n \)-simplex of \( \text{Sing}_\bullet(X) \); we wish to show that \( f \) belongs to \( Y(k) \) for \( k \gg 0 \). Let us identify the topological \( n \)-simplex \( |\Delta^n| \) with the subset of Euclidean space \( V = \mathbb{R}^{n+1} \) given by the convex hull of the standard basis vectors \( \{v_i\}_{0 \leq i \leq n} \). Then the collection of inverse images \( \{f^{-1}(U)\}_{U \in \mathcal{U}} \) can be refined to an open covering of \( |\Delta^n| \). It follows that there exists a positive real number \( \epsilon \) with the property that, for every point \( v \in |\Delta^n| \), the open ball

\[
B_\epsilon(v) = \{ w \in |\Delta^n| : |v - w| < \epsilon \}
\]

is contained in \( f^{-1}(U) \), for some \( U \in \mathcal{U} \). Choose an integer \( k \) satisfying \( (\frac{n}{n+1})^k \text{diam}(|\Delta^n|) < \epsilon \).

It then follows from iterated application of Lemma 3.4.6.7 that the composite map

\[
|\text{St}^k(\Delta^n)| \simeq |\Delta^n| \to X
\]
carries each simplex of \( \text{St}^k(\Delta^n) \) into a subset \( U \subseteq X \) belonging to \( \mathcal{U} \), so that \( f \) belongs to the semisimplicial subset \( Y(k) \subseteq \text{Sing}_\bullet(X) \).

Note that the inclusion \( \iota : \text{Sing}_\bullet(Y(k)) \hookrightarrow \text{Sing}_\bullet(X) \) is a weak homotopy equivalence of simplicial sets if and only if it is a weak homotopy equivalence when regarded as a morphism of semisimplicial sets (Corollary 3.4.5.5). It follows from the preceding argument that, as a morphism of semisimplicial sets, \( \iota \) can be realized as a filtered colimit of the inclusion maps \( \iota(k) : \text{Sing}_\bullet(Y(k)) = Y(0) \hookrightarrow Y(k) \). Since the collection of weak homotopy equivalences is closed under filtered colimits (Remark 3.4.5.2), it will suffice to show that each \( \iota(k) \) is a weak homotopy equivalence. Proceeding by induction on \( k \), we are reduced to showing that each of the inclusion maps \( Y(k) \hookrightarrow Y(k+1) \) is a weak homotopy equivalence. Note that the semisimplicial isomorphism \( \varphi : \text{Sing}_\bullet(X) \simeq \text{Ex}(\text{Sing}_\bullet(X)) \) of Example 3.3.2.9 restricts to a map \( \varphi^k : \text{Sing}_\bullet(Y(k)) \to \text{Ex}(\text{Sing}_\bullet(Y(k))) \) (which is generally not an isomorphism). Unwinding the definitions, we see that the inclusion \( Y(k) \hookrightarrow Y(k+1) \) can be identified with the map \( \text{Ex}^k(\varphi^k) : \text{Ex}^k(\text{Sing}_\bullet(Y(k))) \to \text{Ex}^{k+1}(\text{Sing}_\bullet(Y(k))) \) (see Variant 3.3.2.10). By virtue of Corollary 3.4.5.8 it will suffice to show that \( \varphi^k \) is a weak homotopy equivalence.

Fix an integer \( n \geq 0 \) as above, let \( \text{Chain}[n] \) denote the collection of all nonempty subsets of \( [n] = \{0 < 1 < \cdots < n\} \). Let \( \sigma \) be an \( n \)-simplex of the simplicial set \( \Delta^1 \times \text{Sing}_\bullet(Y(k)) \), which we identify with a pair \((\epsilon, f)\) where \( \epsilon : [n] \to [1] \) is a nondecreasing function and \( f : |\Delta^n| \to X \) is a continuous map of topological spaces. Define a map of sets \( g_\epsilon : \text{Chain}[n] \to |\Delta^n| \) by the formula

\[
g_\epsilon(S) = \begin{cases} \sum_{i \in S} v_i/|S| & \text{if } \epsilon|S = 0 \\ v_{\text{Max}(S)} & \text{otherwise}. \end{cases}
\]

Then \( g_\epsilon \) extends to a continuous map

\[
\overline{g}_\epsilon : |N_\bullet(\text{Chain}[n])| \to |\Delta^n|
\]
which is affine when restricted to each simplex of \(|N_\bullet(\text{Chain}[n])| \simeq |\text{Sd}(\Delta^n)|\). The composite map

\[ |\text{Sd}(\Delta^n)| \xrightarrow{\partial} |\Delta^n| \xrightarrow{f} X \]

can be identified with an \(n\)-simplex of \(\text{Ex}(\text{Sing}^U_\bullet(X))\), which we will denote by \(h(\sigma)\). It is not difficult to see that the construction \(\sigma \mapsto h(\sigma)\) is compatible with face operators, and therefore determines a morphism of semisimplicial sets \(h : \Delta^1 \times \text{Sing}^U_\bullet(X) \to \text{Ex}(\text{Sing}^U_\bullet(X))\). By construction, this morphism fits into a commutative diagram of semisimplicial sets

\[
\begin{array}{ccc}
\Delta^1 \times \text{Sing}^U_\bullet(X) & \xrightarrow{i_0} & \{0\} \times \text{Sing}^U_\bullet(X) \\
\downarrow^{i_1} & & \downarrow^h \\
\{1\} \times \text{Sing}^U_\bullet(X) & \xrightarrow{\rho} & \text{Ex}(\text{Sing}^U_\bullet(X)),
\end{array}
\]

where \(i_0\) and \(i_1\) are the inclusion maps and \(\rho = \rho_{\text{Sing}^U_\bullet(X)}\) is the comparison map of Construction 3.3.4.3. Note that the morphisms \(i_0\), \(i_1\), and \(\rho\) are weak homotopy equivalences of simplicial sets (Theorem 3.3.5.1), and therefore also weak homotopy equivalences of semisimplicial sets (Corollary 3.4.5.5). Invoking the two-out-of-three property (Remark 3.4.5.3), we conclude that \(h\) and \(\varphi^U\) are also weak homotopy equivalences of semisimplicial sets. \(\square\)

### 3.4.7 The Seifert van-Kampen Theorem

Let \(X\) be a topological space containing a pair of subsets \(U, V \subseteq X\). If \(X\) is covered by the interiors \(\hat{U}\) and \(\hat{V}\), then Theorem 3.4.6.1 guarantees that the diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Sing}_\bullet(U \cap V) & \longrightarrow & \text{Sing}_\bullet(U) \\
\downarrow & & \downarrow \\
\text{Sing}_\bullet(V) & \longrightarrow & \text{Sing}_\bullet(X)
\end{array}
\]

is homotopy coCartesian. In this section, we apply this assertion of recover several classical results in algebraic topology.

**Theorem 3.4.7.1 (Seifert-van Kampen).** Let \(X\) be a topological space containing a pair of subsets \(U, V \subseteq X\) which satisfy the following conditions:

1. The topological spaces \(U\), \(V\), and \(U \cap V\) are path connected.
2. The interiors \(\hat{U} \subseteq U\) and \(\hat{V} \subseteq V\) comprise an open covering of \(X\).
Then, for every point $x \in U \cap V$, the diagram

$$
\begin{array}{ccc}
\pi_1(U \cap V, x) & \rightarrow & \pi_1(U, x) \\
\downarrow & & \downarrow \\
\pi_1(V, x) & \rightarrow & \pi_1(X, x)
\end{array}
$$

is a pushout square in the category of groups.

We will deduce Theorem 3.4.7.1 from the following variant of Brown ([3]), which does not require any connectivity hypotheses.

**Theorem 3.4.7.2 (Seifert-van Kampen, Groupoid Version).** Let $X$ be a topological space, and let $U, V \subseteq X$ be subsets whose interiors $\hat{U} \subseteq U$ and $\hat{V} \subseteq V$ comprise an open covering of $X$. Then the diagram of fundamental groupoids

$$
\begin{array}{ccc}
\pi_\leq(U \cap V) & \rightarrow & \pi_\leq(U) \\
\downarrow & & \downarrow \\
\pi_\leq(V) & \rightarrow & \pi_\leq(X)
\end{array}
$$

is a pushout square in the (ordinary) category $\text{Cat}$.

**Proof.** Let $\mathcal{C}$ be a category; we wish to show that the diagram of sets $\sigma :$

$$
\begin{array}{ccc}
\text{Hom}_{\text{Cat}}(\pi_\leq(U \cap V), \mathcal{C}) & \leftarrow & \text{Hom}_{\text{Cat}}(\pi_\leq(U), \mathcal{C}) \\
\uparrow & & \uparrow \\
\text{Hom}_{\text{Cat}}(\pi_\leq(V), \mathcal{C}) & \leftarrow & \text{Hom}_{\text{Cat}}(\pi_\leq(X), \mathcal{C})
\end{array}
$$

is a pullback square. Replacing $\mathcal{C}$ by its core $\mathcal{C}^\sim$ (Construction 1.2.4.4), we may assume without loss of generality that $\mathcal{C}$ is a groupoid. Let $N_\bullet(\mathcal{C})$ denote the nerve of $\mathcal{C}$, so that we can identify $\sigma$ with the diagram

$$
\begin{array}{ccc}
\text{Hom}_{\Delta}(\text{Sing}_\bullet(U \cap V), N_\bullet(\mathcal{C})) & \leftarrow & \text{Hom}_{\Delta}(\text{Sing}_\bullet(U), N_\bullet(\mathcal{C})) \\
\uparrow & & \uparrow \\
\text{Hom}_{\Delta}(\text{Sing}_\bullet(V), N_\bullet(\mathcal{C})) & \leftarrow & \text{Hom}_{\Delta}(\text{Sing}_\bullet(X), N_\bullet(\mathcal{C})).
\end{array}
$$

Let $K$ denote the pushout $\text{Sing}_\bullet(U \coprod_{\text{Sing}_\bullet(U \cap V)} \text{Sing}_\bullet(V)$, which we regard as a simplicial subset of $\text{Sing}_\bullet(X)$. Unwinding the definitions, we must show that every morphism of simplicial sets $f : K \rightarrow N_\bullet(\mathcal{C})$ extends uniquely to a map $\overline{f} : \text{Sing}_\bullet(X) \rightarrow N_\bullet(\mathcal{C})$. Note that the inclusion $K \hookrightarrow \text{Sing}_\bullet(X)$ is a weak homotopy equivalence (Theorem 3.4.6.1) and
therefore anodyne (Corollary 3.3.7.5), so the existence of $\bar{f}$ follows from the observation
that $N_\bullet(C)$ is a Kan complex (Proposition 1.2.4.2). To prove uniqueness, suppose that we
are given a pair of maps $\bar{f}, \bar{f}' : \text{Sing}_\bullet(X) \to N_\bullet(C)$ satisfying $\bar{f}|_K = f = \bar{f}'|_K$. It follows
that there exists a homotopy $h : \Delta^1 \times \text{Sing}_\bullet(X) \to N_\bullet(C)$ which is constant when restricted
to $\Delta^1 \times K$. Note that $\bar{f}$ and $\bar{f}'$ can be identified with functors $F, F' : \pi_{\leq 1}(X) \to C$, and
$h$ with a natural transformation of functors $H : F \to F'$. Since every vertex of $\text{Sing}_\bullet(X)$
is contained in $K$, this natural transformation carries each point $x \in X$ to the identity
morphism $id_{F(x)} : F(x) \to F(x) = F'(x)$. It follows that the functors $F$ and $F'$ are identical,
so that the morphisms $f$ and $f'$ are the same.

Proof of Theorem 3.4.7.1. For every group $G$, let us write $BG$ for the groupoid having a
single object with automorphism group $G$ (Example 1.2.4.3). Fix a point $x \in U \setminus V$. To
show that the diagram

$$
\begin{array}{ccc}
\pi_1(U \setminus V, x) & \longrightarrow & \pi_1(U, x) \\
\downarrow & & \downarrow \\
\pi_1(V, x) & \longrightarrow & \pi_1(X, x)
\end{array}
$$

is a pushout square in the category of groups, it will suffice to show that the diagram $\sigma_0$:

$$
\begin{array}{ccc}
B\pi_1(U \setminus V, x) & \longrightarrow & B\pi_1(U, x) \\
\downarrow & & \downarrow \\
B\pi_1(V, x) & \longrightarrow & B\pi_1(X, x)
\end{array}
$$

is a pushout square in the (ordinary) category Cat.

For each point $y \in X$, choose a continuous path $p_y : [0, 1] \to X$ satisfying $p(0) = x$ and
$p(1) = y$. By virtue of our assumption that $U$, $V$, and $U \cap V$ are path connected, we can
arrange that these paths satisfy the following requirements:

- If $y = x$, then $p_y : [0, 1] \to X$ is the constant map taking the value $x$.
- If $y$ is contained in the intersection $U \cap V$, then the path $p_y$ factors through $U \cap V$.
- If $y$ is contained in $U$, then the path $p_y$ factors through $U$.
- If $y$ is contained in $V$, then the path $p_y$ factors through $V$.

Note that, for $W \in \{X, U, V, U \cap V\}$, we can identify $B\pi_1(W, x)$ with the full subcategory
of $\pi_{\leq 1}(W)$ spanned by the point $x$. Let $r_W : \pi_{\leq 1}(W) \to B\pi_1(W, x)$ be the functor which
carries each point of $W$ to the point $x$, and each morphism $\alpha \in \text{Hom}_{\pi_{\leq 1}(W)}(y, z)$ to the
composition $[p_x]^{-1} \circ \alpha \circ [p_y]$ (where $[p_y]$ and $[p_x]$ denote the homotopy classes of the paths
$p_y$ and $p_x$, regarded as morphisms in the fundamental groupoid $\pi_{\leq 1}(W)$). The functors $r_W$
restrict to the identity on \(B\pi_1(W,x)\) and are compatible as \(W\) varies, and therefore exhibit \(\sigma_0\) as a retract of the diagram \(\sigma\):

\[
\begin{array}{ccc}
\pi_{\leq 1}(U \cap V) & \longrightarrow & \pi_{\leq 1}(U) \\
\downarrow & & \downarrow \\
\pi_{\leq 1}(V) & \longrightarrow & \pi_{\leq 1}(X)
\end{array}
\]

in the category \(\text{Fun}([1] \times [1], \text{Cat})\). Since \(\sigma\) is a pushout square (by virtue of Theorem 3.4.7.2), it follows that \(\sigma_0\) is also a pushout square.

If \(X\) is a topological space and \(U \subseteq X\) is a subspace (not necessarily open), we will write \(H_*(X,U;\mathbb{Z})\) for the relative homology groups of the pair \((X,U)\): that is, the homology groups of the quotient chain complex \(C_*(X;\mathbb{Z})/C_*(U;\mathbb{Z})\) (see Example 2.5.5.3).

**Theorem 3.4.7.3** (Excision for Homology). Let \(X\) be a topological space and let \(U,V \subseteq X\) be subsets whose interiors \(\hat{U} \subseteq U\) and \(\hat{V} \subseteq V\) comprise an open covering of \(X\). Then the inclusion \(U \hookrightarrow X\) induces an isomorphism of relative homology groups

\[
H_*(U,U \setminus V;\mathbb{Z}) \rightarrow H_*(X,V;\mathbb{Z}).
\]

**Proof.** Let \(K\) denote the pushout \(\text{Sing}_*(U) \amalg_{\text{Sing}_*(U \setminus V)} \text{Sing}_*(V)\). We then have a commutative diagram of short exact sequences of chain complexes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(V;\mathbb{Z}) & \longrightarrow & C_*(K;\mathbb{Z}) & \longrightarrow & C_*(U;\mathbb{Z})/C_*(U \cap V;\mathbb{Z}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \theta' & & \downarrow \theta & & \\
0 & \longrightarrow & C_*(V;\mathbb{Z}) & \longrightarrow & C_*(X;\mathbb{Z}) & \longrightarrow & C_*(X;\mathbb{Z})/C_*(V;\mathbb{Z}) & \longrightarrow & 0
\end{array}
\]

Consequently, to show that \(\theta\) is a quasi-isomorphism, it will suffice to show that \(\theta'\) is a quasi-isomorphism (Remark 2.5.1.7). This is a special case of Proposition 3.1.5.17 since the inclusion \(K \hookrightarrow \text{Sing}_*(X)\) is a weak homotopy equivalence of simplicial sets (Theorem 3.4.6.1).

**Remark 3.4.7.4** (The Mayer-Vietoris Sequence). Let \(X\) be a topological space, let \(U,V \subseteq X\) be subsets whose interiors \(\hat{U} \subseteq U\) and \(\hat{V} \subseteq V\) comprise an open covering of \(X\), and set \(K = \text{Sing}_*(U) \amalg_{\text{Sing}_*(U \cap V)} \text{Sing}_*(V)\). Then the inclusion \(K \hookrightarrow \text{Sing}_*(X)\) induces a quasi-isomorphism \(C_*(K;\mathbb{Z}) \hookrightarrow C_*(X;\mathbb{Z})\) (by virtue of Theorem 3.4.6.1 and Proposition 3.1.5.17), and we have a short exact sequence of chain complexes

\[
0 \rightarrow C_*(U \cap V;\mathbb{Z}) \rightarrow C_*(U;\mathbb{Z}) \oplus C_*(V;\mathbb{Z}) \rightarrow C_*(K;\mathbb{Z}) \rightarrow 0.
\]
Passing to homology groups (see Construction [?]), we obtain a long exact sequence of abelian groups

\[
\cdots \to H_{s+1}(X; \mathbb{Z}) \xrightarrow{\delta} H_s(U \cap V; \mathbb{Z}) \to H_s(U; \mathbb{Z}) \oplus H_s(V; \mathbb{Z}) \to H_s(X; \mathbb{Z}) \to \cdots
\]

which we refer to as the Mayer-Vietoris sequence of the covering \( \{U, V\} \). The existence of this sequence is essentially equivalent to the statement of Theorem 3.4.7.3.

### 3.5 Comparison with Topological Spaces

Let \( \text{Set}_\Delta \) denote the category of simplicial sets and let \( \text{Top} \) denote the category of topological spaces. In §1.1.7 and §1.1.8, we constructed a pair of adjoint functors

\[
\text{Set}_\Delta \xrightarrow{\ast} \text{Sing} \xleftarrow{\text{Sing} \cdot} \text{Top}.
\]

Our goal in this section is to prove that, after passing to homotopy categories, these functors are not far from being (mutually inverse) equivalences:

**Theorem 3.5.0.1.** The geometric realization functor \( |\bullet| : \text{Set}_\Delta \to \text{Top} \) induces an equivalence from the homotopy category \( h\text{Kan} \) to the full subcategory of \( h\text{Top} \) spanned by those topological spaces \( X \) which have the homotopy type of a CW complex.

Theorem 3.5.0.1 is essentially due to Milnor (see [32]). We give a proof in §3.5.5 which has three main steps. The first of these is of a technical nature: we must show that geometric realization is well-defined at the level of homotopy categories (see Construction 3.5.5.1). Let \( X \) and \( Y \) be Kan complexes, and suppose that we are given a pair of morphisms \( f_0, f_1 : X \to Y \). If \( f_0 \) is homotopic to \( f_1 \) (in the category of Kan complexes), then there exists a morphism of simplicial sets \( h : \Delta^1 \times X \to Y \) satisfying \( f_0 = h|_{\{0\} \times X} \) and \( f_1 = h|_{\{1\} \times X} \). Passing to geometric realizations, we obtain a continuous function \( |h| : \Delta^1 \times |X| \to |Y| \). We would like to interpret \( |h| \) as a homotopy from \( |f_0| \) to \( |f_1| \) (in the category of topological spaces). For this, we need to know that the comparison map

\[
|\Delta^1 \times X| \to |\Delta^1| \times |X| \simeq [0,1] \times |X|
\]

is a homeomorphism. In §3.5.2, we prove a more general assertion: for any pair of simplicial sets \( A \) and \( B \), the comparison map \( |A \times B| \to |A| \times |B| \) is a bijection (Theorem 3.5.2.1), which is a homeomorphism if either \( A \) or \( B \) is finite (that is, if either \( A \) or \( B \) has only finitely many nondegenerate simplices; see Corollary 3.5.2.2).

The second step in the proof of Theorem 3.5.0.1 is to show that the geometric realization functor \( |\bullet| : h\text{Kan} \to h\text{Top} \) is fully faithful (Proposition 3.5.5.2). This is equivalent to the...
assertion that for any Kan complex $X$, the unit map $u_X : X \to \text{Sing}_\bullet(|X|)$ is a homotopy equivalence. More generally, we show in §3.5.4 that for any simplicial set $X$, the unit map $u_X : X \to \text{Sing}_\bullet(|X|)$ is a weak homotopy equivalence (Theorem 3.5.4.1). Our strategy is to reduce to the case where the simplicial set $X$ is finite, and to proceed by induction on the number of nondegenerate simplices of $X$. The inductive step will make use of excision (Theorem 3.4.6.1) to analyze the homotopy type of the Kan complex $\text{Sing}_\bullet(|X|)$.

To complete the proof of Theorem 3.5.0.1, we must show that if $Y$ is a topological space, then the counit map $v_Y : |\text{Sing}_\bullet(Y)| \to Y$ is a homotopy equivalence if and only if $Y$ has the homotopy type of a CW complex (Proposition 3.5.5.3). It follows formally from the preceding step that the map $v_Y$ is always a weak homotopy equivalence: that is, it induces a bijection on path components and an isomorphism on homotopy groups for any choice of base point (Corollary 3.5.4.2). We will complete the proof using a result of Whitehead which asserts that any weak homotopy equivalence between CW complexes is a homotopy equivalence (see Proposition 3.5.3.8 and Corollary 3.5.3.10), which we prove in §3.5.3.

### 3.5.1 Digression: Finite Simplicial Sets

We now introduce a finiteness condition on simplicial sets.

**Definition 3.5.1.1.** We say that a simplicial set $X$ is *finite* if it satisfies the following pair of conditions:

- For every integer $n \geq 0$, the set of $n$-simplices $X_n \simeq \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, X)$ is finite.
- The simplicial set $X$ is finite-dimensional (Definition 1.1.3.9): that is, there exists an integer $m$ such that every nondegenerate simplex has dimension $\leq m$.

**Example 3.5.1.2.** For each integer $n \geq 0$, the standard $n$-simplex $\Delta^n$ is finite.

**Remark 3.5.1.3.** Let $X$ be a finite simplicial set. Then any simplicial subset $Y \subseteq X$ is also finite. In particular, any retract of $X$ is finite.

**Remark 3.5.1.4.** If $X$ and $Y$ are finite simplicial sets, then the coproduct $X \amalg Y$ is also finite.

**Remark 3.5.1.5.** Let $f : X \to Y$ be an epimorphism of simplicial sets. If $X$ is finite, then $Y$ is also finite.

**Remark 3.5.1.6.** Let $X$ and $Y$ be finite simplicial sets. Then the product $X \times Y$ is finite (see Proposition 1.1.3.11).

**Proposition 3.5.1.7.** Let $X$ be a simplicial set. The following conditions are equivalent:

(a) The simplicial set $X$ has only finitely many nondegenerate simplices.
(b) There exists an epimorphism of simplicial sets \( f : Y \to X \), where \( Y \simeq \coprod_{i \in I} \Delta^{n_i} \) is a finite coproduct of standard simplices.

(c) The simplicial set \( X \) is finite (Definition 3.5.1.1).

Proof. If \( X \) is finite, then it has dimension \( \leq n \) for some integer \( n \gg 0 \). It follows that every nondegenerate simplex of \( X \) has dimension \( \leq n \). Since \( X \) has only finitely many (nondegenerate) simplices of each dimension, it follows that \( X \) has only finitely many nondegenerate simplices. This proves that \( (c) \Rightarrow (a) \). The implication \( (b) \Rightarrow (c) \) follows from Example 3.5.1.2 together with Remarks 3.5.1.4 and 3.5.1.5. We will complete the proof by showing that \( (a) \Rightarrow (b) \). Let \( \{ \sigma_i : \Delta^{n_i} \to X \}_{i \in I} \) be the collection of all nondegenerate simplices of \( X \), and amalgamate the morphisms \( \sigma_i \) to a single map \( f : Y = \coprod_{i \in I} \Delta^{n_i} \to X \).

By construction, every nondegenerate simplex of \( X \) belongs to the image of \( f \) and therefore every simplex of \( f \) belongs to the image of \( f \) (see Proposition 1.1.3.4). It follows that \( f \) is an epimorphism of simplicial sets. If condition \( (a) \) is satisfied, then the set \( I \) is finite, so that \( f : Y \to X \) satisfies the requirements of \( (b) \).

Remark 3.5.1.8. Every simplicial set \( X \) can be realized as a union \( \bigcup_{X' \subseteq X} X' \), where \( X' \) ranges over the collection of finite simplicial subsets of \( X \) (to prove this, we observe that every \( n \)-simplex \( \sigma \) is contained in a finite simplicial subset \( X' \subseteq X \): in fact, we can take \( X' \) to be the image of \( \sigma : \Delta^n \to X \). Moreover, the collection of finite simplicial subsets of \( X \) is closed under finite unions. It follows that realization \( X \simeq \bigcup_{X' \subseteq X} X' \) exhibits \( X \) as a filtered direct limit of its finite simplicial subsets.

Let \( X \) be a simplicial set having geometric realization \( |X| \). For every simplicial subset \( X' \subseteq X \), the inclusion of \( X' \) into \( X \) induces a homeomorphism from \( |X'| \) onto a closed subset of \( |X| \). In what follows, we will abuse notation by identifying \( |X'| \) with its image in \( |X| \).

Proposition 3.5.1.9. Let \( X \) be a simplicial set. Then a subset \( K \subseteq |X| \) is compact if and only if it is closed and contained in \( |X'| \subseteq |X| \), for some finite simplicial subset \( X' \subseteq X \).

Corollary 3.5.1.10. A simplicial set \( X \) is finite if and only if the topological space \( |X| \) is compact.

The proof of Proposition 3.5.1.9 is based on the following observation:

Lemma 3.5.1.11. Let \( X \) be a simplicial set and let \( S \) be a subset of the geometric realization \( |X| \). Suppose that, for every nondegenerate \( n \)-simplex \( \sigma \) of \( X \), the inverse image of \( S \) under the composite map \( |\Delta^n| \to |X| \) contains only finitely many points of the interior \( |\Delta^n| \subseteq |\Delta^n| \).

Then \( S \) is closed.

Proof. The geometric realization \( |X| \) can be described as the colimit \( \lim_{\Delta^n \to X} |\Delta^n| \), indexed by the category of simplices of \( X \) (see Construction 1.1.8.19). Consequently, to show that
the subset \( S \subseteq |X| \) is closed, it will suffice to show that the inverse image \( |\sigma|^{-1}(S) \subseteq |\Delta^n| \) is closed, for every \( n \)-simplex \( \sigma : \Delta^n \to X \). We proceed by induction on \( n \). Using Proposition \ref{1.1.3.4}, we can reduce to the case where \( \sigma \) is nondegenerate. In this case, our inductive hypothesis guarantees that \( |\sigma|^{-1}(S) \) has closed intersection with the boundary \( |\partial\Delta^n| \subseteq |\Delta^n| \).

\[ \text{Proof of Proposition 3.5.1.9.} \]

Let \( X \) be a simplicial set. If \( X' \subseteq X \) is a finite simplicial subset, then the geometric realization \( |X'| \) is a continuous image of a finite disjoint union \( \coprod_{i \in I} |\Delta^{n_i}| \) (Proposition \ref{3.5.1.7}), and is therefore compact. It follows that any closed subset \( K \subseteq |X'| \) is also compact. Conversely suppose that \( K \subseteq |X| \) is compact. Since \( |X| \) is Hausdorff, the set \( K \) is closed. We wish to show that \( K \) is contained in \( |X'| \) for some finite simplicial subset \( X' \subseteq X \). Suppose otherwise. Then we can choose an infinite collection of nondegenerate simplices \( \{\sigma_j : \Delta^{n_j} \to X\}_{j \in J} \) for which each of the corresponding cells \( |\Delta^{n_j}| \hookrightarrow |X| \) contains some point \( s_j \in S \). Applying Lemma \ref{3.5.1.11} we deduce that for every subset \( J' \subseteq J \), the set \( \{s_j\}_{j \in J'} \) is closed in \( |X| \). In particular, \( \{s_j\}_{j \in J} \) is an infinite closed subset of \( S \) endowed with the discrete topology, contradicting our assumption that \( S \subseteq |X| \) is compact.

\[ \text{Proof of Proposition 3.5.2.1.} \]

Our goal in this section is to study the exactness properties of the geometric realization functor \( X \mapsto |X| \) of Definition \ref{1.1.8.1}. Our main result can be stated as follows:

\[ \text{Theorem 3.5.2.1.} \]

The geometric realization functor

\[ \text{Set}_\Delta \to \text{Set} \quad X \mapsto |X| \]

preserves finite limits. In particular, for every diagram of simplicial sets \( X \to Z \leftarrow Y \), the induced map \( |X \times_Z Y| \to |X| \times |Y| \) is a bijection.

Before giving the proof of Theorem \ref{3.5.2.1} let us collect some consequences.

\[ \text{Corollary 3.5.2.2.} \]

Let \( X \) and \( Y \) be simplicial sets. Then the canonical map \( \theta_{X,Y} : |X \times Y| \to |X| \times |Y| \) is a bijection. If either \( X \) or \( Y \) is finite, then \( \theta \) is a homeomorphism.

\[ \text{Proof.} \]

The first assertion follows immediately from Theorem \ref{3.5.2.1}. If \( X \) and \( Y \) are both finite, then the product \( X \times Y \) is also finite (Remark \ref{3.5.1.6}), so that the geometric realizations \( |X|, |Y|, \) and \( |X \times Y| \) are compact Hausdorff spaces (Corollary \ref{3.5.1.10}). In this case, \( \theta_{X,Y} \) is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.
CHAPTER 3. KAN COMPLEXES

Now suppose that $X$ is finite and $Y$ is arbitrary. Let $M = \text{Hom}_{\text{Top}}(|X|, |X \times Y|)$ denote the set of all continuous functions from $|X|$ to $|X \times Y|$, endowed with the compact-open topology. For every finite simplicial subset $Y' \subseteq Y$, the composite map

$$|X| \times |Y'| \xrightarrow{\theta_{X,Y'}} |X \times Y'| \hookrightarrow |X \times Y|,$$

determines a continuous function $\rho_{Y'} : |Y'| \to M$. Writing the geometric realization $|Y|$ as a colimit $\lim_{Y' \subset Y} |Y'|$ (see Remark 3.5.1.8), we can amalgamate the functions $f_{Y'}$ to a single continuous function $\rho : |Y| \to M$. Our assumption that $X$ guarantees that the topological space $|X|$ is compact and Hausdorff, so the evaluation map

$$\text{ev} : |X| \times M \to |X \times Y| \quad (x,f) \mapsto f(x)$$

is continuous (see Theorem [?]). We complete the proof by observing that the bijection $\theta_{X,Y'}^{-1}$ is a composition of continuous functions

$$|X| \times |Y| \xrightarrow{id \times \rho} |X| \times M \xrightarrow{\text{ev}} |X \times Y|,$$

and is therefore continuous.

Warning 3.5.2.3. Let $X$ and $Y$ be simplicial sets. In general, the comparison map $\theta_{X,Y} : |X \times Y| \to |X| \times |Y|$ need not be a homeomorphism if neither $X$ or $Y$ is assumed to be finite. For an explicit counterexample, we refer the reader to Section 5 of [9].

Remark 3.5.2.4. Let $X$ and $Y$ be simplicial sets having at most countably many simplices of each dimension. Then the comparison map $\theta_{X,Y} : |X \times Y| \to |X| \times |Y|$ is a homeomorphism. For a proof, we refer the reader to [32].

Example 3.5.2.5. Let $X$ be a simplicial set and let $Y$ be a topological space, and let $\text{Hom}_{\text{Top}}(|X|, Y_\bullet)$ be the simplicial set defined in Example 2.4.1.5. For each $n \geq 0$, precomposition with the homeomorphism $|X \times \Delta^n| \to |X| \times |\Delta^n|$ induces a bijection

$$\text{Hom}_{\text{Top}}(|X|, Y_\bullet)_n = \text{Hom}_{\text{Top}}(|X| \times |\Delta^n|, Y)$$

$$\simeq \text{Hom}_{\text{Top}}(|X \times \Delta^n|, Y)$$

$$\simeq \text{Hom}_{\text{Set}_\Delta}(X \times \Delta^n, \text{Sing}_\bullet(Y))$$

$$= \text{Fun}(X, \text{Sing}_\bullet(Y))_n.$$

These bijections are compatible with face and degeneracy operators, and therefore determine an isomorphism of simplicial sets $\text{Hom}_{\text{Top}}(|X|, Y_\bullet) \to \text{Fun}(X, \text{Sing}_\bullet(Y))$.

We now turn to the proof of Theorem 3.5.2.1. Our proof will make use of an explicit description of the underlying set of a geometric realization $|X|$ (see Remark 3.5.2.10) which given by Drinfeld in [10] (and also appears in unpublished work of Besser and Grayson).
Construction 3.5.2.6. Let $S$ be a finite subset of the unit interval $[0,1]$, and assume that $0, 1 \in S$. For each $n \geq 0$, we let $|\Delta^n|_S$ denote the subset of the topological $n$-simplex

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}_{\geq 0}^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}$$

consisting of those tuples $(t_0, t_1, \ldots, t_n)$ having the property that each of the partial sums $t_0 + t_1 + \cdots + t_i$ belongs to $S$. Note that these subsets are stable under the coface and codegeneracy operators of the cosimplicial topological space $|\Delta^\bullet|$, so we can regard the construction $[n] \mapsto |\Delta^n|_S$ as a cosimplicial set.

By virtue of Proposition 1.1.8.22, the functor

$$\textbf{Set} \rightarrow \textbf{Set} \quad Y \mapsto ([n] \mapsto \text{Hom}_{\text{Set}}(|\Delta^n|_S, Y))$$

admits a left adjoint, which we will denote by $|\bullet|_S : \Delta \rightarrow \textbf{Set}$ and refer to as the $S$-partial geometric realization. Concretely, this functor carries a simplicial set $X$ to the colimit $|X|_S = \lim_{\Delta^n \rightarrow X} |\Delta^n|_S$, where the colimit is indexed by the category of simplices $\Delta_X$ of Construction 1.1.8.19.

Remark 3.5.2.7. For each integer $n \geq 0$, the topological $n$-simplex $|\Delta^n|$ can be identified with the filtered direct limit $\lim_{\rightarrow S} |\Delta^n|_S$, where $S$ ranges over the collection of all finite subsets of $[0,1]$ which contain the endpoints 0 and 1 (which we regard as a partially ordered set with respect to inclusion). We therefore obtain a canonical isomorphism of cosimplicial sets $\lim_{\rightarrow S} |\Delta^\bullet|_S \cong |\Delta^\bullet|$. It follows that, for every simplicial set $X$, the canonical map $\lim_{\rightarrow S} |X|_S \rightarrow |X|$ is a bijection.

Notation 3.5.2.8. Let $\text{Lin}_{\neq \emptyset}$ denote the category whose objects are nonempty finite linearly ordered sets, and whose morphisms are nondecreasing functions. Note that, if $S$ is a finite subset of the unit interval $[0,1]$, then the complement $[0,1] \setminus S$ has finitely many connected components. Moreover, there is a unique linear ordering on the set $\pi_0([0,1] \setminus S)$ for which the quotient map

$$([0,1] \setminus S) \rightarrow \pi_0([0,1] \setminus S)$$

is nondecreasing. We can therefore regard $\pi_0([0,1] \setminus S)$ as an object of the category $\text{Lin}_{\neq \emptyset}$.

Proposition 3.5.2.9. Let $S$ be a finite subset of the unit interval $[0,1]$ which contains 0 and 1. Then the cosimplicial set

$$|\Delta^\bullet|_S : \Delta \rightarrow \textbf{Set} \quad [n] \mapsto |\Delta^n|_S$$

is a corepresentable functor. More precisely, there exists a functorial bijection $|\Delta^n|_S \simeq \text{Hom}_{\text{Lin}_{\neq \emptyset}}(\pi_0([0,1] \setminus S), [n])$. 
Proof. Let \( S = \{ 0 = s_0 < s_1 < \cdots < s_k = 1 \} \) be a finite subset of the unit interval \([0, 1]\) which contains 0 and 1. Let \( n \) be a nonnegative integer and let \((t_0, \ldots, t_n)\) be a point of \( |\Delta^n|_S \). For every real number \( u \in [0, 1] \setminus S \), there exists a unique integer \( 0 \leq i \leq n \) satisfying

\[
t_0 + t_1 + \cdots + t_{i-1} < u < t_0 + t_1 + \cdots + t_i.
\]

The construction \( u \mapsto i \) defines a continuous nondecreasing function \(([0, 1] \setminus S) \to [n]\). This observation induces a bijection \( |\Delta^n|_S \simeq \{ \text{Continuous nondecreasing functions } f : [0, 1] \setminus S \to [n] \} \simeq \text{Hom}_{\text{Lin}^\not\emptyset} (\pi_0([0, 1] \setminus S), [n]) \).

Explicitly, the inverse bijection carries a continuous nondecreasing function \( f : [0, 1] \setminus S \to [n] \) to the sequence

\[
(\mu(f^{-1}\{0\}), \mu(f^{-1}\{1\}), \ldots, \mu(f^{-1}\{n\})),
\]

where

\[
\mu(f^{-1}\{i\}) = \sum_{(s_{j-1}, s_j) \subseteq f^{-1}\{i\}} (s_j - s_{j-1})
\]
denotes the measure of the inverse image \( f^{-1}\{i\} \).

\[
\square
\]

Proof of Theorem 3.5.2.1. Let \( U : \text{Top} \to \text{Set} \) denote the forgetful functor. We wish to show that the composite functor \( \text{Set} \xrightarrow{\Delta} \text{Top} \xrightarrow{U} \text{Set} \) preserves finite limits. By virtue of Remark 3.5.2.7, we can write this composite functor as a filtered colimit of functors of the form \( X \mapsto |X|_S \), where \( S \) ranges over all finite subsets of the unit interval \([0, 1]\) which contain 0 and 1. It will therefore suffice to show that each of the functors \( X \mapsto |X|_S \) preserves finite limits. Using Proposition 3.5.2.9, see that \( X \mapsto |X|_S \) can be identified with the evaluation functor \( X \mapsto X_m \), where \( m \) is chosen so that there is an isomorphism of linearly ordered sets \([m] \simeq \pi_0([0, 1] \setminus S)\).

\[
\square
\]

Remark 3.5.2.10. Let \( X \) be a simplicial set, which we view as a functor from \( \Delta^\text{op} \) to the category of sets. Then \( X \) admits a canonical extension to a functor \( \text{Lin}^\not\emptyset \to \text{Set} \), given on objects by the construction \((I = \{i_0 < i_1 < \cdots < i_n\}) \mapsto X_n \). Let us write \( X(I) \) for the value of this extension on an object \( I \in \text{Lin}^\not\emptyset \). Arguing as in the proof of Theorem 3.5.2.1, we obtain a canonical bijection

\[
\lim_{S} \frac{X([0, 1] \setminus S)}{S'} \simeq \lim_{S} \frac{|X|_S}{S} \simeq X,
\]

where the (filtered) colimit is taken over the collection of all finite subsets \( S \subseteq [0, 1] \) containing 0 and 1.
3.5.3 Weak Homotopy Equivalences in Topology

Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a continuous function. Recall that $f$ is a homotopy equivalence if there exists a continuous function $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps $\text{id}_X$ and $\text{id}_Y$, respectively. In other words, $f$ is a homotopy equivalence if its homotopy class $[f]$ is invertible when regarded as a morphism in the homotopy category of topological spaces $\text{hTop}$ (see Example 2.4.6.5). For some purposes, it is convenient to consider a somewhat weaker condition.

Definition 3.5.3.1. Let $X$ and $Y$ be topological spaces. We say that a continuous function $f : X \to Y$ is a weak homotopy equivalence if the induced map of singular simplicial sets $\text{Sing}_\bullet(f) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y)$ is a homotopy equivalence (Definition 3.1.5.1).

Remark 3.5.3.2. Let $f : X \to Y$ be a continuous function between topological spaces. Then $f$ is a weak homotopy equivalence of topological spaces if and only if $\text{Sing}_\bullet(f)$ is a weak homotopy equivalence of simplicial sets. This is a special case of Proposition 3.1.5.12, since the simplicial sets $\text{Sing}_\bullet(X)$ and $\text{Sing}_\bullet(Y)$ are Kan complexes (Proposition 1.1.9.8).

Example 3.5.3.3. Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a homotopy equivalence. Then $f$ is a weak homotopy equivalence. This is a reformulation of Example 3.1.5.2.

Remark 3.5.3.4. Let $f : X \to Y$ be a continuous function between topological spaces. Then $f$ is a weak homotopy equivalence if and only if it satisfies the following pair of conditions:

- The induced map of path components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.
- For every point $x \in X$ and every $n \geq 1$, the map of homotopy groups $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y,f(x))$ is an isomorphism.

This follows by applying Theorem 3.2.6.1 to the map of Kan complexes $\text{Sing}_\bullet(f) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y)$ (see Example 3.2.2.7).

Example 3.5.3.5. We say that a topological space $X$ is weakly contractible if the projection map $f : X \to \ast$ is a weak homotopy equivalence (in other words, $X$ is weakly contractible if the singular simplicial set $\text{Sing}_\bullet(X)$ is a contractible Kan complex). Using Remark 3.5.3.4 we see that $X$ is weakly contractible if and only if it is path connected (that is, the set $\pi_0(X)$ is a singleton) and the homotopy groups $\pi_n(X,x)$ are trivial for $n > 0$ and any choice of base point $x \in X$ (assuming that $X$ is path connected, this condition is independent of the choice of base point).
Remark 3.5.3.6. Recall that a topological space $X$ is contractible if the projection map $X \to *$ is a homotopy equivalence. Equivalently, $X$ is contractible if the identity map $\text{id}_X : X \to X$ is homotopic to the constant function $X \to \{x\} \to X$, for some base point $x \in X$. It follows from Example 3.5.3.3 that every contractible topological space is weakly contractible. In particular, for each $n \geq 0$, the standard simplex $|\Delta^n|$ is weakly contractible.

Example 3.5.3.7. Let $X$ be a topological space with the property that every continuous path $p : [0, 1] \to X$ is constant (this condition is satisfied, for example, if $X$ is totally disconnected). Let $X'$ denote the topological space whose underlying set coincides with $X$, but endowed with the discrete topology. Then the identity map $f : X' \to X$ induces an isomorphism of singular simplicial sets $\text{Sing}_\bullet(X') \to \text{Sing}_\bullet(X)$, and is therefore a weak homotopy equivalence of topological spaces. However, $f$ is a homotopy equivalence if and only if the topology on $X$ is discrete (since any homotopy inverse of $f$ must coincide with the identity map $f^{-1} : X \to X'$).

Example 3.5.3.7 illustrates that the notions of homotopy equivalence and weak homotopy equivalence are not the same in general. However, they agree for sufficiently nice topological spaces.

Proposition 3.5.3.8. Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces. Assume that both $X$ and $Y$ have the homotopy type of a CW complex (that is, there exist homotopy equivalences $X' \to X$ and $Y' \to Y$, where $X'$ and $Y'$ are CW complexes). Then $f$ is a homotopy equivalence.

Warning 3.5.3.9. In the formulation of Proposition 3.5.3.8, the hypothesis that $X$ and $Y$ have the homotopy type of a CW complex cannot be omitted. For any topological space $Y$, the counit map $v : |\text{Sing}_\bullet(Y)| \to Y$ is a weak homotopy equivalence (Corollary 3.5.4.2), whose domain is a CW complex (Remark 1.1.8.14). If $Y$ satisfies the conclusion of Proposition 3.5.3.8 then $v$ is a homotopy equivalence, so $Y$ has the homotopy type of a CW complex.

Corollary 3.5.3.10 (Whitehead’s Theorem for Topological Spaces). Let $X$ and $Y$ be topological spaces having the homotopy type of CW complexes, and let $f : X \to Y$ be a continuous function. Then $f$ is a homotopy equivalence if and only if it satisfies the following pair of conditions:

- The induced map of path components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.
- For every point $x \in X$ and every $n \geq 1$, the map of homotopy groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.

Proof. Combine Remark 3.5.3.4 with Proposition 3.5.3.8 and Example 3.5.3.3. □
3.5. COMPARISON WITH TOPOLOGICAL SPACES

We will deduce Proposition 3.5.3.8 from the following:

**Lemma 3.5.3.11.** Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces, let $K$ be a CW complex, and let $g : K \to Y$ be a continuous function. Then there exists a continuous function $\overline{g} : K \to X$ such that $g$ is homotopic to $f \circ \overline{g}$.

*Proof.* For each $n \geq -1$, let $\text{sk}_n(K)$ denote the $n$-skeleton of $K$ (with respect to some fixed cell decomposition), so that $\text{sk}_{-1}(K) = \emptyset$. To prove Lemma 3.5.3.11, it will suffice to construct a compatible sequence of continuous functions $\overline{g}_n : \text{sk}_n(K) \to X$ and homotopies $h_n : [0,1] \times \text{sk}_n(K) \to Y$ from $\overline{g}_n$ to $g|_{\text{sk}_n(K)}$. We proceed by recursion. Assume that $n \geq 0$ and that the pair $(\overline{g}_{n-1}, h_{n-1})$ has already been constructed. Let $S$ denote the collection of $n$-cells of $K$. For each $s \in S$, let $b_s : |\partial \Delta^n| \to \text{sk}_{n-1}(K)$ denote the corresponding attaching map. To construct the pair $(\overline{g}_n, h_n)$, it will suffice to show that each composition $\overline{g}_{n-1} \circ b_s$ can be extended to a continuous map $u_s : |\Delta^n| \to X$ and that each composition $h_{n-1} \circ (b_s \times \text{id}_{[0,1]})$ can be extended to a homotopy from $u_s$ to $g|_{|\Delta^n|}$. Unwinding the definitions, we can rephrase this as a lifting problem

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \text{Sing}_\bullet(X) \times_{\text{Fun}(\{0\}, \text{Sing}_\bullet(Y))} \text{Fun}(\Delta^1, \text{Sing}_\bullet(Y)) \\
& & \theta \\
\Delta^n & \longrightarrow & \text{Fun}(\{1\}, \text{Sing}_\bullet(Y))
\end{array}
$$

in the category of simplicial sets. Here the morphism $\theta$ is the path fibration of Example 3.1.6.10 (associated to the map of Kan complexes $\text{Sing}_\bullet(f) : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(Y)$). Our assumption that $f$ is a weak homotopy equivalence guarantees that $\text{Sing}_\bullet(f)$ is a homotopy equivalence of Kan complexes, so that $\theta$ is also a homotopy equivalence. Applying Corollary 3.2.6.9 we deduce that $\theta$ is a trivial Kan fibration, so that the lifting problem admits a solution as desired. □

*Proof of Proposition 3.5.3.8.* In what follows, we denote the homotopy class of a continuous function $f : X \to Y$ by $[f]$. Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces, and suppose that there exists a homotopy equivalence $u : Y' \to Y$, where $Y'$ is a CW complex. Using Lemma 3.5.3.11 we deduce that $[u] = [f] \circ [\overline{u}]$ for some continuous function $\overline{u} : Y' \to X$. Let $v : Y \to Y'$ be a homotopy inverse to $u$ and set $g = \overline{u} \circ v$. Then

$$
[f] \circ [g] = [f \circ \overline{u}] \circ [v] = [u] \circ [v] = [\text{id}_Y],
$$

so $g$ is a right homotopy inverse to $f$. Since $f$ is a weak homotopy equivalence, it follows that $g$ is also a weak homotopy equivalence. If $X$ also has the homotopy type of a CW complex, then we can apply the same reasoning to deduce that $g$ admits a right homotopy inverse $f' : X \to Y$. Then

$$
[g] \circ [f] = [g] \circ [f] \circ [\text{id}_X] = [g] \circ [f] \circ [g] \circ [f'] = [g] \circ [\text{id}_Y] \circ [f'] = [g] \circ [f'] = [\text{id}_X].
$$
It follows that \( g \) is also a left homotopy inverse to \( f \), so that \( f \) is a homotopy equivalence (with homotopy inverse \( g \)).

### 3.5.4 The Unit Map \( u : X \to \text{Sing}_\bullet(|X|) \)

Our goal in this section is to prove the following result:

**Theorem 3.5.4.1** (Milnor). Let \( X \) be a simplicial set. Then the unit map \( u_X : X \to \text{Sing}_\bullet(|X|) \) is a weak homotopy equivalence of simplicial sets.

Theorem 3.5.4.1 was proved by Milnor in [32]. It is closely related to the following earlier result of Giever ([18]):

**Corollary 3.5.4.2.** Let \( X \) be a topological space. Then the counit map \( v_X : |\text{Sing}_\bullet(X)| \to X \) is a weak homotopy equivalence of topological spaces.

**Proof.** We must show that \( \text{Sing}_\bullet(v_X) : \text{Sing}_\bullet(|\text{Sing}_\bullet(X)|) \to \text{Sing}_\bullet(X) \) is a homotopy equivalence of Kan complexes. This is clear, since \( \text{Sing}_\bullet(v_X) \) is left inverse to the unit map \( u_{\text{Sing}_\bullet(X)} : \text{Sing}_\bullet(X) \to \text{Sing}_\bullet(|\text{Sing}_\bullet(X)|) \), which is a weak homotopy equivalence by virtue of Theorem 3.5.4.1 (and therefore a homotopy equivalence, since both \( \text{Sing}_\bullet(X) \) and \( \text{Sing}_\bullet(|\text{Sing}_\bullet(X)|) \) are Kan complexes).

**Proof of Theorem 3.5.4.1.** Let \( X \) be a simplicial set. By virtue of Remark 3.5.1.8 we can write \( X \) as a filtered colimit of finite simplicial subsets \( X' \subseteq X \). It follows from Proposition 3.5.1.9 that, for any compact topological space \( K \), every continuous function \( f : K \to |X| \) factors through \( |X'| \subseteq |X| \) for some finite simplicial subset \( X' \subseteq X \). Applying this observation in the case \( K = |\Delta^n| \), we conclude that the natural map \( \lim_{X' \subseteq X} \text{Sing}_\bullet(|X'|) \to \text{Sing}_\bullet(|X|) \) is an isomorphism of simplicial sets. It follows that the unit map \( u_X : X \to \text{Sing}_\bullet(|X|) \) can be realized as filtered colimit of unit maps \( u_{X'} : X' \to \text{Sing}_\bullet(|X'|) \). Since the collection of weak homotopy equivalences is closed under filtered colimits (Proposition 3.2.7.3), it will suffice to show that each of the morphisms \( u_{X'} \) is a weak homotopy equivalence. Replacing \( X \) by \( X' \), we are reduced to proving Theorem 3.5.4.1 under the additional assumption that the simplicial set \( X \) is finite.

We now proceed by induction on the dimension of \( X \). If \( X \) is empty, then \( u_X \) is an isomorphism and the result is obvious. Otherwise, let \( n \geq 0 \) be the dimension of \( X \). We proceed by induction on the number of nondegenerate \( n \)-simplices of \( X \). Using Proposition 1.1.3.13 we can choose a pushout diagram

\[
\begin{array}{ccc}
\partial \Delta^n & \to & \Delta^n \\
\downarrow & & \downarrow \\
X' & \to & X,
\end{array}
\]

(3.24)
where $X'$ is a simplicial subset of $X$ with a smaller number of nondegenerate $n$-simplices. Since the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ is a monomorphism, the diagram (3.24) is also a homotopy pushout square (Proposition 3.4.2.6). By virtue of our inductive hypotheses, the unit morphisms $u_{X'}$ and $u_{\partial\Delta^n}$ are weak homotopy equivalences. Since the simplicial sets $\Delta^n$ and $\operatorname{Sing}_\bullet(|\Delta^n|)$ are contractible (Remark 3.2.6.5), the unit map $u_{\Delta^n}$ is also a (weak) homotopy equivalence. Invoking Proposition 3.4.2.4, we see that $u_X$ is a homotopy equivalence if and only if the diagram of simplicial sets

$$
\begin{array}{ccc}
\operatorname{Sing}_\bullet(|\partial\Delta^n|) & \longrightarrow & \operatorname{Sing}_\bullet(|\Delta^n|) \\
\downarrow & & \downarrow \\
\operatorname{Sing}_\bullet(|X'|) & \longrightarrow & \operatorname{Sing}_\bullet(|X|),
\end{array}
$$

is also homotopy pushout square.

Let $V = |\Delta^n| \setminus |\partial\Delta^n|$ be the interior of the topological $n$-simplex, and fix a point $v \in V$ having image $x \in |X|$. We then have a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\operatorname{Sing}_\bullet(V \setminus \{v\}) & \longrightarrow & \operatorname{Sing}_\bullet(V) \\
\downarrow & & \downarrow \\
\operatorname{Sing}_\bullet(|\partial\Delta^n|) & \longrightarrow & \operatorname{Sing}_\bullet(|\Delta^n| \setminus \{v\}) \longrightarrow \operatorname{Sing}_\bullet(|\Delta^n|) \\
\downarrow & & \downarrow \\
\operatorname{Sing}_\bullet(|X'|) & \longrightarrow & \operatorname{Sing}_\bullet(|X| \setminus \{x\}) \longrightarrow \operatorname{Sing}_\bullet(|X|).
\end{array}
$$

Note that the left horizontal maps and the upper vertical maps are homotopy equivalences, since they are obtained from homotopy equivalences of topological spaces

$$
|X'| \hookrightarrow |X| \setminus \{x\} \quad |\partial\Delta^n| \hookrightarrow |\Delta^n| \setminus \{v\} \hookrightarrow V \setminus \{v\} \quad |\Delta^n| \hookrightarrow V
$$

(see Example 3.5.3.3). It follows that the upper square and left square in diagram (3.26) are homotopy coCartesian (Proposition 3.4.2.5). Moreover, the outer rectangle on the right is homotopy coCartesian by virtue of Theorem 3.4.6.1. Applying Proposition 3.4.1.9, we deduce that the lower right square and bottom rectangle are also homotopy coCartesian.

### 3.5.5 Comparison of Homotopy Categories

Our goal in this section is to carry out the proof of Theorem 3.5.0.1. We begin with an elementary application of the results of §3.5.2.

**Construction 3.5.5.1** (Geometric Realization as a Simplicial Functor). Let $X$ and $Y$ be simplicial sets and let $\sigma$ be an $n$-simplex of the simplicial set $\operatorname{Fun}(X,Y)$, which we identify
with a morphism $\Delta^n \times X \to Y$. By virtue of Corollary 3.5.2.2, the geometric realization of $\sigma$ can be identified with a continuous function

$$|\sigma| : |\Delta^n| \times |X| \to |Y|,$$

which we can view as an $n$-simplex of the simplicial set $\text{Hom}_{\text{Top}}(|X|, |Y|)_\bullet$ parametrizing continuous functions from $X$ to $Y$ (see Example 2.4.1.5). This construction is compatible with face and degeneracy operators, and therefore determines a morphism of simplicial sets $\text{Fun}(X,Y) \to \text{Hom}_{\text{Top}}(|X|, |Y|)_\bullet$. Allowing $X$ and $Y$ to vary, we obtain a simplicial structure on the geometric realization functor $|\bullet| : \text{Set} \to \text{Top}$.

**Proposition 3.5.5.2.** Let $X$ and $Y$ be simplicial sets. If $Y$ is a Kan complex, then the comparison map

$$\theta : \text{Fun}(X,Y) \to \text{Hom}_{\text{Top}}(|X|, |Y|)_\bullet$$

of Construction 3.5.5.1 is a homotopy equivalence of Kan complexes.

**Proof.** Using Example 3.5.2.5 we can identify $\theta$ with the morphism

$$\text{Fun}(X,Y) \to \text{Fun}(X, \text{Sing}_\bullet(|Y|))$$

given by postcomposition with the unit map $u_Y : Y \to \text{Sing}_\bullet(|Y|)$. By virtue of Theorem 3.5.4.1 the map $u_Y$ is a weak homotopy equivalence. Since $Y$ and $\text{Sing}_\bullet(|Y|)$ are Kan complexes, we conclude that $u_Y$ is a homotopy equivalence (Proposition 3.1.5.12). It follows that $\theta$ is also a homotopy equivalence (it admits a homotopy inverse, given by postcomposition with any homotopy inverse to $u_Y$).

**Proposition 3.5.5.3.** Let $X$ be a topological space. The following conditions are equivalent:

1. The counit map $|\text{Sing}_\bullet(X)| \to X$ is a homotopy equivalence of topological spaces.
2. There exists a Kan complex $Y$ and a homotopy equivalence of topological spaces $|Y| \to X$.
3. There exists a simplicial set $Y$ and a homotopy equivalence of topological spaces $|Y| \to X$.
4. There exists a homotopy equivalence of topological spaces $X' \to X$, where $X'$ is a CW complex.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from the observation that $\text{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.1.9.8), the implication (2) $\Rightarrow$ (3) is trivial, and the implication (3) $\Rightarrow$ (4) follows from Remark 1.1.8.14. To complete the proof, it will suffice to show that if $X$ has the homotopy type of a CW complex, then the counit map $v : |\text{Sing}_\bullet(X)| \to X$ is a homotopy equivalence. By virtue of Proposition 3.5.3.8 it will suffice to show that $v$ is a weak homotopy equivalence, which follows from Corollary 3.5.4.2. \qed
3.5. COMPARISON WITH TOPOLOGICAL SPACES

Proof of Theorem 3.5.0.1. Using Construction 3.5.5.1, we see that the geometric realization functor \( |\bullet| : \text{Set} \rightarrow \text{Top} \) induces a functor of homotopy categories \( |\bullet| : \text{hKan} \rightarrow \text{hTop} \). It follows from Proposition 3.5.5.2 that this functor is fully faithful, and from Proposition 3.5.5.3 that its essential image consists of those topological spaces \( X \) which have the homotopy type of a CW complex.

Remark 3.5.5.4. Proposition 3.5.5.2 implies a stronger version of Theorem 3.5.0.1: the simplicially enriched functor \( |\bullet| : \text{Kan} \rightarrow \text{Top} \) induces a fully faithful embedding of \( \infty \)-categories \( S = N^{hc}(\text{Kan}) \rightarrow N^{hc}(\text{Top}) \) (Example [?]).

Using Theorem 3.5.4.1, we can also give a purely topological characterization of the homotopy category \( \text{hKan} \) (which does not make reference to the theory of simplicial sets).

Corollary 3.5.5.5. Let \( C \) be a category, and let \( \mathcal{E}' \subseteq \text{Fun}(\text{Top}, C) \) be the full subcategory spanned by those functors \( F : \text{Top} \rightarrow C \) which carry weak homotopy equivalences of topological spaces to isomorphisms in the category \( C \). Then:

(a) For every functor \( F \in \mathcal{E}' \), the composite functor

\[
\text{Kan} \xrightarrow{|\bullet|} \text{Top} \xrightarrow{F} C
\]

factors uniquely as a composition \( \text{Kan} \rightarrow \text{hKan} \xrightarrow{F} C \).

(b) The construction \( F \mapsto F \) induces an equivalence of categories \( \mathcal{E}' \rightarrow \text{Fun}(\text{hKan}, C) \).

We can state Corollary 3.5.5.5 more informally as follows: the homotopy category \( \text{hKan} \) of Kan complexes can be obtained from the category of topological spaces \( \text{Top} \) by formally adjoining inverses to all weak homotopy equivalences.

Proof of Corollary 3.5.5.5. Let \( \mathcal{E} \subseteq \text{Fun}(\text{Kan}, C) \) be the full subcategory spanned by those functors \( F : \text{Kan} \rightarrow C \) which carry homotopy equivalences of Kan complexes to isomorphisms in \( C \). By virtue of Corollary 3.1.6.7 it will suffice to show that precomposition with the geometric realization functor \( |\bullet| : \text{Kan} \rightarrow \text{Top} \) induces an equivalence of categories \( \mathcal{E}' \rightarrow \mathcal{E} \).

We claim that this functor has a homotopy inverse \( \mathcal{E} \rightarrow \mathcal{E}' \), given by precomposition with the functor \( \text{Sing}_\bullet : \text{Kan} \rightarrow \text{Top} \). This follows from the following pair of observations:

- For every functor \( F : \text{Top} \rightarrow C \), the counit map \( \overline{F} \circ \text{Sing}_\bullet \rightarrow F \) is an isomorphism when \( F \) belongs to \( \mathcal{E}' \) (since, for every topological space \( X \), the counit map \( |\text{Sing}_\bullet(X)| \rightarrow X \) is a weak homotopy equivalence; see Corollary 3.5.4.2).

- For every functor \( F_0 : \text{Kan} \rightarrow C \), the unit map \( F_0 \rightarrow F_0 \circ \text{Sing}_\bullet \) is an isomorphism (since, for every simplicial set \( Y \), the unit map \( Y \rightarrow |\text{Sing}_\bullet(|Y|)| \) is a weak homotopy equivalence of simplicial sets, and therefore induces a homotopy equivalence of topological spaces \( |Y| \rightarrow |\text{Sing}_\bullet(|Y|)| \)).
CHAPTER 3. KAN COMPLEXES
Chapter 4

The Homotopy Theory of ∞-Categories

Let \( q : X \to S \) be a morphism of simplicial sets. Recall that \( q \) is a Kan fibration if and only if it has the right lifting property with respect to every horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \) for \( n > 0 \) and \( 0 \leq i \leq n \). The theory of Kan fibrations can be viewed as a relativization of the theory of Kan complexes, which plays an essential role in the classical homotopy theory of simplicial sets (as in Chapter 3). In this chapter, we study several weaker notions of fibration, which will play an analogous role in the study of ∞-categories:

- We say that a morphism of simplicial sets \( q : X \to S \) is an inner fibration if it has the right lifting property with respect to every inner horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \), \( 0 < i < n \) (Definition 4.1.1.1). If this condition is satisfied, then for each vertex \( s \in S \), the fiber \( X_s = \{s\} \times_S X \) is an ∞-category (Remark 4.1.1.6). Consequently, the theory of inner fibrations can be regarded as a relative version of the theory of ∞-categories, which we study in §4.1.

- We say that a morphism of simplicial sets \( q : X \to S \) is a left fibration if it has the right lifting property with respect to the horn inclusions \( \Lambda^n_i \hookrightarrow \Delta^n \) for \( 0 \leq i < n \), and a right fibration if it has the right lifting property with respect to the horn inclusions \( \Lambda^n_i \hookrightarrow \Delta^n \) for \( 0 < i \leq n \) (Definition 4.2.1.1). If either of these conditions are satisfied, then the fiber \( X_s = \{s\} \times_S X \) is a Kan complex for each vertex \( s \in S \) (Corollary 4.4.2.2). We will see later that the construction \( s \mapsto X_s \) is covariantly functorial when \( q \) is a left fibration, and contravariantly functorial when \( q \) is a right fibration (see §5.1.1). In §4.2, we develop some basic formal properties of left and right fibrations; we will carry out a more detailed analysis in Chapter 5.

- We say that a morphism of simplicial sets \( q : X \to S \) is an isofibration if it has the
right lifting property with respect to every inclusion of simplicial sets $A \hookrightarrow B$ which is a categorical equivalence (Definition 4.5.7.1). This condition is primarily useful in the case where $X$ and $S$ are $\infty$-categories, in which case it is equivalent to the requirement that $q$ is an inner fibration which satisfies a lifting property with respect to isomorphisms (Proposition 4.5.6.1). We study isofibrations between $\infty$-categories in §4.4 and between general simplicial sets in §4.5.7.

If $q : X \rightarrow S$ is a morphism of simplicial sets, we have the following diagram of implications:

\[
\begin{array}{ccc}
q \text{ is a trivial Kan fibration} & \rightarrow & q \text{ is a Kan fibration} \\
\downarrow & & \downarrow \\
q \text{ is a left fibration} & \leftrightarrow & q \text{ is a right fibration} \\
\downarrow & & \downarrow \\
q \text{ is an isofibration} & \leftrightarrow & q \text{ is an inner fibration.}
\end{array}
\]

Beware that, in general, none of these implications is reversible.

In §4.3, we consider some prototypical examples of left and right fibrations which arise frequently in practice. Let $\mathcal{C}$ be an $\infty$-category. To each object $X \in \mathcal{C}$, one can associate a simplicial set $\mathcal{C}/X$, whose $n$-simplices are given by maps $\sigma : \Delta^{n+1} \rightarrow \mathcal{C}$ which carry the final vertex of $\Delta^{n+1}$ to the object $X \in \mathcal{C}$. In particular, vertices of $\mathcal{C}/X$ can be identified with morphisms $f : Y \rightarrow X$ in $\mathcal{C}$ having target $X$, and edges of $\mathcal{C}/X$ can be identified with commutative diagrams

\[
\begin{array}{ccc}
Z & \rightarrow & Y \\
\downarrow & & \downarrow \\
X.
\end{array}
\]

in the $\infty$-category $\mathcal{C}$ (see Notation 4.3.5.6 for a precise definition). The simplicial set $\mathcal{C}/X$ is itself an $\infty$-category, which we will refer to as the slice $\infty$-category of $\mathcal{C}$ over the object $X$. Moreover, the evident forgetful functor $\mathcal{C}/X \rightarrow \mathcal{C}$ (given on objects by the construction $(f : Y \rightarrow X) \mapsto Y$) is a right fibration (Proposition 4.3.6.1). A dual version of this construction produces another $\infty$-category $\mathcal{C}_{/X}$ whose objects are morphisms $f : X \rightarrow Y$ in the $\infty$-category $\mathcal{C}$, which we refer to as the coslice $\infty$-category of $\mathcal{C}$ under the object $X$. 
The slice and coslice constructions (and generalizations thereof) provide a rich supply of right and left fibrations between simplicial sets, and will play an essential role in the theory developed in this book.

Our final goal in this chapter is to generalize the notion of equivalence to the \(\infty\)-categorical setting. Let \(F : C \to D\) be a functor of \(\infty\)-categories. We will say that \(F\) is an equivalence of \(\infty\)-categories if there exists a functor \(G : D \to C\) such that \(G \circ F\) and \(F \circ G\) are isomorphic to \(\text{id}_C\) and \(\text{id}_D\) as objects of the \(\infty\)-categories \(\text{Fun}(C, C)\) and \(\text{Fun}(D, D)\), respectively (Definition 4.5.1.9). In §4.5 we study equivalences of \(\infty\)-categories in detail, as well as the more general notion of categorical equivalence between arbitrary simplicial sets (Definition 4.5.2.1).

### 4.1 Inner Fibrations

Recall that a simplicial set \(X\) is an \(\infty\)-category if, for every pair of integers \(0 < i < n\), every morphism of simplicial sets \(\sigma_0 : \Lambda^n_i \to X\) can be extended to an \(n\)-simplex of \(X\) (Definition 1.3.0.1). The goal of this section is to introduce and study a relative version of this condition. We say that a morphism of simplicial sets \(q : X \to S\) is an inner fibration if it has the right lifting property with respect to the horn inclusion \(\Lambda^n_i \to \Delta^n\) for \(0 < i < n\) (Definition 4.1.1.1). In the special case \(S = \Delta^0\), this reduces to the assumption that \(X\) is an \(\infty\)-category (Example 4.1.1.2). More generally, we will see in §4.1.1 that a morphism \(q : X \to S\) is an inner fibration if and only if the inverse image of every simplex of \(S\) is an \(\infty\)-category (Remark 4.1.1.12). Let \(C\) be an \(\infty\)-category. We will say that a simplicial subset \(C' \subseteq C\) is a subcategory of \(C\) if the inclusion map \(C' \hookrightarrow C\) is an inner fibration (Definition 4.1.2.2). In this case, \(C'\) is also an \(\infty\)-category, whose homotopy category \(\text{h}C'\) can be identified with a subcategory of \(\text{h}C\) (in the sense of classical category theory). In §4.1.2 we show that every subcategory of \(\text{h}C\) can be obtained (uniquely) in this way: more precisely, the construction \(C \mapsto \text{h}C'\) induces a bijection from the set of subcategories of \(C\) to the set of subcategories of \(\text{h}C\) (Proposition 4.1.2.10).

Recall that a morphism of simplicial sets \(i : A \to B\) is said to be inner anodyne if it can be constructed from horn inclusions \(\Lambda^n_i \to \Delta^n\) using pushouts, retracts, and transfinite composition (Definition 1.4.6.4). It follows immediately from the definitions that a morphism of simplicial sets \(q : X \to S\) is an inner fibration if and only if it has the right lifting property with respect to all inner anodyne morphisms (Proposition 4.1.3.1). In §4.1.3 we use a version of Quillen’s small object argument (Proposition 4.1.3.2) to show that, conversely, a morphism \(i : A \to B\) is inner anodyne if and only if it has the left lifting property with respect to every inner fibration (Corollary 4.1.3.4).

If \(C\) is an \(\infty\)-category and \(K\) is an arbitrary simplicial set, then the simplicial set \(\text{Fun}(K, C)\) is also an \(\infty\)-category (Theorem 1.4.3.7). In §4.1.4 we establish a relative form of this result: if \(q : X \to S\) is an inner fibration of simplicial sets, then postcomposition...
with \( q \) induces another inner fibration \( \text{Fun}(K, X) \to \text{Fun}(K, S) \) (Corollary 4.1.4.3). This is a special case of a more general result (Proposition 4.1.4.1), which is essentially equivalent to the stability of inner anodyne morphisms under the formation of pushout-products (see Lemma 1.4.7.5).

4.1.1 Inner Fibrations of Simplicial Sets

We now introduce the primary objects of interest in this section.

**Definition 4.1.1.1.** Let \( q : X \to S \) be a morphism of simplicial sets. We say that \( q \) is an **inner fibration** if, for every pair of integers \( 0 < i < n \), every lifting problem

\[
\begin{array}{ccc}
\Lambda^n_i & \overset{\sigma_0}{\longrightarrow} & X \\
\downarrow & & \downarrow q \\
\Delta^n & \overset{\sigma}{\longrightarrow} & S
\end{array}
\]

admits a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets \( \sigma_0 : \Lambda^n_i \to X \) and every \( n \)-simplex \( \sigma : \Delta^n \to S \) extending \( q \circ \sigma_0 \), we can extend \( \sigma_0 \) to an \( n \)-simplex \( \sigma : \Delta^n \to X \) satisfying \( q \circ \sigma = \sigma \).

**Example 4.1.1.2.** Let \( X \) be a simplicial set. Then the projection map \( X \to \Delta^0 \) is an inner fibration if and only if \( X \) is an \( \infty \)-category.

**Remark 4.1.1.3.** Let \( q : X \to S \) be a morphism of simplicial sets. Then \( q \) is an inner fibration if and only if the opposite morphism \( q^{\text{op}} : X^{\text{op}} \to S^{\text{op}} \) is an inner fibration.

**Remark 4.1.1.4.** The collection of inner fibrations is closed under retracts. That is, given a diagram of simplicial sets

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow q & & \downarrow q' & \downarrow q \\
S & \longrightarrow & S'
\end{array}
\]

where both horizontal compositions are the identity, if \( q' \) is an inner fibration, then so is \( q \).
Remark 4.1.1.5. The collection inner fibrations is closed under pullback. That is, given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow &   & \downarrow \\
S' & \rightarrow & S
\end{array}
\]

where \(q\) is an inner fibration, the morphism \(q'\) is also an inner fibration. Conversely, if \(q'\) is an inner fibration and \(f\) is surjective, then \(q\) is an inner fibration.

Remark 4.1.1.6. Let \(q : X \rightarrow S\) be an inner fibration of simplicial sets. Then, for every vertex \(s \in S\), the fiber \(X_s = \{s\} \times_S X\) is an \(\infty\)-category (this follows from Remark 4.1.1.5 and Example 4.1.1.2).

Remark 4.1.1.7. The collection of inner fibrations is closed under filtered colimits. That is, if \(\{q_\alpha : X_\alpha \rightarrow S_\alpha\}\) is a filtered diagram in the arrow category \(\text{Fun}([1], \text{Set}_\Delta)\) having colimit \(q : X \rightarrow S\), and each \(q_\alpha\) is an inner fibration of simplicial sets, then \(q\) is also an inner fibration of simplicial sets.

Remark 4.1.1.8. Let \(p : X \rightarrow Y\) and \(q : Y \rightarrow Z\) be inner fibrations of simplicial sets. Then the composite map \((q \circ p) : X \rightarrow Z\) is an inner fibration of simplicial sets.

Remark 4.1.1.9. Let \(q : X \rightarrow Y\) be an inner fibration of simplicial sets. If \(Y\) is an \(\infty\)-category, then \(X\) is also an \(\infty\)-category (this follows by combining Remark 4.1.1.8 with Example 4.1.1.2).

Proposition 4.1.1.10. Let \(C\) be a category, and let \(q : X \rightarrow N\bullet(C)\) be a morphism of simplicial sets. Then \(q\) is an inner fibration if and only if \(X\) is an \(\infty\)-category.

Proof. If \(q\) is an inner fibration, then Remark 4.1.1.9 guarantees that \(X\) is an \(\infty\)-category. Conversely, suppose that \(X\) is an \(\infty\)-category and that we are given a lifting problem

\[
\begin{array}{ccc}
\Lambda^n_i & \rightarrow & X \\
\downarrow & \sigma_0 & \downarrow q \\
\Delta^n & \rightarrow & N\bullet(C)
\end{array}
\]

for integers \(0 < i < n\). Our assumption that \(X\) is an \(\infty\)-category guarantees that \(\sigma_0\) can be extended to an \(n\)-simplex \(\sigma : \Delta^n \rightarrow X\). The equality \(q \circ \sigma = \sigma\) is automatic by virtue of Proposition 1.2.3.1.
Example 4.1.1.11. Let $\mathcal{C}$ be an $\infty$-category and let $h\mathcal{C}$ denote its homotopy category (Construction 1.3.5.1). Then the canonical map $\mathcal{C} \to N_{\bullet}(h\mathcal{C})$ is an inner fibration.

Remark 4.1.1.12. Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism $q$ is an inner fibration.
2. For every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ is an inner fibration.
3. For every simplex $\sigma : \Delta^n \to S$, the fiber product $\Delta^n \times_S X$ is an $\infty$-category.

The equivalence (1) $\iff$ (2) is immediate from the definition, and the equivalence (2) $\iff$ (3) follows from Proposition 4.1.1.10.

4.1.2 Subcategories of $\infty$-Categories

Let $\mathcal{C}$ be a category, and let $\text{Ob}(\mathcal{C})$ be the set of objects of $\mathcal{C}$. Suppose that we are given a subset $\text{Ob}'(\mathcal{C}) \subseteq \text{Ob}(\mathcal{C})$ and, for every pair of objects $X, Y \in \text{Ob}'(\mathcal{C})$, a subset $\text{Hom}'_{\mathcal{C}}(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$ satisfying the following conditions:

- For every object $X \in \text{Ob}'(\mathcal{C})$, the identity morphism $\text{id}_X$ belongs to $\text{Hom}'_{\mathcal{C}}(X, X)$.
- For every triple of objects $X, Y, Z \in \text{Ob}'(\mathcal{C})$ and every pair of morphisms $f \in \text{Hom}'_{\mathcal{C}}(X, Y), g \in \text{Hom}'_{\mathcal{C}}(Y, Z)$, the composition $g \circ f$ belongs to $\text{Hom}'_{\mathcal{C}}(X, Z)$.

In this case, we can construct a category $\mathcal{C}'$ by setting $\text{Ob}(\mathcal{C}') = \text{Ob}'(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}'}(X, Y) = \text{Hom}'_{\mathcal{C}}(X, Y)$ for every pair of objects $X, Y \in \text{Ob}(\mathcal{C}')$ (where the composition of morphisms in $\mathcal{C}'$ agrees with their composition in $\mathcal{C}$). In this situation, we refer to $\mathcal{C}'$ as the subcategory of $\mathcal{C}$ spanned by the objects $\text{Ob}'(\mathcal{C})$ and the morphisms $\{\text{Hom}'_{\mathcal{C}}(X, Y)\}_{X, Y \in \text{Ob}'(\mathcal{C})}$.

Remark 4.1.2.1. Let $\mathcal{C}$ be a category. We will say that a category $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if it arises from the construction described above (for some collection of objects $\text{Ob}'(\mathcal{C})$ and collections of morphisms $\{\text{Hom}'_{\mathcal{C}}(X, Y)\}_{X, Y \in \text{Ob}'(\mathcal{C})}$). Phrased differently, a category $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the following conditions are satisfied:

- The set of objects $\text{Ob}(\mathcal{C}')$ is a subset of the set of objects $\text{Ob}(\mathcal{C})$.
- For every pair of objects $X, Y \in \text{Ob}(\mathcal{C}') \subseteq \text{Ob}(\mathcal{C})$, the set of morphisms $\text{Hom}_{\mathcal{C}'}(X, Y)$ is a subset of the set of morphisms $\text{Hom}_{\mathcal{C}}(X, Y)$.
- There is a functor $\mathcal{C}' \to \mathcal{C}$ which is the identity on objects and morphisms.

We write $\mathcal{C}' \subseteq \mathcal{C}$ to indicate that $\mathcal{C}'$ is a subcategory of $\mathcal{C}$.
4.1. INNER FIBRATIONS

We now generalize the notion of subcategory to the setting of ∞-categories.

**Definition 4.1.2.2.** Let $\mathcal{C}$ be an ∞-category. A subcategory of $\mathcal{C}$ is a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ for which the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an inner fibration.

**Remark 4.1.2.3.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Then $\mathcal{C}'$ is also an ∞-category (Remark 4.1.1.9).

**Example 4.1.2.4.** Let $\mathcal{C}$ be an ordinary category and let $N_{\bullet}(\mathcal{C})$ be its nerve. For every subcategory $\mathcal{C}' \subseteq \mathcal{C}$, the nerve $N_{\bullet}(\mathcal{C}')$ can be viewed as a subcategory of $N_{\bullet}(\mathcal{C})$ (the inclusion map $N_{\bullet}(\mathcal{C}') \hookrightarrow N_{\bullet}(\mathcal{C})$ is automatically an inner fibration, by virtue of Proposition 4.1.1.10). We will see in a moment that every subcategory of $N_{\bullet}(\mathcal{C})$ arises in this way (Corollary 4.1.2.11). In other words, when restricted to (the nerves of) ordinary categories, Definition 4.1.2.2 reduces to the classical notion of subcategory.

**Warning 4.1.2.5.** The terminology of Definition 4.1.2.2 has the potential to cause confusion. If $\mathcal{C}$ is an ∞-category and $\mathcal{C}' \subseteq \mathcal{C}$ is a subcategory, then $\mathcal{C}'$ need not be (isomorphic to the nerve of) an ordinary category. Our use of the term “subcategory” (rather than the more technically correct “sub-∞-category”) is intended to avoid awkward language.

**Remark 4.1.2.6** (Pullbacks of Subcategories). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let $\mathcal{D}'$ be a subcategory of $\mathcal{D}$. Then the inverse image $F^{-1}(\mathcal{D}') \subseteq \mathcal{C}$ is a subcategory of $\mathcal{C}$ (see Remark 4.1.1.5).

**Remark 4.1.2.7.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Suppose that $\mathcal{C}$ contains a 2-simplex $\sigma :$

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow h & & \downarrow f \\
Z & \rightarrow & \text{ } \\
\end{array}
\]

which witnesses $h$ as a composition of $g$ and $f$ (Definition 1.3.4.1). If $f$ and $g$ belong to the subcategory $\mathcal{C}'$, then the 2-simplex $\sigma$ also belongs to the subcategory $\mathcal{C}'$ (since the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has the right lifting property with respect to the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$). In particular, if $f$ and $g$ belong to $\mathcal{C}'$, then $h$ also belongs to $\mathcal{C}'$.

**Remark 4.1.2.8.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Suppose we are given a pair of morphisms $f, g : X \to Y$ in $\mathcal{C}$ having the same source and target. If $f$ and $g$ are homotopic as morphisms in the ∞-category $\mathcal{C}$ and $f$ belongs to the subcategory $\mathcal{C}'$, then $g$ also belongs to the subcategory $\mathcal{C}'$ and the morphisms $f$ and $g$ are homotopic in the ∞-category $\mathcal{C}'$. This a special case of Remark 4.1.2.7 (note that $f$ and $g$ are homotopic if and only if $g$ is a composition of $f$ with the identity morphism $\text{id}_Y$; see Definition 1.3.3.1).
Remark 4.1.2.9. Let $C$ be an $\infty$-category, let $C' \subseteq C$ be a subcategory, let $\sigma: \Delta^n \to C$ be an $n$-simplex of $C$ for $n > 0$. The following conditions are equivalent:

1. The $n$-simplex $\sigma$ is contained in the subcategory $C'$.

2. For every pair of integers $0 \leq i < j \leq n$, the edge
   \[ \Delta^1 \simeq N_\bullet(\{i < j\}) \hookrightarrow \Delta^n \xrightarrow{\sigma} C \]
   is contained in the subcategory $C'$.

3. For every integer $1 \leq j \leq n$, the edge
   \[ \Delta^1 \simeq N_\bullet(\{j - 1 < j\}) \hookrightarrow \Delta^n \xrightarrow{\sigma} C \]
   is contained in the subcategory $C'$.

The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are immediate from the definitions, and the implication $(3) \Rightarrow (1)$ follows from fact that the inclusion $C' \hookrightarrow C$ has the right lifting property with respect to the inner anodyne morphism $\text{Spine}[n] \hookrightarrow \Delta^n$ (see Example 1.4.7.7 and Proposition 4.1.3.1).

Proposition 4.1.2.10. Let $C$ be an $\infty$-category, let $\text{h}C$ be its homotopy category, and let $F: C \to N_\bullet(\text{h}C)$ be the canonical map. Then the construction $(D \subseteq \text{h}C) \mapsto (F^{-1}(N_\bullet(D)) \subseteq \text{h}C)$ induces a bijection

\[ \{\text{Subcategories of the ordinary category } \text{h}C\} \simeq \{\text{Subcategories of the } \infty\text{-category } C\} \]

Proof. We first observe that if $D$ is a subcategory of the homotopy category $\text{h}C$, then the nerve $N_\bullet(D)$ is a subcategory of the $\infty$-category $N_\bullet(\text{h}C)$ (Example 4.1.2.4), so that $F^{-1}(N_\bullet(D))$ is a subcategory of the $\infty$-category $C$ (Remark 4.1.2.6). Moreover, subcategory $D$ is uniquely determined by its inverse image $F^{-1}(N_\bullet(D))$: this follows from the fact that $F: C \to N_\bullet(\text{h}C)$ is an epimorphism of simplicial sets (Remark 1.4.7.9). To complete the proof, it will suffice to show that every subcategory $C' \subseteq C$ arises in this way. Note that the inclusion map $C' \hookrightarrow C$ induces a functor of homotopy categories $G: \text{h}C' \hookrightarrow \text{h}C$, which is obviously injective at the level of objects. In addition, for every pair of objects $X, Y \in \text{h}C'$, the functor $G$ induces a monomorphism $\text{Hom}_{\text{h}C'}(X, Y) \to \text{Hom}_{\text{h}C}(X, Y)$: this follows from the observation that a pair of morphisms $f, g: X \to Y$ are homotopic in the $\infty$-category $C'$ if and only if they are homotopic in the $\infty$-category $C$ (Remark 4.1.2.8). It follows that the functor $G$ induces an isomorphism in $\text{h}C'$ onto a subcategory $D \subseteq \text{h}C$. We therefore have an inclusion $C' \subseteq F^{-1}(N_\bullet(\text{h}D))$. To complete the proof, it will suffice to show that this inclusion is an equality. In other words, we must show that if an $n$-simplex $\sigma: \Delta^n \to C$ is contained in $C'$ if and only if the induced map $[n] \to \text{h}C$ factors through the subcategory...
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Let $\mathcal{D} \subseteq hC$. Without loss of generality, we may assume that $n > 0$ (the case $n = 0$ is trivial). Using Remark 4.1.2.9, we can reduce to the case where $n = 1$, so that $\sigma$ can be identified with a morphism $g : X \to Y$ in the $\infty$-category $C$. Our assumption that $F(\sigma)$ belongs to $N_* (\mathcal{D})$ guarantees that $g$ is homotopic to a morphism $f : X \to Y$ which belongs to the subcategory $C' \subseteq C$ (and, in particular, that the objects $X$ and $Y$ belong to $C'$). Invoking Remark 4.1.2.8, we conclude that $g$ also belongs to the subcategory $C'$, as desired. □

**Corollary 4.1.2.11.** Let $C$ be an ordinary category. Then the construction $C' \mapsto N_* (C')$ induces a bijection

$\{ \text{Subcategories of the ordinary category } C \} \simeq \{ \text{Subcategories of the } \infty\text{-category } N_* (C) \}$

*Proof.* Combine Proposition 4.1.2.10 with Example 1.3.5.4. □

**Corollary 4.1.2.12.** Let $C$ be an $\infty$-category, let $S$ be a collection of objects of $C$, and let $T$ be a collection of morphisms of $C$. The following conditions are equivalent:

- There exists a subcategory $C' \subseteq C$ whose objects are the elements of $S$ and whose morphisms are the elements of $T$.

- The collections $S$ and $T$ satisfy the following conditions:

  1. For each object $X \in S$, the identity morphism $\text{id}_X$ belongs to $T$.
  2. For each morphism $f : X \to Y$ of $C$ which belongs to $T$, the objects $X$ and $Y$ belong to $S$.
  3. If $f : X \to Y$ is an morphism of $C$ which belongs to $T$ and $g : X \to Y$ is a morphism of $C$ which is homotopic to $f$, then $g$ also belongs to $C$.
  4. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $C$ which belong to $T$, then some composition $(g \circ f) : X \to Z$ also belongs to $T$.

Moreover, if these conditions are satisfied, then the subcategory $C' \subseteq C$ is uniquely determined by $S$ and $T$.

*Proof.* The necessity of conditions (1) and (2) is immediate, and the necessity of (3) and (4) follow from Remark 4.1.2.8 and Remark 4.1.2.7. Conversely, suppose that conditions (1) through (4) are satisfied. Using (1), (2), and (4), we deduce that there exists a subcategory $\mathcal{D} \subseteq hC$ whose objects are the elements of $S$ and whose morphisms are the homotopy classes of morphisms belonging to $T$. Let $C' \subseteq C$ be the inverse image of the subcategory $N_* (\mathcal{D}) \subseteq N_* (hC)$. It follows immediately from the definition that an object of $C$ belongs to the subcategory $C'$ if and only if it is an element of $S$, and from (3) that a morphism of $C$ belongs to the subcategory $C'$ if and only if it is an element of $T$. The uniqueness of the subcategory $C'$ follows from Proposition 4.1.2.10. □
Chapter 4. The Homotopy Theory of ∞-Categories

Definition 4.1.2.13. Let \( \mathcal{C} \) be an \( \infty \)-category. Suppose we are given a collection \( S \) of objects of \( \mathcal{C} \) and a collection \( T \) of morphisms of \( \mathcal{C} \) satisfying the assumptions of Corollary 4.1.2.12, so that there exists a unique subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) whose objects are the elements of \( S \) and whose morphisms are the elements of \( T \). In this case, we will refer to \( \mathcal{C}' \) as the subcategory of \( \mathcal{C} \) spanned by the objects of \( S \) and the morphisms of \( T \).

Remark 4.1.2.14. Let \( \mathcal{C} \) be an \( \infty \)-category, and let \( \mathcal{C}' \subseteq \mathcal{C} \) be the subcategory spanned by the collection of objects \( S \) of \( \mathcal{C} \) and a collection of morphisms \( T \) of \( \mathcal{C} \). Then a morphism of simplicial sets \( f : K \to \mathcal{C} \) factors through the subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) if and only if it carries each vertex of \( K \) to an element of \( T \) and each edge of \( K \) to an element of \( T \).

Let \( \mathcal{C} \) be an ordinary category. Recall that a subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) is full if, for every pair of objects \( X, Y \in \mathcal{C}' \), the inclusion map \( \text{Hom}_{\mathcal{C}'}(X, Y) \hookrightarrow \text{Hom}_{\mathcal{C}}(X, Y) \) is bijective. This definition has an obvious counterpart in the \( \infty \)-categorical setting:

Definition 4.1.2.15. Let \( \mathcal{C} \) be an \( \infty \)-category. We say that a subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) is full if it satisfies the following condition:

- Let \( X \) and \( Y \) be objects of \( \mathcal{C}' \) and let \( f : X \to Y \) be a morphism in the \( \infty \)-category \( \mathcal{C} \). Then \( f \) is also contained in \( \mathcal{C}' \).

Proposition 4.1.2.16. Let \( \mathcal{C} \) be an \( \infty \)-category and let \( S \) be a collection of objects of \( \mathcal{C} \). Then there exists a unique full subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) having object set \( S \).

Proof. Take \( T \) to be the collection of all morphisms \( f : X \to Y \) in the \( \infty \)-category \( \mathcal{C} \) for which \( X \) and \( Y \) belong to \( S \), and apply Corollary 4.1.2.12.

Definition 4.1.2.17. Let \( \mathcal{C} \) be an \( \infty \)-category and let \( S \) be a collection of objects of \( \mathcal{C} \). By virtue of Proposition 4.1.2.16, there exists a unique subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) having object set \( S \). We will refer to \( \mathcal{C}' \) as the full subcategory of \( \mathcal{C} \) spanned by \( S \).

Remark 4.1.2.18. Let \( \mathcal{C} \) be an \( \infty \)-category, and let \( \mathcal{C}' \subseteq \mathcal{C} \) be the full subcategory of \( \mathcal{C} \) spanned by a set of objects \( S \) of \( \mathcal{C} \). Then a morphism of simplicial sets \( f : K \to \mathcal{C} \) factors through the full subcategory \( \mathcal{C}' \subseteq \mathcal{C} \) if and only if, for every vertex \( x \in K \), the image \( f(x) \in \mathcal{C} \) belongs to \( S \).

Remark 4.1.2.19. Let \( \mathcal{C} \) be an ordinary category. Then the construction \( \mathcal{C}' \mapsto N_\bullet(\mathcal{C}') \) induces a bijection

\[ \{\text{Full subcategories of } \mathcal{C}\} \simeq \{\text{Full subcategories of } N_\bullet(\mathcal{C})\} . \]
### 4.1.3 Inner Anodyne Morphisms

By definition, a morphism of simplicial sets $q : X \to S$ is an inner fibration if it has the right lifting property with respect to every inner horn inclusion $\Lambda^n_i \to \Delta^n$. From this, one can immediately deduce a stronger lifting property.

**Proposition 4.1.3.1.** Let $q : X \to S$ be a morphism of simplicial sets. Then $q$ is an inner fibration if and only if it satisfies the following condition:

(*) For every square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow & & \downarrow q \\
B & \xrightarrow{} & S
\end{array}
\]

where $i$ is inner anodyne, there exists a dotted arrow rendering the diagram commutative.

**Proof.** The “if” direction is immediate from the definition, since the horn inclusion $\Lambda^n_i \to \Delta^n$ is inner anodyne for $0 < i < n$. The reverse implication follows from Proposition 1.4.4.16.

**Proposition 4.1.3.2.** Let $f : X \to Y$ be a morphism of simplicial sets. Then $f$ can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where $f''$ is an inner fibration and $f'$ is inner anodyne. Moreover, the simplicial set $Q(f)$ (and the morphisms $f'$ and $f''$) can be chosen to depend functorially on $f$, in such a way that the functor

\[
\text{Fun}([1], \text{Set}_\Delta) \to \text{Set}_\Delta \quad (f : X \to Y) \mapsto Q(f)
\]

commutes with filtered colimits.

**Proof.** We proceed as in the proof of Proposition 3.1.6.1. We construct a sequence of simplicial sets $\{X(m)\}_{m \geq 0}$ and morphisms $f(m) : X(m) \to Y$ by recursion. Set $X(0) = X$ and $f(0) = f$. Assuming that $f(m) : X(m) \to Y$ has been defined, let $S(m)$ denote the set of all commutative diagrams $\sigma:$

\[
\begin{array}{ccc}
\Lambda^n & \xrightarrow{f(m)} & X(m) \\
\downarrow & & \downarrow f(m) \\
\Delta^n & \xrightarrow{u_\sigma} & Y,
\end{array}
\]

where $\sigma$ is an inner anodyne morphism.
where $0 < i < n$ and the left vertical map is the inclusion. For every such commutative diagram $\sigma$, let $C_\sigma = \Lambda^n_i$ denote the upper left hand corner of the diagram $\sigma$, and $D_\sigma = \Delta^n$ the lower left hand corner. Form a pushout diagram

$$
\begin{array}{ccc}
\prod_{\sigma \in S(m)} C_\sigma & \longrightarrow & X(m) \\
\downarrow & & \downarrow \\
\prod_{\sigma \in S(m)} D_\sigma & \longrightarrow & X(m + 1)
\end{array}
$$

and let $f(m + 1) : X(m + 1) \to Y$ be the unique map whose restriction to $X(m)$ is equal to $f(m)$ and whose restriction to each $D_\sigma$ is equal to $u_\sigma$. By construction, we have a direct system of inner anodyne morphisms

$$X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

Set $Q(f) = \lim_{\to m} X(m)$. Then the natural map $f' : X \to Q(f)$ is inner anodyne (since the collection of inner anodyne maps is closed under transfinite composition), and the system of morphisms $\{f(m)\}_{m \geq 0}$ can be amalgamated to a single map $f'' : Q(f) \to Y$ satisfying $f = f'' \circ f'$. It is clear from the definition that the construction $f \mapsto Q(f)$ is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that $f''$ is a inner fibration: that is, that every lifting problem

$$
\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & Q(f) \\
\downarrow & \nearrow & \downarrow f'' \\
\Delta^n & \longrightarrow & Y
\end{array}
$$

admits a solution (provided that $0 < i < n$). Let us abuse notation by identifying each $X(m)$ with its image in $Q(f)$. Since $\Lambda^n_i$ is a finite simplicial set, its image under $v$ is contained in $X(m)$ for some $m \gg 0$. In this case, we can identify $\sigma$ with an element of the set $S(m)$, so that the lifting problem

$$
\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & X(m + 1) \\
\downarrow & \nearrow & \downarrow f(m + 1) \\
\Delta^n & \longrightarrow & Y
\end{array}
$$

admits a solution by construction.
Applying Proposition 4.1.3.2 in the special case $Y = \Delta^0$, we obtain the following:

**Corollary 4.1.3.3.** Let $X$ be a simplicial set. Then there exists an inner anodyne morphism $f : X \to Q(X)$, where $Q(X)$ is an $\infty$-category. Moreover, the $\infty$-category $Q(X)$ (and the morphism $f$) can be chosen to depend functorially on $X$, in such a way that the functor $X \mapsto Q(X)$ commutes with filtered colimits.

Using Proposition 4.1.3.2, we obtain the following counterpart of Proposition 4.1.3.1:

**Corollary 4.1.3.4.** Let $i : A \to B$ be a morphism of simplicial sets. Then $i$ is inner anodyne if and only if it satisfies the following condition:

\[ (*) \text{ For every square diagram of simplicial sets } \]

\[
\begin{array}{c}
A \\
\downarrow^i \\
B \end{array} \quad \begin{array}{c}
X \\
\downarrow^q \\
S \end{array} \quad \begin{array}{c}
\rightarrow \\
\nearrow \\
\end{array}
\]

where $q$ is an inner fibration, there exists a dotted arrow rendering the diagram commutative.

**Proof.** The “if” direction follows from Proposition 4.1.3.1. For the converse, suppose that condition $(*)$ is satisfied. Using Proposition 4.1.3.2, we can factor $i$ as a composition $A \xrightarrow{i'} Q \xrightarrow{q} B$, where $i'$ is inner anodyne and $q$ is an inner fibration. If the lifting problem

\[
\begin{array}{c}
A \\
\downarrow^i \\
B \end{array} \quad \begin{array}{c}
Q \\
\downarrow^q \\
B \end{array} \quad \begin{array}{c}
\rightarrow \\
\nearrow^r \\
\end{array}
\]

admits a solution, then the morphism $r$ exhibits $i$ as a retract of $i'$ (in the arrow category $\text{Fun}([1], \text{Set}_\Delta)$). Since the collection of inner anodyne morphisms is closed under retracts, it follows that $i$ is inner anodyne.

\[ \square \]

### 4.1.4 Exponentiation for Inner Fibrations

Recall that, if $C$ is an $\infty$-category and $B$ is an arbitrary simplicial set, then the simplicial set $\text{Fun}(B, C)$ is also an $\infty$-category (Theorem 1.4.3.7). We now record a relative version of this result.
Proposition 4.1.4.1. Let \( q : X \to S \) be an inner fibration of simplicial sets, and let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets. Then the restriction map

\[
\rho : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)
\]

is also an inner fibration of simplicial sets.

Proof. By virtue of Proposition 4.1.3.1 it will suffice to show that every lifting problem

\[
\begin{array}{ccc}
A' & \rightarrow & \text{Fun}(B, X) \\
\downarrow^{i'} & \nearrow & \downarrow^{\rho} \\
B' & \rightarrow & \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\end{array}
\]

admits a solution, provided that \( i' \) is inner anodyne. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \rightarrow & X \\
\downarrow & \nearrow & \downarrow^{q} \\
B \times B' & \rightarrow & S
\end{array}
\]

admits a solution. This follows from Proposition 4.1.3.1 since the left vertical map is inner anodyne (Lemma 1.4.7.5) and \( q \) is an inner fibration.

Corollary 4.1.4.2. Let \( C \) be an \( \infty \)-category and let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets. Then the restriction functor \( \text{Fun}(B, C) \to \text{Fun}(A, C) \) is an inner fibration.

Proof. Apply Proposition 4.1.4.1 in the special case \( S = \Delta^0 \).

Corollary 4.1.4.3. Let \( q : X \to S \) be an inner fibration of simplicial sets and let \( B \) be an arbitrary simplicial set. Then composition with \( q \) induces an inner fibration \( \text{Fun}(B, X) \to \text{Fun}(B, S) \).

Proof. Apply Proposition 4.1.4.1 in the special case \( A = \emptyset \).

We now record an analogous generalization of Proposition 1.4.7.6.

Proposition 4.1.4.4. Let \( q : X \to S \) be an inner fibration of simplicial sets, and let \( i : A \hookrightarrow B \) be an inner anodyne morphism of simplicial sets. Then the restriction map

\[
\rho : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)
\]

is a trivial Kan fibration.
4.1. INNER FIBRATIONS

Proof. We wish to show that every lifting problem

\[
\begin{array}{ccc}
A' & \longrightarrow & \text{Fun}(B, X) \\
\uparrow \varphi' & & \uparrow \rho \\
B' & \longrightarrow & \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\end{array}
\]

admits a solution, provided that \( \varphi' \) is a monomorphism of simplicial sets. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \longrightarrow & X \\
\downarrow & & \downarrow q \\
B \times B' & \longrightarrow & S
\end{array}
\]

admits a solution. This follows from Proposition 4.1.3.1, since the left vertical map is inner anodyne (Lemma 1.4.7.5) and \( q \) is an inner fibration.

Proposition 4.1.4.4 admits the following converse (generalizing Theorem 1.4.6.1):

**Proposition 4.1.4.5.** Let \( q : X \to S \) be a morphism of simplicial sets. Then \( q \) is an inner fibration if and only if the induced map

\[
\rho : \text{Fun}(\Delta^2, X) \to \text{Fun}(\Lambda^2_1, X) \times_{\text{Fun}(\Lambda^2_1, S)} \text{Fun}(\Delta^2, S)
\]

is a trivial Kan fibration.

Proof. The “only if” direction follows from Proposition 4.1.4.4. For the converse, we observe that \( \rho \) is a trivial Kan fibration if and only if \( q \) has the right lifting property with respect to the inclusion map

\[
(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2
\]

for every nonnegative integer \( m \). Since the collection of inner anodyne morphisms is generated (as a weakly saturated class) by such inclusions (Lemma 1.4.6.9), it follows that \( q \) has the right lifting property with respect to all inner anodyne morphisms (Proposition 1.4.4.16) and is therefore an inner fibration (Proposition 4.1.3.1).

**Proposition 4.1.4.6.** Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow i & & \downarrow q \\
B & \xrightarrow{g} & S
\end{array}
\]
of simplicial sets, where \(i\) is a monomorphism and \(q\) is an inner fibration. Then the simplicial set \(\text{Fun}_{A/\mathcal{S}}(B, X)\) of Construction 3.1.3.7 is an \(\infty\)-category. Moreover, if \(i\) is inner anodyne, then \(\text{Fun}_{A/\mathcal{S}}(B, X)\) is a contractible Kan complex.

**Proof.** By virtue of Remark 3.1.3.11, the simplicial set \(\text{Fun}_{A/\mathcal{S}}(B, X)\) can be identified with a fiber of the restriction map

\[
\theta : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, X).
\]

Proposition 4.1.4.1 asserts that \(\theta\) is an inner fibration of simplicial sets, so its fibers are \(\infty\)-categories (Remark 4.1.1.6). If \(i\) is inner anodyne, then Proposition 4.1.4.4 guarantees that \(\theta\) is a trivial Kan fibration, so its fibers are contractible Kan complexes.

**Corollary 4.1.4.7.** Let \(B\) be a simplicial set, let \(A \subseteq B\) be a simplicial subset, and let \(f : A \to C\) be a morphism of simplicial sets. If \(C\) is an \(\infty\)-category, then the simplicial set \(\text{Fun}_{A/(B, C)}\) is an \(\infty\)-category. Moreover, if the inclusion \(A \hookrightarrow B\) is inner anodyne, then \(\text{Fun}_{A/(B, C)}\) is a contractible Kan complex.

**Proof.** Apply Proposition 4.1.4.6 in the special case \(S = \Delta^0\).

**Corollary 4.1.4.8.** Let \(q : X \to S\) be an inner fibration of simplicial sets and let \(g : B \to S\) be any morphism of simplicial sets. Then the simplicial set \(\text{Fun}_{S/B}(X)\) is an \(\infty\)-category.

**Proof.** Apply Proposition 4.1.4.6 in the special case \(A = \emptyset\).

### 4.2 Left and Right Fibrations

Let \(q : X \to S\) be a morphism of simplicial sets. Recall that \(q\) is a **Kan fibration** if and only if it has the right lifting property with respect to every horn inclusion \(\Lambda^n_i \hookrightarrow \Delta^n\) for \(n > 0\) and \(0 \leq i \leq n\) (Definition 3.1.1.1). In particular, if \(q\) is a Kan fibration, then it has the right lifting property with respect to both of the inclusion maps \(\{0\} \hookrightarrow \Delta^1 \hookrightarrow \{1\}\). Concretely, this translates into the following pair of assertions:

**(Left Path Lifting Property):** Let \(q : X \to S\) be a Kan fibration of simplicial sets, let \(x\) be a vertex of \(X\), and let \(\overline{e} : q(x) \to \overline{y}\) be an edge of \(S\) originating at the vertex \(q(x)\). Then there exists an edge \(e : x \to y\) in \(X\) which originates at the vertex \(x\) and satisfies \(q(e) = \overline{e}\).

**(Right Path Lifting Property):** Let \(q : X \to S\) be a Kan fibration of simplicial sets, let \(y\) be a vertex of \(X\), and let \(\overline{e} : \overline{x} \to q(y)\) be an edge of \(S\) terminating at the vertex \(q(y)\). Then there exists an edge \(e : x \to y\) in \(X\) which terminates at the vertex \(y\) and satisfies \(q(e) = \overline{e}\).
In §4.2.1, we introduce stronger versions of these conditions. We say that a morphism of simplicial sets $q : X \to S$ is a left fibration if it has the right lifting property with respect to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i < n$, and a right fibration if it has the right lifting property with respect to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i \leq n$ (Definition 4.2.1.1). Setting $n = 1$, we see that every left fibration satisfies the left path lifting property, and that every right fibration satisfies the right path lifting property. Moreover, this assertion has a partial converse. Note that evaluation at the vertices of $\Delta^1$ induces morphisms of simplicial sets

$$ev_0 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{0\}, X) \times_{\text{Fun}(\{0\}, S)} \text{Fun}(\Delta^1, S) \simeq X \times_S \text{Fun}(\Delta^1, S)$$

$$ev_1 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{1\}, X) \times_{\text{Fun}(\{1\}, S)} \text{Fun}(\Delta^1, S) \simeq X \times_S \text{Fun}(\Delta^1, S).$$

In §4.2.4, we show that $f$ is a left fibration if and only if the evaluation map $ev_0$ is a trivial Kan fibration, and that $f$ is a right fibration if and only if $ev_1$ is a trivial Kan fibration (Proposition 4.2.4.1). The “only if” direction of this assertion is a special case of general stability properties of left and right fibrations under exponentiation, which we prove in §4.2.3 (Propositions 4.2.3.1 and 4.2.3.4). Our proofs will make use of some basic facts about left anodyne and right anodyne morphisms of simplicial sets, which we establish in §4.2.2.

The notions of left and right fibration introduced in this section have a counterpart in classical category theory. In §4.2.5, we recall the notion a fibration in groupoids (Definition 4.2.5.1), and show that a functor of ordinary categories $F : C \to D$ is a fibration in groupoids if and only if the induced map of nerves $N_\bullet(F) : N_\bullet(C) \to N_\bullet(D)$ is a right fibration of simplicial sets (Proposition 4.2.5.12). Similarly, a functor of ordinary categories $F : C \to D$ is an opfibration in groupoids if only if the map $N_\bullet(F)$ is a left fibration of simplicial sets.

### 4.2.1 Left and Right Fibrations of Simplicial Sets

We begin by introducing some terminology.

**Definition 4.2.1.1.** Let $f : X \to S$ be a morphism of simplicial sets. We will say that $f$ is a left fibration if, for every pair of integers $0 \leq i < n$, every lifting problem

$$\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\sigma_0} & X \\
\downarrow{\sigma} & & \downarrow{f} \\
\Delta^n & \xrightarrow{\sigma} & S
\end{array}$$

has a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ and every $n$-simplex $\sigma : \Delta^n \to S$ extending $f \circ \sigma_0$, we can extend $\sigma_0$ to an $n$-simplex $\sigma : \Delta^n \to X$ satisfying $f \circ \sigma = \sigma$. 
We say that \( f \) is a right fibration if, for every pair of integers \( 0 < i \leq n \), every lifting problem

\[
\begin{array}{ccc}
\Lambda^n_i & \to & X \\
\downarrow & \nearrow \sigma_0 \\
\Delta^n & \to & S
\end{array}
\]

has a solution.

**Example 4.2.1.2.** Any isomorphism of simplicial sets is both a left fibration and a right fibration.

**Remark 4.2.1.3.** Let \( f : X \to S \) be a morphism of simplicial sets. Then \( f \) is a left fibration if and only if the opposite morphism \( f^{\text{op}} : X^{\text{op}} \to S^{\text{op}} \) is a right fibration.

**Remark 4.2.1.4.** Let \( f : X \to S \) be a morphism of simplicial sets. If \( f \) is either a left fibration or a right fibration, then it is an inner fibration. In this case, if \( S \) is an \( \infty \)-category, then \( X \) is also an \( \infty \)-category (Remark 4.1.1.9).

**Example 4.2.1.5.** A morphism of simplicial sets \( f : X \to S \) is a Kan fibration if and only if it is both a left fibration and a right fibration.

**Warning 4.2.1.6.** In the statement of Example 4.2.1.5, both hypotheses are necessary: a left fibration of simplicial sets need not be a right fibration and vice versa. For example, the inclusion map \( \{1\} \hookrightarrow \Delta^1 \) is a left fibration, but not a right fibration (and therefore not a Kan fibration).

**Remark 4.2.1.7.** The collection of left and right fibrations is closed under retracts. That is, suppose we are given a diagram of simplicial sets

\[
\begin{array}{ccc}
X & \to & X' \\
\downarrow f & & \downarrow f' \\
S & \to & S'
\end{array}
\]

where both horizontal compositions are the identity. If \( f' \) is a left fibration, then \( f \) is a left fibration. If \( f' \) is a right fibration, then \( f \) is a right fibration.
Remark 4.2.1.8. The collections of left and right are closed under pullback. That is, suppose we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}
\]

If \( f \) is a left fibration, then \( f' \) is also a left fibration. If \( f \) is a right fibration, then \( f' \) is a right fibration.

Remark 4.2.1.9. Let \( f : X \to S \) be a map of simplicial sets. Suppose that, for every simplex \( \sigma : \Delta^n \to S \), the projection map \( \Delta^n \times_S X \to \Delta^n \) is a left fibration (right fibration). Then \( f \) is a left fibration (right fibration). Consequently, if we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
S' & \longrightarrow & S
\end{array}
\]

where \( g \) is surjective and \( f' \) is a left fibration (right fibration), then \( f \) is also a left fibration (right fibration).

Remark 4.2.1.10. The collections of left and right are closed under filtered colimits. That is, suppose we are given a filtered diagram \( \{ f_\alpha : X_\alpha \to S_\alpha \} \) in the arrow category \( \text{Fun}([1], \text{Set}_\Delta) \), having colimit \( f : X \to S \). If each \( f_\alpha \) is a left fibration, then \( f \) is also a left fibration. If each \( f_\alpha \) is a right fibration, then \( f \) is also a right fibration.

Remark 4.2.1.11. Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of simplicial sets. If both \( f \) and \( g \) are left fibrations, then the composite map \( (g \circ f) : X \to Z \) is a left fibration. If both \( f \) and \( g \) are right fibrations, then \( g \circ f \) is a right fibration.

4.2.2 Left Anodyne and Right Anodyne Morphisms

To study left and right fibrations between simplicial sets, it is useful to consider the following counterpart of Definitions 3.1.2.1 and 1.4.6.4:

Definition 4.2.2.1 (Left Anodyne Morphisms). Let \( T_L \) be the smallest collection of morphisms in the category \( \text{Set}_\Delta \) with the following properties:
• For each $n > 0$ and each $0 < i < n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to $T_L$.

• The collection $T_L$ is weakly saturated (Definition 1.4.4.15). That is, $T_L$ is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets $f : A \to B$ is left anodyne if it belongs to $T_L$.

**Variant 4.2.2.2** (Right Anodyne Morphisms). Let $T_R$ be the smallest collection of morphisms in the category $\text{Set}_\Delta$ with the following properties:

• For each $n > 0$ and each $0 < i \leq n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to $T_R$.

• The collection $T_R$ is weakly saturated (Definition 1.4.4.15). That is, $T_R$ is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets $f : A \to B$ is right anodyne if it belongs to $T_R$.

**Remark 4.2.2.3.** Let $f : A \to B$ be a morphism of simplicial sets. Then $f$ is left anodyne if and only if the opposite morphism $f^{\text{op}} : A^{\text{op}} \to B^{\text{op}}$ is right anodyne.

**Remark 4.2.2.4.** Let $f : A \to B$ be a morphism of simplicial sets. If $f$ is either left or right anodyne, then it is anodyne (Definition 3.1.2.1). In particular, any left or right anodyne morphism of simplicial sets is a monomorphism (Remark 3.1.2.3) and a weak homotopy equivalence (Proposition 3.1.5.13). Conversely, if $f$ is inner anodyne (Definition 1.4.6.4), then it is both left anodyne and right anodyne. That is, we have inclusions

$$
\{\text{Inner anodyne morphisms}\} \subset \{\text{Left anodyne morphisms}\} \subset \{\text{Right anodyne morphisms}\} \subset \{\text{Anodyne morphisms}\}.
$$

All of these inclusions are strict (see Example 4.2.2.6).

**Proposition 4.2.2.5.** Let $q : X \to S$ be a morphism of simplicial sets. Then:

1. The morphism $q$ is a left fibration if and only if, for every square diagram of simplicial sets

$$
\begin{array}{ccc}
A & \to & X \\
\downarrow^i & & \downarrow^q \\
B & \to & S
\end{array}
$$

where $i$ is left anodyne, there exists a dotted arrow rendering the diagram commutative.
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(2) The morphism \( q \) is a right fibration if and only if, for every square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow^i & & \downarrow^q \\
B & \rightarrow & S
\end{array}
\]

where \( i \) is right anodyne, there exists a dotted arrow rendering the diagram commutative.

Proof. The “only if” directions are immediate from the definitions, and the “if” directions follow from Proposition 1.4.4.16. \( \square \)

Example 4.2.2.6. The inclusion map \( i_0 : \{0\} \hookrightarrow \Delta^1 \) is left anodyne (and therefore anodyne). However, it is not right anodyne (and therefore not inner anodyne). This follows from Proposition 4.2.2.5, since the lifting problem

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & \{0\} \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & \Delta^1
\end{array}
\]

does not admit a solution (note that the inclusion map \( i_0 : \{0\} \hookrightarrow \Delta^1 \) is a right fibration; see Warning 4.2.1.6).

Proposition 4.2.2.7. Let \( f : X \rightarrow Y \) be a morphism of simplicial sets. Then \( f \) can be factored as a composition \( X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y \), where \( f'' \) is a left fibration and \( f' \) is left anodyne. Moreover, the simplicial set \( Q(f) \) (and the morphisms \( f' \) and \( f'' \)) can be chosen to depend functorially on \( f \), in such a way that the functor

\[
\text{Fun}([1], \text{Set}_\Delta) \rightarrow \text{Set}_\Delta \quad (f : X \rightarrow Y) \rightarrow Q(f)
\]

commutes with filtered colimits.

Proof. We proceed as in the proof of Proposition 3.1.6.1. We construct a sequence of simplicial sets \( \{X(m)\}_{m \geq 0} \) and morphisms \( f(m) : X(m) \rightarrow Y \) by recursion. Set \( X(0) = X \) and \( f(0) = f \). Assuming that \( f(m) : X(m) \rightarrow Y \) has been defined, let \( S(m) \) denote the set.
of all commutative diagrams $\sigma$:

\[
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{v} & X(m) \\
| & & | \\
\downarrow & & \downarrow f(m) \\
\Delta^n & \xrightarrow{u_\sigma} & Y,
\end{array}
\]

where $0 \leq i < n$ and the left vertical map is the inclusion. For every such commutative diagram $\sigma$, let $C_\sigma = \Lambda^n_i$ denote the upper left hand corner of the diagram $\sigma$, and $D_\sigma = \Delta^n$ the lower left hand corner. Form a pushout diagram

\[
\begin{array}{ccc}
\Pi_{\sigma \in S(m)} C_\sigma & \xrightarrow{\pi} & X(m) \\
| & & | \\
\downarrow & & \downarrow \\
\Pi_{\sigma \in S(m)} D_\sigma & \xrightarrow{\pi} & X(m+1)
\end{array}
\]

and let $f(m+1) : X(m+1) \to Y$ be the unique map whose restriction to $X(m)$ is equal to $f(m)$ and whose restriction to each $D_\sigma$ is equal to $u_\sigma$. By construction, we have a direct system of left anodyne morphisms

\[
X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots
\]

Set $Q(f) = \lim_{\longrightarrow} X(m)$. Then the natural map $f' : X \to Q(f)$ is left anodyne (since the collection of left anodyne maps is closed under transfinite composition), and the system of morphisms $\{f(m)\}_{m \geq 0}$ can be amalgamated to a single map $f'' : Q(f) \to Y$ satisfying $f = f'' \circ f'$. It is clear from the definition that the construction $f \mapsto Q(f)$ is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that $f''$ is a left fibration: that is, that every lifting problem $\sigma$:

\[
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{v} & Q(f) \\
| & & | \\
\downarrow & & \downarrow f'' \\
\Delta^n & \xrightarrow{} & Y
\end{array}
\]

admits a solution (provided that $0 \leq i < n$). Let us abuse notation by identifying each $X(m)$ with its image in $Q(f)$. Since $\Lambda^n_i$ is a finite simplicial set, its image under $v$ is contained in
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$X(m)$ for some $m \gg 0$. In this case, we can identify $\sigma$ with an element of the set $S(m)$, so that the lifting problem

\[
\begin{array}{ccc}
\Lambda^n & \xrightarrow{v} & X(m + 1) \\
\downarrow & & \downarrow \ \ f(m+1) \\
\Delta^n & \rightarrow & Y
\end{array}
\]

admits a solution by construction.

\[\square\]

**Variant 4.2.2.8.** Let $f : X \to Y$ be a morphism of simplicial sets. Then $f$ can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where $f''$ is a right fibration and $f'$ is right anodyne. Moreover, the simplicial set $Q(f)$ (and the morphisms $f'$ and $f''$) can be chosen to depend functorially on $f$, in such a way that the functor

\[
\text{Fun([1], Set}_\Delta) \to \text{Set}_\Delta \quad (f : X \to Y) \to Q(f)
\]

commutes with filtered colimits.

Using Proposition 4.2.2.7 (and Variant 4.2.2.8), we obtain the following converse of Proposition 4.2.2.5:

**Corollary 4.2.2.9.** Let $i : A \to B$ be a morphism of simplicial sets. Then:

1. The morphism $i$ is left anodyne if and only if, for every square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow & & \downarrow f \\
B & \xrightarrow{f} & S
\end{array}
\]

where $f$ is left fibration, there exists a dotted arrow rendering the diagram commutative.

2. The morphism $i$ is right anodyne if and only if, for every square diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow & & \downarrow f \\
B & \xrightarrow{f} & S
\end{array}
\]

where $f$ is right fibration, there exists a dotted arrow rendering the diagram commutative.
Proof. We will prove (1); the proof of (2) is similar. Using Proposition 4.2.2.7, we can factor $i$ as a composition $A \xrightarrow{i'} Q \xrightarrow{f} B$, where $i'$ is left anodyne and $f$ is a left fibration. If the lifting problem

![Diagram](https://example.com/diagram.png)

admits a solution, then the map $r$ exhibits $i$ as a retract of $i'$ (in the arrow category Fun([1], Set_{Δ})). Since the collection of anodyne morphisms is closed under retracts, it follows that $i$ is anodyne. This proves the “if” direction of (1); the reverse implication follows from Proposition 4.2.2.5.

4.2.3 Exponentiation for Left and Right Fibrations

We now establish a stability property for left and right fibrations under exponentiation.

**Proposition 4.2.3.1.** Let $f: X \to S$ and $i: A \hookrightarrow B$ be morphisms of simplicial sets, where $i$ is a monomorphism, and let

$$
\rho: \text{Fun}(B, X) \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
$$

be the induced map. If $f$ is a left fibration, then $\rho$ is a left fibration. If $f$ is a right fibration, then $\rho$ is a right fibration.

**Corollary 4.2.3.2.** Let $f: X \to S$ be a morphism of simplicial sets, let $B$ be an arbitrary simplicial set, and let $\rho: \text{Fun}(B, X) \to \text{Fun}(B, S)$ be the morphism induced by composition with $f$. If $f$ is a left fibration, then $\rho$ is a left fibration. If $f$ is a right fibration, then $\rho$ is a right fibration.

Proposition 4.2.3.1 is essentially equivalent to the following stability property of left and right anodyne morphisms:

**Proposition 4.2.3.3.** Let $f: A \hookrightarrow B$ and $f': A' \hookrightarrow B'$ be monomorphisms of simplicial sets. If $f$ is left anodyne, then the induced map

$$
\theta: (A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'
$$

is left anodyne. If $f$ is right anodyne, then $\theta$ is right anodyne.
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Proof. We will prove the second assertion (the first follows by a similar argument). We proceed as in the proof of Proposition 3.1.2.7. Let us first regard the monomorphism \( f' : A' \hookrightarrow B' \) as fixed, and let \( T \) be the collection of all maps \( f : A \rightarrow B \) for which the induced map

\[
\theta_{f,f'} : (A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'
\]

is right anodyne. We wish to show that every right anodyne morphism belongs to \( T \). Since \( T \) is weakly saturated, it will suffice to show that every horn inclusion \( f : \Lambda^n_i \hookrightarrow \Delta^n \) belongs to \( T \) for \( 0 < i \leq n \). In this case, Lemma 3.1.2.8 guarantees that \( f \) is a retract of the morphism \( g : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Delta^n} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n \). It will therefore suffice to show that \( g \) belongs to \( T \). Replacing \( f' \) by the monomorphism \((\Lambda^n_i \times B') \coprod_{\Lambda^n_i \times A'} (\Delta^n \times A') \hookrightarrow \Delta^n \times B' \), we are reduced to showing that the inclusion \( \{1\} \hookrightarrow \Delta^1 \) belongs to \( T \).

Let \( T' \) denote the collection of all morphisms of simplicial sets \( f'' : A'' \rightarrow B'' \) for which the map \((\{1\} \times B'') \coprod_{\{1\} \times A''} (\Delta^1 \times A'') \hookrightarrow \Delta^1 \times B'' \) is right anodyne. We will complete the proof by showing that \( T' \) contains all monomorphisms of simplicial sets. By virtue of Proposition 1.4.5.12, it will suffice to show that \( T'' \) contains the inclusion map \( \partial \Delta^m \hookrightarrow \Delta^m \), for each \( m > 0 \). In other words, we are reduced to showing that the inclusion \((\{1\} \times \Delta^m) \coprod_{\{1\} \times \partial \Delta^m} (\Delta^1 \times \partial \Delta^m) \hookrightarrow \Delta^1 \times \Delta^m \) is right anodyne, which follows from Lemma 3.1.2.9. \( \square \)

Proof of Proposition 4.2.3.1. Let \( f : X \rightarrow S \) be a left fibration of simplicial sets and let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets. We wish to show that the restriction map

\[
\rho : \text{Fun}(B, X) \rightarrow \text{Fun}(B, S) \times_{\text{Fun}(A, S) \times \text{Fun}(A, X)} \text{Fun}(A, X)
\]

is also a left fibration (the dual assertion about right fibrations follows by passing to opposite simplicial sets). By virtue of Proposition 4.2.2.5, this is equivalent to the assertion that every lifting problem

\[
\begin{array}{ccc}
A' & \longrightarrow & \text{Fun}(B, X) \\
\downarrow & & \downarrow \rho \\
B' & \longrightarrow & \text{Fun}(B, S) \times_{\text{Fun}(A, S) \times \text{Fun}(A, X)} \text{Fun}(A, X)
\end{array}
\]

admits a solution, provided that \( i' \) is left anodyne. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \longrightarrow & X \\
\downarrow & & \downarrow f \\
B \times B' & \longrightarrow & S
\end{array}
\]

admits a solution, provided that \( i' \) is left anodyne. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \longrightarrow & X \\
\downarrow & & \downarrow f \\
B \times B' & \longrightarrow & S
\end{array}
\]
admits a solution. This follows from Proposition 4.2.2.5 since the left vertical map is left anodyne (Proposition 4.2.3.3) and the right vertical map is a left fibration.

Proposition 4.2.3.3 has another application, which will be useful in the next section:

**Proposition 4.2.3.4.** Let \( f : X \to S \) and \( i : A \to B \) be morphisms of simplicial sets, and let

\[
\rho : \text{Fun}(B, X) \to \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\]

be the induced map. If \( f \) is a left fibration and \( i \) is left anodyne, then \( \rho \) is a trivial Kan fibration. If \( f \) is a right fibration and \( i \) is right anodyne, then \( \rho \) is a trivial Kan fibration.

**Proof.** We proceed as in the proof of Proposition 4.2.3.1. Assume that \( f \) is a left fibration and that \( i \) is left anodyne; we will show that \( \rho \) is a trivial Kan fibration (the dual assertion for right fibrations follows by a similar argument). Fix a monomorphism of simplicial sets \( i' : A' \to B' \); we wish to show that every lifting problem

\[
\begin{array}{ccc}
A' & \longrightarrow & \text{Fun}(B, X) \\
\downarrow i' & & \downarrow \rho \\
B' & \longrightarrow & \text{Fun}(B, S) \times_{\text{Fun}(A, S)} \text{Fun}(A, X)
\end{array}
\]

admits a solution. Equivalently, we must show that every lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{A \times A'} (B \times A') & \longrightarrow & X \\
\downarrow & & \downarrow f \\
B \times B' & \longrightarrow & S
\end{array}
\]

admits a solution. This follows from Proposition 4.2.2.5 since the left vertical map is left anodyne (Proposition 4.2.3.3) and the right vertical map is a left fibration. □

### 4.2.4 The Homotopy Extension Lifting Property

We now show that left and right fibrations can be characterized by homotopy lifting properties.

**Proposition 4.2.4.1.** Let \( f : X \to S \) be a morphism of simplicial sets. Then:

- The morphism \( f \) is a left fibration if and only if the evaluation map

\[
\text{ev}_0 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{0\}, X) \times_{\text{Fun}(\{0\}, \text{S})} \text{Fun}(\Delta^1, \text{S})
\]

is a trivial Kan fibration.
The morphism $f$ is a right fibration if and only if the evaluation map

$$ev_1 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{1\}, X) \times_{\text{Fun}(\{0\}, S)} \text{Fun}(\Delta^1, S)$$

is a trivial Kan fibration.

**Proof.** We prove the second assertion; the first follows by passing to opposite simplicial sets. If $f$ is a right fibration, then the evaluation map $ev_1$ is a trivial Kan fibration by virtue of Proposition 4.2.3.4 (since the inclusion $\{1\} \hookrightarrow \Delta^1$ is right anodyne). Conversely, suppose that $ev_1$ is a trivial Kan fibration. Then every lifting problem

$$(\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \rightarrow X$$

admits a solution. In other words, $f$ has the right lifting property with respect to the inclusion map

$$u : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n.$$ If $0 < i \leq n$, then the horn inclusion $u_0 : \Lambda^n_i \hookrightarrow \Delta^n$ is a retract of $u$ (Lemma 3.1.2.8). It follows that $f$ also has the right lifting property with respect to $u_0$ (Proposition 1.4.4.9): that is, every lifting problem

$$(\Lambda^n_i \rightarrow X)$$

admits a solution. □

**Corollary 4.2.4.2.** Let $f : X \rightarrow S$ be a morphism of simplicial sets. Then $f$ is a Kan fibration if and only if both of the evaluation maps

$$ev_0 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{0\}, X) \times_{\text{Fun}(\{0\}, S)} \text{Fun}(\Delta^1, S)$$

$$ev_1 : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{1\}, X) \times_{\text{Fun}(\{1\}, S)} \text{Fun}(\Delta^1, S)$$

are trivial Kan fibrations.

**Proof.** Combine Proposition 4.2.4.1 with Example 4.2.1.5 □
Remark 4.2.4.3 (The Homotopy Extension Lifting Property). Let \( f : X \to S \) be a morphism of simplicial sets. Unwinding the definitions, we see that the following conditions are equivalent:

- The morphism \( f \) is a left fibration.
- For every monomorphism of simplicial sets \( i : A \hookrightarrow B \), every lifting problem

\[
\begin{array}{ccc}
A & \to & \text{Fun}(\Delta^1, X) \\
\downarrow & & \downarrow \text{ev}_0 \\
B & \to & \text{Fun}(\{0\}, X) \times_{\text{Fun}(\{0\}, S)} \text{Fun}(\Delta^1, S)
\end{array}
\]

admits a solution (indicated by the dotted arrow in the diagram).
- For every monomorphism of simplicial sets \( i : A \hookrightarrow B \), every lifting problem

\[
\begin{array}{ccc}
(\Delta^1 \times A) \coprod_{\{0\} \times A}(\{0\} \times B) & \to & X \\
\downarrow & & \downarrow f \\
\Delta^1 \times B & \to & S
\end{array}
\]

admits a solution (indicated by the dotted arrow in the diagram).
- Let \( u : B \to X \) be a map of simplicial sets and let \( \overline{h} : \Delta^1 \times B \to S \) be a map satisfying \( \overline{h}|_{\{0\} \times B} = f \circ u \): that is, \( \overline{h} \) is a homotopy from \( f \circ u \) to another map \( \overline{v} = \overline{h}|_{\{1\} \times B} \). Then we can choose a map of simplicial sets \( h : \Delta^1 \times B \to X \) satisfying \( f \circ h = \overline{h} \) and \( h|_{\{0\} \times B} = u \): in other words, \( \overline{h} \) can be lifted to a homotopy \( h \) from \( u \) to another map \( v = h|_{\{1\} \times B} \). Moreover, given any simplicial subset \( A \subseteq B \) and any map \( h_0 : \Delta^1 \times A \to X \) satisfying \( f \circ h_0 = \overline{h}|_{\Delta^1 \times A} \) and \( h_0|_{\{0\} \times A} = u|_A \), we can arrange that \( h \) is an extension of \( h_0 \).

In the special case where \( B = \Delta^0 \) and \( A = \emptyset \), each of these assertions reduces to the left path lifting property of \( f \).

4.2.5 Example: Fibrations in Groupoids

Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. Then \( F \) induces a morphism of simplicial sets \( N_\bullet(F) : N_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{D}) \). In this case, the requirement that \( N_\bullet(F) \) is a right fibration can be formulated directly in terms of the functor \( F \), without reference to the theory of simplicial sets (see Proposition 4.2.5.12 below).

Definition 4.2.5.1. Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. We say that \( F \) is a *fibration in groupoids* if the following conditions are satisfied:
(A) For every object $Y \in \mathcal{C}$ and every morphism $\overline{u} : \overline{X} \to F(Y)$ in $\mathcal{D}$, there exists a morphism $u : X \to Y$ in $\mathcal{C}$ with $\overline{X} = F(X)$ and $\overline{u} = F(u)$.

(B) For every morphism $v : Y \to Z$ in $\mathcal{C}$ and every object $X \in \mathcal{C}$, the diagram of sets

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(X,Y) & \xrightarrow{v \circ} & \text{Hom}_\mathcal{C}(X,Z) \\
\downarrow F & & \downarrow F \\
\text{Hom}_\mathcal{D}(F(X),F(Y)) & \xrightarrow{F(v) \circ} & \text{Hom}_\mathcal{D}(F(X),F(Z))
\end{array}
\]

is a pullback square.

In this case, we will also say that $\mathcal{C}$ is fibered in groupoids over $\mathcal{D}$.

Remark 4.2.5.2. The notion of a fibration in groupoids was introduced by Grothendieck in [19] (Exposé 6).

Remark 4.2.5.3. Condition (B) of Definition 4.2.5.1 can be rephrased as follows: given any commutative diagram

in the category $\mathcal{D}$ and any partially defined lift

\[
\begin{array}{ccc}
X & \xrightarrow{w} & Z \\
\uparrow u & & \uparrow w \\
Y & \xrightarrow{v} & \overline{Y}
\end{array}
\]

to a diagram in $\mathcal{C}$ (so that $F(u) = \overline{u}$ and $F(w) = \overline{w}$), there exists a unique extension as indicated (that is, a unique morphism $u : X \to Y$ in $\mathcal{C}$ satisfying $F(u) = \overline{u}$).

Warning 4.2.5.4. The requirement that a functor $F : \mathcal{C} \to \mathcal{D}$ is a fibration in groupoids is not invariant under equivalence. For example, an equivalence of categories need not be a fibration in groupoids.
**Variant 4.2.5.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that $F$ is an *opfibration in groupoids* if the opposite functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is a fibration in groupoids. In other words, $F$ is an opfibration in groupoids if and only if it satisfies the following conditions:

(A') For every object $X \in \mathcal{C}$ and every morphism $\overline{u} : F(X) \to Y$ in $\mathcal{D}$, there exists a morphism $u : X \to Y$ in $\mathcal{C}$ with $Y = F(Y)$ and $\overline{u} = F(u)$.

(B') For every morphism $u : X \to Y$ in $\mathcal{C}$ and every object $Z \in \mathcal{C}$, the diagram of sets

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(Y, Z) & \xrightarrow{\circ u} & \text{Hom}_{\mathcal{C}}(X, Z) \\
\downarrow F & & \downarrow F \\
\text{Hom}_{\mathcal{D}}(F(Y), F(Z)) & \xrightarrow{\circ F(u)} & \text{Hom}_{\mathcal{D}}(F(X), F(Z))
\end{array}
$$

is a pullback square.

**Warning 4.2.5.6.** Some authors use the term *cofibration in groupoids* to refer to what we call an opfibration in groupoids. We will avoid the use of the word “cofibration” in this context, since it appears often in homotopy theory with a very different meaning.

**Example 4.2.5.7.** Let $\mathcal{C}$ be a category, let $[0]$ denote the category having a single object and a single morphism, and let $F : \mathcal{C} \to [0]$ be the unique functor. Then $F$ automatically satisfies condition (A) of Definition 4.2.5.1. Condition (B) asserts that for every morphism $v : Y \to Z$ in $\mathcal{C}$, the composition map

$$
\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Z) \quad u \mapsto v \circ u
$$

is bijective for every object $X \in \mathcal{C}$: that is, $v$ is an isomorphism. It follows that $F$ is a fibration in groupoids if and only if the category $\mathcal{C}$ is a groupoid. Similarly, $F$ is an opfibration in groupoids if and only if $\mathcal{C}$ is a groupoid.

**Remark 4.2.5.8.** Suppose we are given a pullback diagram

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{F'} & \mathcal{C} \\
\downarrow F' & & \downarrow F \\
\mathcal{D}' & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

in the ordinary category $	ext{Cat}$ (so that the category $\mathcal{C}'$ is isomorphic to the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$). If $F$ is a fibration in groupoids, then so is $F'$. Similarly, if $F$ is an opfibration in groupoids, then so is $F'$. 
Remark 4.2.5.9. Let $F : C \to D$ be a functor between categories. For each object $D \in D$, let $C_D = \{D\} \times_D C$ denote the corresponding fiber of $F$ (more concretely, $C_D$ is the subcategory of $C$ spanned by those objects $C \in C$ satisfying $F(C) = D$, and those morphisms $u : C \to C'$ satisfying $F(u) = \text{id}_D$). It follows from Remark 4.2.5.8 that if $F$ is a fibration in groupoids, then the projection map $C_D \to \{D\}$ is also a fibration in groupoids, so that the category $C_D$ is a groupoid (Example 4.2.5.7). This observation motivates the terminology of Definition 4.2.5.1: if $F$ is a fibration in groupoids, then one can think of the category $C$ as the total space of a “family” of groupoids $\{C_D\}_{D \in D}$ which is parametrized by the category $D$.

Warning 4.2.5.10. The converse of Remark 4.2.5.9 is generally false: if $F : C \to D$ is a functor having the property that each fiber $C_D$ is a groupoid, then $F$ need not be a fibration in groupoids. For example, this condition is also satisfied whenever $F$ is an opfibration in groupoids, but an opfibration in groupoids need not be a fibration in groupoids (Exercise 4.2.5.11). Roughly speaking, one can think of a fibration in groupoids $F : C \to D$ as encoding a family of groupoids $\{C_D\}$ having a contravariant dependence on the object $D \in D$, and an opfibration in groupoids $F : C \to D$ as encoding a family of groupoids $\{C_D\}$ having a covariant dependence on the object $D \in D$ (for a more precise formulation of this idea, we refer the reader to §[?]).

Exercise 4.2.5.11. Define a category $\text{Set}_*$ as follows:

- The objects of $\text{Set}_*$ are pairs $(X, x)$, where $X$ is a set and $x \in X$ is an element.

- A morphism from $(X, x)$ to $(Y, y)$ in $\text{Set}_*$ is a function $f : X \to Y$ satisfying $f(x) = y$.

We will refer to $\text{Set}_*$ as the category of pointed sets. Let $F : \text{Set}_* \to \text{Set}$ denote the forgetful functor, given on objects by the construction $(X, x) \mapsto X$. Show that $F$ is an opfibration in groupoids, but not a fibration in groupoids.

The notion of a fibration (opfibration) in groupoids can be regarded as a special case of the notion of a right (left) fibration between simplicial sets:

Proposition 4.2.5.12. Let $C$ and $D$ be categories. Then:

1. A functor $F : C \to D$ is a fibration in groupoids if and only if the induced map $N_* (F) : N_* (C) \to N_* (D)$ is a right fibration of simplicial sets.

2. A functor $F : C \to D$ is an opfibration in groupoids if and only if the induced map $N_* (F) : N_* (C) \to N_* (D)$ is a left fibration of simplicial sets.
Proof. We will prove (1); the proof of (2) is similar. Assume first that $F$ is a fibration in groupoids; we wish to show that for every pair of integers $0 < i \leq n$, every lifting problem

$$
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{\sigma_0} & N_\bullet(C) \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\tau} & N_\bullet(D)
\end{array}
$$

admits a solution. If $0 < i < n$, then $\sigma_0$ admits a unique extension $\sigma : \Delta^n \to N_\bullet(C)$ (Proposition 1.2.3.1). Moreover, since $N_\bullet(F) \circ \sigma$ and $\tau$ coincide on the simplicial subset $\Lambda^n_i \subseteq \Delta^n$, they automatically coincide (again by Proposition 1.2.3.1). We may therefore assume without loss of generality that $i = n$. We consider four cases:

- If $n = 1$, then the existence of a solution to the lifting problem (4.1) is equivalent to condition (A) of Definition 4.2.5.1, and is therefore ensured by our assumption that $F$ is a fibration in groupoids.

- If $n = 2$, then the existence of a solution to the lifting problem (4.1) follows from condition (B) of Definition 4.2.5.1 (see Remark 4.2.5.3), and is again ensured by our assumption that $F$ is a fibration in groupoids.

- If $n = 3$, then the morphism $\sigma_0$ encodes a collection of objects $\{X_j\}_{0 \leq j \leq 3}$ and morphisms $\{f_{kj} : X_j \to X_k\}_{0 \leq j < k \leq 3}$ in the category $C$, which satisfy the identities

  $$
  f_{30} = f_{31} \circ f_{10} \quad f_{30} = f_{32} \circ f_{20} \quad f_{31} = f_{32} \circ f_{21}.
  $$

  To extend $\sigma_0$ to a 3-simplex $\sigma$ of $N_\bullet(C)$, we must show that $f_{20} = f_{21} \circ f_{10}$ (note that any such extension automatically satisfies $\tau = N_\bullet(F) \circ \sigma$, since the horn $\Lambda^3_3$ contains the 1-skeleton of $\Delta^3$). Invoking our assumption that $F$ is a fibration in groupoids, we deduce that the map

  $$
  \text{Hom}_C(X_0, X_2) \to \text{Hom}_C(X_0, X_3) \times \text{Hom}_D(F(X_0), F(X_2)) \quad u \mapsto (f_{32} \circ u, F(u))
  $$

  is injective. Using the calculation

  $$
  f_{32} \circ f_{20} = f_{30} = f_{31} \circ f_{10} = (f_{32} \circ f_{21}) \circ f_{10} = f_{32} \circ (f_{21} \circ f_{10}),
  $$

  we are reduced to proving that $F(f_{20})$ is equal to $F(f_{21} \circ f_{10}) = F(f_{21}) \circ F(f_{10})$, which follows from the existence of the 2-simplex $\tau$.

- If $n \geq 4$, then the horn $\Lambda^n_i$ contains the 2-skeleton of $\Delta^n$. It follows that $\sigma_0$ admits a unique extension to a map $\sigma : \Delta^n \to N_\bullet(C)$, which automatically satisfies $\tau = N_\bullet(F) \circ \sigma$. 

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We now prove the converse. Assume that \( N_\bullet(F) \) is a right fibration of simplicial sets; we wish to show that \( F \) is a fibration in groupoids. As above, we note that condition (A) of Definition 4.2.5.1 follows from the solvability of the lifting problem (4.1) in the special case \( i = n = 1 \). To verify condition (B), we must show that for every diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{v} & Z \\
\downarrow & & \\
X & \xrightarrow{w} & Z
\end{array}
\]

in the category \( \mathcal{C} \) and every compatible extension

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{F(v)} & F(Z) \\
\downarrow^{F(w)} & & \\
F(X) & \xrightarrow{F(u)} & F(Z)
\end{array}
\]

in the category \( \mathcal{D} \), there exists a unique morphism \( u : X \to Y \) in \( \mathcal{C} \) satisfying \( F(u) = \pi \) and \( v \circ u = w \). The existence of \( u \) follows from the solvability of the lifting problem (4.1) in the special case \( i = n = 2 \). To prove uniqueness, suppose we are given a pair of morphisms \( u, u' : X \to Y \) in \( \mathcal{C} \) satisfying \( F(u) = \pi = F(u') \) and \( v \circ u = w = v \circ u' \). Consider the not-necessarily-commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{id_Y} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{w} & Z
\end{array}
\]

in the category \( \mathcal{C} \). Every triangle in this diagram commutes with the possible exception of the upper left, so it determines a map of simplicial sets \( \sigma_0 : \Lambda_3^2 \to N_\bullet(\mathcal{C}) \). Moreover, the equation \( F(u) = F(u') \) guarantees that \( N_\bullet(F) \circ \sigma_0 \) extends to a 3-simplex \( \tau \) of \( N_\bullet(\mathcal{D}) \). Invoking the solvability of the lifting problem (4.1) in the case \( i = n = 3 \), we conclude that \( \sigma_0 \) can be extended to a 3-simplex of \( \mathcal{C} \), which proves that \( u' = id_Y \circ u = u \). \( \square \)

4.3 The Slice and Join Constructions
Let $F : \mathcal{K} \to \mathcal{C}$ be a functor between categories. A cone over $F$ is an object $C \in \mathcal{C}$ together with a collection of morphisms $\{ \alpha_K : C \to F(K) \}_{K \in \mathcal{K}}$ with the following property: for every morphism $\beta : K \to K'$ of the category $\mathcal{K}$, the diagram

\[\begin{array}{ccc}
C & \xrightarrow{\alpha_K} & F(K) \\
\alpha_{K'} & & F(\beta) \\
\downarrow & & \downarrow \\
F(K') & \xrightarrow{\alpha_{K'}} & F(K')
\end{array}\]

commutes. The collection of cones $(C, \{ \alpha_K \}_{K \in \mathcal{K}})$ can be organized into a category, which we will denote by $\mathcal{C}/F$ and refer to as the slice category of $\mathcal{C}$ over $F$ (Construction 4.3.1.7). This construction plays an important role in category theory: for example, a limit of the diagram $F$ is (by definition) a final object of the category $\mathcal{C}/F$.

Our goal in this section is to generalize the construction $(F : \mathcal{K} \to \mathcal{C}) \mapsto \mathcal{C}/F$ to the setting of $\infty$-categories. Our first step is to show that the slice category $\mathcal{C}/F$ can be characterized by a universal property. In §4.3.2, we associate to every pair of categories $\mathcal{D}$ and $\mathcal{K}$ a new category $\mathcal{D} \star \mathcal{K}$, which we refer to as the join of $\mathcal{D}$ and $\mathcal{K}$ (Definition 4.3.2.1). This is a new category which contains $\mathcal{D}$ and $\mathcal{K}$ as full subcategories, having a unique morphism from each object of $\mathcal{D}$ to each object of $\mathcal{K}$ (and no morphisms in the opposite direction). We then show the datum of a functor $\mathcal{D} \to \mathcal{C}/F$ is equivalent to the datum of a functor $F : \mathcal{D} \star \mathcal{K} \to \mathcal{C}$ satisfying $F|_{\mathcal{K}} = F$ (Proposition 4.3.2.10).

In §4.3.3, we extend the join construction to the setting of $\infty$-categories. To every pair of simplicial sets $X$ and $Y$, we associate a new simplicial set $X \star Y$ (Construction 4.3.3.13), which contains $X$ and $Y$ as (disjoint) simplicial subsets. This construction has the following features:

- For every pair of categories $\mathcal{C}$ and $\mathcal{D}$, there is a canonical isomorphism of simplicial sets $N_* \mathcal{C} \star N_* \mathcal{D} \simeq N_* ( \mathcal{C} \star \mathcal{D} )$ (Example 4.3.3.20). Consequently, the join operation on simplicial sets can be regarded as a generalization of the join operation on categories.

- For every pair of $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, the join $\mathcal{C} \star \mathcal{D}$ is an $\infty$-category (Proposition 4.3.3.21).

- For every pair of simplicial sets $X$ and $Y$, the join $X \star Y$ is equipped with a continuous bijection $|X \star Y| \simeq X \coprod_{(X \times \{0\} \times Y)} (X \times [0,1] \times Y) \coprod_{(X \times \{1\} \times Y)} Y$, which is a homeomorphism if either $X$ or $Y$ is finite (Proposition 4.3.4.11 and Corollary 4.3.4.12).
Let \( f : K \to X \) be any morphism of simplicial sets. In \[4.3.5\] we introduce a new simplicial set \( X_f \), which we will refer to as the slice of \( X \) over \( f \) (Construction \[4.3.5.1\]). The simplicial set \( X_f \) is characterized (up to isomorphism) by the following universal mapping property: for any simplicial set \( Y \), the datum of a morphism of simplicial sets \( Y \to X_f \) is equivalent to the datum of a morphism of simplicial sets \( Y \star K \to X \) satisfying \( f|_K = f \) (Proposition \[4.3.5.14\]). Moreover, we will show that it has the following additional properties:

- If \( F : K \to C \) is a functor between categories and \( N_\bullet(F) : N_\bullet(K) \to N_\bullet(C) \) is the associated map of simplicial sets, then there is a canonical isomorphism of simplicial sets \( N_\bullet(C)/N_\bullet(F) \cong N_\bullet(C/F) \) (Example \[4.3.5.7\]). Consequently, the slice operation on simplicial sets can be regarded as a generalization of the slice operation on categories.

- If \( C \) is an \( \infty \)-category and \( f : K \to C \) is a morphism of simplicial sets, then the simplicial set \( C/f \) is also an \( \infty \)-category. Moreover, the evident forgetful functor \( C/f \to C \) is a right fibration of \( \infty \)-categories (Proposition \[4.3.6.1\]).

- If \( q : X \to S \) is a left fibration of simplicial sets and \( f : K \to X \) is any morphism of simplicial sets, then the natural map \( X_f \to X \times_S S_{(q\circ f)} \) is a Kan fibration of simplicial sets (Corollary \[4.3.7.3\]).

- If \( q : X \to S \) is a right fibration of simplicial sets and \( x \in X \) is a vertex (which we identify with a map of simplicial sets \( \Delta^0 \to X \) having image \( s \in S \)), then the induced map \( X_x \to S_s \) is a trivial Kan fibration of simplicial sets (Corollary \[4.3.7.12\]).

### 4.3.1 Slices of Categories

We begin by discussing the slice construction in a special case.

**Construction 4.3.1.1 (Slice Categories over Objects).** Let \( C \) be a category containing an object \( S \). We define a category \( C/S \) as follows:

- The objects of \( C/S \) are pairs \((X, f)\), where \( X \) is an object of \( C \) and \( f : X \to S \) is a morphism in \( C \).

- If \((X, f)\) and \((Y, g)\) are objects of \( C/S \), then a morphism from \((X, f)\) to \((Y, g)\) in the category \( C/S \) is a morphism \( u : X \to Y \) in the category \( C \) satisfying \( f = g \circ u \). In other words, morphisms from \((X, f)\) to \((Y, g)\) are given by commutative diagrams:

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{g} \\
S \\
\end{array}
\]
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in the category $C$.

- Composition of morphisms in the category $C/S$ is given by composition of morphisms in the category $C$.

We will refer to $C/S$ as the *slice category of $C$ over $S$.*

**Example 4.3.1.2.** Let $\text{Set}$ denote the category of sets, and let $S \in \text{Set}$ be a set. Then the construction

$$(f : X \to S) \mapsto \{X_s = f^{-1}\{s\}\}_{s \in S}$$

induces an equivalence of categories $\text{Set}\!/S \to \prod_{s \in S} \text{Set}$. 

**Remark 4.3.1.3.** Let $C$ be a category which admits finite limits and let $\ast$ denote a final object of $C$. For any object $S \in C$, one can adapt the construction of Example 4.3.1.2 to define a functor

$$F : C/S \to \prod_{s : \ast \to S} C \quad F(X \to S) = \{\ast \times_s X\}_{s : \ast \to S}.$$ 

Motivated by this observation, it is often useful to think of objects of the slice category $C/S$ as “families” of objects of $C$ which are parametrized by $S$. Beware that the functor $F$ is usually not an equivalence of categories.

**Variant 4.3.1.4 (Coslice Categories under Objects).** Let $C$ be a category containing an object $S$. We define a category $C_{S/}$ as follows:

- The objects of $C_{S/}$ are pairs $(X, f)$, where $X$ is an object of $C$ and $f : S \to X$ is a morphism in $C$.

- If $(X, f)$ and $(Y, g)$ are objects of $C_{S/}$, then a morphism from $(X, f)$ to $(Y, g)$ in the category $C/S$ is a morphism $u : X \to Y$ in the category $C$ satisfying $g = f \circ u$. In other words, morphisms from $(X, f)$ to $(Y, g)$ are given by commutative diagrams

$$\begin{array}{ccc} S & \xrightarrow{f} & X \\ \downarrow{g} & & \downarrow{u} \\ \ast & \xrightarrow{} & Y \end{array}$$

in the category $C$.

- Composition of morphisms in the category $C_{S/}$ is given by composition of morphisms in the category $C$. 
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We will refer to $C_S/\cdot as the \textit{coslice category of } C \textit{ under } S.

**Remark 4.3.1.5.** Variant 4.3.1.4 is formally dual to Construction 4.3.1.1. More precisely, if $S$ is an object of a category $C$, then we have a canonical isomorphism of categories

$$(C/S)^{\text{op}} \cong (C^{\text{op}})_S,$$

where we view $S$ also as an object of the opposite category $C^{\text{op}}$.

**Remark 4.3.1.6 (Slice Categories as Comma Categories).** Let $C$ be a category and let $\text{Fun}([1],C)$ denote the arrow category of $C$, so that the elements $0, 1 \in [1]$ determine evaluation functors

$$\text{ev}_0 : \text{Fun}([1],C) \to \text{Fun}([0],C) \simeq C \quad \text{ev}_1 : \text{Fun}([1],C) \to \text{Fun}([1],C) \simeq C.$$ 

For each object $S \in C$, the slice category $C/S$ can be identified with the fiber of the evaluation functor $\text{ev}_1$ over $S$, and the coslice category $C_S/\cdot$ can be identified with the fiber of the evaluation functor $\text{ev}_0$ over $S$. That is, we have pullback diagrams

$$
\begin{array}{ccc}
C_S/ & \longrightarrow & \text{Fun}([1],C) \\
\downarrow \text{ev}_0 & & \downarrow \\
\{S\} & \longrightarrow & C \\
\end{array}
\quad
\begin{array}{ccc}
C/\cdot & \longrightarrow & \text{Fun}([1],C) \\
\downarrow & & \downarrow \\
\{S\} & \longrightarrow & C .
\end{array}
$$

In other words, we can identify $C/S$ with the comma category $C \downarrow S$ of Notation 2.1.4.19 (here we identify the object $S$ with the constant functor $[0] \to C$ taking the value $S$), and $C_S/\cdot$ with the comma category $S \downarrow C$.

For many applications it is useful to consider a generalization of Construction 4.3.1.1, which associates a slice category $C/F$ to an arbitrary diagram $F : \mathcal{K} \to C$ (instead of a single object $S \in C$).

**Construction 4.3.1.7 (Slice Categories over Diagrams).** Let $\mathcal{K}$ and $C$ be categories. For each object $C \in \mathcal{C}$, we let $\underline{C} : \mathcal{K} \to C$ denote the associated constant functor (carrying each object of $\mathcal{K}$ to the object $C$ and each morphism of $\mathcal{K}$ to the identity morphism $\text{id}_C$). The construction $C \mapsto \underline{C}$ determines a functor $C \to \text{Fun}(\mathcal{K},C)$.

For every functor $F : \mathcal{K} \to C$, we let $C/F$ denote the fiber product $C \times_{\text{Fun}(\mathcal{K},C)} \text{Fun}(\mathcal{K},C)/_F$, where $\text{Fun}(\mathcal{K},C)/_F$ is the slice category of Construction 4.3.1.1. Similarly, we let $C/F\cdot$ denote the fiber product $C \times_{\text{Fun}(\mathcal{K},C)} \text{Fun}(\mathcal{K},C)_{/F}$, where $\text{Fun}(\mathcal{K},C)_{/F}$ denotes the coslice category of Variant 4.3.1.4. We will refer to $C/F$ as the \textit{slice category of } $C \textit{ over } F$, and to $C/F\cdot$ as the \textit{coslice category of } $C \textit{ under } F$. 
Remark 4.3.1.8. The slice and coslice constructions of Construction 4.3.1.7 are mutually dual. More precisely, if $F : K \to C$ is a functor between categories and $F^{op} : K^{op} \to C^{op}$ is the induced functor between opposite categories, then we have canonical isomorphisms
\[(C/F)^{op} \simeq (C^{op})_{F^{op}}/ \quad (C_{F/})^{op} \simeq (C^{op})/F^{op} .\]

Example 4.3.1.9. Let $[0]$ denote the category having a single object and a single morphism. For any category $C$, the diagonal map $\delta : C \to \text{Fun}([0], C)$ is an isomorphism of categories. It follows that, for any object $S \in C$, we have canonical isomorphisms
\[C/S \simeq C_{/S} \quad C_{S/} \simeq C_{S/} .\]
Consequently, we can view Construction 4.3.1.1 and Variant 4.3.1.4 as special cases of Construction 4.3.1.7.

Remark 4.3.1.10. Let $F : K \to C$ be a functor between categories. Using Remark 4.3.1.6, we see that the slice and coslice categories of Construction 4.3.1.7 are given by the formulae
\[C/F \simeq C \times \text{Fun}([0] \times K, C) \times \text{Fun}([1] \times K, C) \times \{F\} , \quad C_{F/} \simeq \{F\} \times \text{Fun}([0] \times K, C) \times \text{Fun}([1] \times K, C) \times \text{C} .\]

Remark 4.3.1.11. Let $C$ be a category and let $F : K \to C$ be a diagram in $C$. If $F$ admits a limit $S = \lim_{\leftarrow I \in K} F(I)$, then the slice category $C/F$ is isomorphic to $C_{/S}$. Similarly, if $F$ admits a colimit $S' = \lim_{\rightarrow I \in K} F(I)$, then the coslice category $C_{F/}$ is isomorphic to $C_{S'}$. We will return to this point later (see Remark [?]).

4.3.2 Joins of Categories

Our next goal is to characterize the slice categories of Construction 4.3.1.7 by a universal mapping property.

Definition 4.3.2.1 (Joins of Categories). Let $C$ and $D$ be categories. We define a category $C \star D$ as follows:

- The set of objects $\text{Ob}(C \star D)$ is the disjoint union of $\text{Ob}(C)$ with $\text{Ob}(D)$.

- Given a pair of objects $X, Y \in \text{Ob}(C \star D)$, we have
  \[\text{Hom}_{C \star D}(X, Y) = \begin{cases} 
  \text{Hom}_C(X, Y) & \text{if } X, Y \in \text{Ob}(C) \\
  \text{Hom}_D(X, Y) & \text{if } X, Y \in \text{Ob}(D) \\
  * & \text{if } X \in \text{Ob}(C), Y \in \text{Ob}(D) \\
  \emptyset & \text{if } X \in \text{Ob}(D), Y \in \text{Ob}(C). 
  \end{cases} \]
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- Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms in \( C \star D \). If \( X, Y, Z \in \text{Ob}(C) \), then \( g \circ f \in \text{Hom}_{C \star D}(X, Z) \) is given by the composition of morphisms in \( C \). If \( X, Y, Z \in \text{Ob}(D) \), then \( g \circ f \) is given by composition of morphisms in \( D \). Otherwise, we let \( g \circ f \) denote the unique morphism from \( X \) to \( Z \) (note that in this case, we necessarily have \( X \in \text{Ob}(C) \) and \( Z \in \text{Ob}(D) \)).

We will refer to \( C \star D \) as the join of \( C \) with \( D \).

Remark 4.3.2.2. In the situation of Definition 4.3.2.1, we will generally abuse notation by identifying \( C \) and \( D \) with full subcategories of the join \( C \star D \).

Remark 4.3.2.3. Let \( F : C \to C' \) and \( G : D \to D' \) be functors. Then \( F \) and \( G \) induce a functor
\[
(F \star G) : C \star D \to C' \star D',
\]
which is uniquely determined by the requirement that it coincides with \( F \) on the full subcategory \( C \subseteq C \star D \) and with \( G \) on the full subcategory \( D \subseteq C \star D \). We can therefore regard the join construction as a functor
\[
\star : \text{Cat} \times \text{Cat} \to \text{Cat} \quad (C, D) \mapsto C \star D,
\]
where \( \text{Cat} \) denotes the category of (small) categories.

Example 4.3.2.4. Let \( C \) and \( D \) be categories. If \( D \) is empty, then the inclusion map \( C \hookrightarrow C \star D \) is an isomorphism of categories.

Example 4.3.2.5 (Cones). Let \([0]\) denote the category having a single object and a single morphism, and let \( C \) be an arbitrary category. We let \( C^a \) denote the join \([0] \star C \), and \( C^p \) the join \( C \star [0] \). We refer to \( C^a \) as the left cone of \( C \), and to \( C^p \) as the right cone on \( C \).

More informally, we can describe the left cone \( C^a \) as the category obtained from \( C \) by adjoining a new object \( X_0 \) satisfying
\[
\text{Hom}_{C^a}(X_0, Y) = * \quad \text{Hom}_{C^a}(X_0, X_0) = * \quad \text{Hom}_{C^a}(Y, X_0) = \emptyset
\]
for \( Y \in C \). Note that \( X_0 \) is an initial object of the category \( C^a \), which we will refer to as the cone point of \( C^a \). Similarly, the right cone \( C^p \) is obtained from \( C \) by adjoining a new object which we refer to as the cone point of \( C^p \) (and which is a final object of \( C^p \)).

Remark 4.3.2.6. Let \( C, D, \) and \( E \) be categories. Then there is a canonical isomorphism of iterated joins
\[
\alpha : C \star (D \star E) \simeq (C \star D) \star E,
\]
characterized by the requirement that it restricts to the identity on \( C, D, \) and \( E \) (which we can regard as full subcategories of both \( C \star (D \star E) \) and \( (C \star D) \star E \), by means of Remark 4.3.2.2).
Remark 4.3.2.7. Let $\text{Cat}$ denote the category of (small) categories. Then $\text{Cat}$ admits a monoidal structure, where the tensor product is given by the join functor

$$\star : \text{Cat} \times \text{Cat} \to \text{Cat} \quad (C, D) \mapsto C \star D$$

of Remark 4.3.2.3 and the associativity constraints are the isomorphisms of Remark 4.3.2.6. The unit for this monoidal structure is the empty category $\emptyset \in \text{Cat}$ (Example 4.3.2.4).

Warning 4.3.2.8. The join operation of Definition 4.3.2.1 is not commutative. For example, if $C$ is a category, then the left cone $C^\oplus$ need not be isomorphic (or even equivalent) to the right cone $C^\oplus$. However, we do have canonical isomorphisms

$$(C \star D)^\text{op} \simeq D^\text{op} \star C^\text{op},$$

depending functorially on $C$ and $D$.

We now relate the join construction of Definition 4.3.2.1 with the slice categories of Construction 4.3.1.7. We begin with a simple observation.

Lemma 4.3.2.9. Let $C$ and $D$ be categories, and let $\iota_C : C \hookrightarrow C \star D$ and $\iota_D : D \hookrightarrow C \star D$ denote the inclusion maps. Then:

1. The inclusion functor $\iota_C$ factors uniquely as a composition

$$C \xrightarrow{\iota_C} (C \star D)_{/\iota_D} \to C \star D.$$

2. The inclusion functor $\iota_D$ factors uniquely as a composition

$$D \xrightarrow{\iota_D} (C \star D)_{/\iota_C} \to C \star D.$$

Proof. Let $\pi_C : C \times D \to C$ and $\pi_D : C \times D \to D$ denote the projection maps. Using Remark 4.3.1.10, we see that both (1) and (2) are equivalent to the assertion that there is a unique natural transformation $u$ from $\iota_C \circ \pi_C$ to $\iota_D \circ \pi_D$ (as functors from the product category $C \times D$ to the join category $C \star D$). Concretely, this natural transformation carries each object $(C, D) \in C \times D$ to the unique element of $\text{Hom}_{C \star D}(C, D)$. □

Proposition 4.3.2.10. Let $C$ be a category and let $G : D \to E$ be a functor between categories. For every functor $U : C \star D \to E$ extending $G$, let $F(U)$ denote the composite functor

$$C \xrightarrow{\iota_C} (C \star D)_{/\iota_D} \xrightarrow{U|_{(C \star D)_{/\iota_D}}} E_{/(U \circ \iota_D)} = E_{/G}.$$ 

Then the construction $U \mapsto F(U)$ induces a bijection

$$\{\text{Functors } U : C \star D \to E \text{ satisfying } U|_{D} = G\} \to \{\text{Functors } F : C \to E_{/G}\}.$$
Example 4.3.2.11. Let $G : \mathcal{D} \to \mathcal{E}$ be a functor of categories. Applying Proposition 4.3.2.10 in the case $\mathcal{C} = \{0\}$, we see that objects of the slice category $\mathcal{E}_{/[G]}$ can be identified with functors $U : \mathcal{D}^\triangleright \to \mathcal{E}$ satisfying $U|_\mathcal{D} = G$.

Example 4.3.2.12. Let $\mathcal{C}$ and $\mathcal{E}$ be categories and let $S$ be an object of $\mathcal{E}$. Applying Proposition 4.3.2.10 in the case $\mathcal{D} = \{0\}$, we see that functors from $\mathcal{C}$ to the slice category $\mathcal{E}_{/S}$ can be identified with functors $U : \mathcal{C}^\triangleright \to \mathcal{E}$ which carry the cone point of $\mathcal{C}^\triangleright$ to the object $S$.

In the situation of Proposition 4.3.2.10, we can use Remark 4.3.1.10 to identify functors $F : \mathcal{C} \to \mathcal{E}_{/[G]}$ with ordered pairs $(F,v)$, where $F : \mathcal{C} \to \mathcal{E}$ is a functor (given by the composition of $F$ with the forgetful functor $\mathcal{E}_{/[G]} \to \mathcal{E}$) and $v$ is a natural transformation from $F \circ \pi_C$ to $G \circ \pi_D$ (regarded as functors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$). Note that, in the case where $F = F(U)$ is obtained from a functor $U : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, we have $F = U|_{\mathcal{C}}$. We can therefore reformulate Proposition 4.3.2.10 in a more symmetric fashion:

Proposition 4.3.2.13. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and suppose we are given functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. Let $u : \iota_C \circ \pi_C \to \iota_D \circ \pi_D$ be the natural transformation appearing in the proof of Lemma 4.3.2.9. Then evaluation on $u$ induces a bijection

$$\{ \text{Functors } U : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \text{ with } U|_{\mathcal{C}} = F \text{ and } U|_{\mathcal{D}} = G \}.$$  

Proof. Let $v$ be a natural transformation from $F \circ \pi_C$ to $G \circ \pi_D$, carrying each object $(C,D) \in \mathcal{C} \times \mathcal{D}$ to a morphism $v_{C:D} : F(C) \to G(D)$ in the category $\mathcal{E}$. We wish to show that there is a unique functor $U : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ satisfying $U|_{\mathcal{C}} = F$, $U|_{\mathcal{D}} = G$, and $U(u_{C:D}) = v_{C:D}$ for $(C,D) \in \mathcal{C} \times \mathcal{D}$. These requirements uniquely determine the value of $U$ on all objects and morphisms of the category $\mathcal{C} \times \mathcal{D}$. To complete the proof, it will suffice to show that $U$ is compatible with composition: that is, for every pair of morphisms $s : X \to Y$ and $t : Y \to Z$ in $\mathcal{C} \times \mathcal{D}$, we have $U(t \circ s) = U(t) \circ U(s)$. We consider four cases:

- If $X$, $Y$, and $Z$ belong to $\mathcal{C}$, then we have $U(t \circ s) = F(t \circ s) = F(t) \circ F(s) = U(t) \circ U(s)$.
- If $X$ and $Y$ belong to $\mathcal{C}$ and $Z$ belongs to $\mathcal{D}$, then we have $U(t \circ s) = v_{X,Z} = v_{Y,Z} \circ F(s) = U(t) \circ U(s)$, where the second equality follows from the naturality of $v$ in the first variable.
- If $Y$ and $Z$ belong to $\mathcal{D}$ and $X$ belongs to $\mathcal{C}$, then we have $U(t \circ s) = v_{X,Z} = G(t) \circ v_{X,Y} = U(t) \circ U(s)$, where the second equality follows from the naturality of $v$ in the second variable.
• If $X$, $Y$, and $Z$ belong to $D$, then we have $U(t \circ s) = G(t \circ s) = G(t) \circ G(s) = U(t) \circ U(s)$.

\begin{proof}
\end{proof}

**Remark 4.3.2.14.** Stated more informally, Proposition 4.3.2.13 asserts that the join $C \star D$ is universal among categories $E$ which are equipped with a pair of functors $C \overset{F}{\rightarrow} E \overset{G}{\leftarrow} D$ and a natural transformation $v : (F \circ \pi_C) \rightarrow (G \circ \pi_D)$. More precisely, there is a pushout square

$$
\begin{array}{ccc}
(C \times \{0\} \times D) \coprod (C \times \{1\} \times D) & \rightarrow & C \times [1] \times D \\
\downarrow & & \downarrow \\
(C \times \{0\}) \coprod (\{1\} \times D) & \rightarrow & C \star D
\end{array}
$$

in the (ordinary) category $\text{Cat}$, where the right vertical map encodes the natural transformation $u : i_C \circ \pi_C \rightarrow i_D \circ \pi_D$ appearing in the proof of Lemma 4.3.2.9.

**Example 4.3.2.15 (The Universal Property of a Cone).** Let $C$ be a category. Applying Remark 4.3.2.14 in the special case $D = [0]$, we obtain a pushout diagram of categories

$$
\begin{array}{ccc}
C \times \{1\} & \rightarrow & C \times [1] \\
\downarrow & & \downarrow \\
[0] & \rightarrow & C^\circ,
\end{array}
$$

where the bottom horizontal map carries the unique object of $[0]$ to the cone point of $C^\circ$. This is essentially a reformulation of Examples 4.3.2.11 and 4.3.2.12. Stated more informally, the right cone $C^\circ$ is obtained from the product $[1] \times C$ by “collapsing” the full subcategory $\{1\} \times C$ to the cone point. Similarly, the left cone of a category $D$ is characterized by the existence of a pushout diagram

$$
\begin{array}{ccc}
\{0\} \times D & \rightarrow & [1] \times D \\
\downarrow & & \downarrow \\
[0] & \rightarrow & D^\circ.
\end{array}
$$

For completeness, we record the dual of Proposition 4.3.2.10, which supplies a universal property of coslice categories (and is also a reformulation of Proposition 4.3.2.13).
Corollary 4.3.2.16. Let \( \mathcal{D} \) be a category and let \( F : \mathcal{C} \to \mathcal{E} \) be a functor between categories. For every functor \( U : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \) extending \( F \), let \( \overline{G}(U) \) denote the composite functor
\[
\mathcal{D} \xrightarrow{\mathcal{C}} (\mathcal{C} \times \mathcal{D})_{i_U^l} \xrightarrow{U} \mathcal{E}_{(i_U \circ i_l)} = \mathcal{E}_{\mathcal{F}^l}.
\]
Then the construction \( U \mapsto \overline{G}(U) \) induces a bijection
\[
\{ \text{Functors } U : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \text{ satisfying } U|_{\mathcal{C}} = F \} \to \{ \text{Functors } \overline{G} : \mathcal{D} \to \mathcal{E}_{\mathcal{F}^l} \}.
\]

Corollary 4.3.2.17.

- For any category \( \mathcal{D} \), the join functor
\[
\text{Cat} \to \text{Cat}_{\mathcal{D}^l} / \quad \mathcal{C} \mapsto \mathcal{C} \times \mathcal{D}
\]
admits a right adjoint, given on objects by the slice construction \([G : \mathcal{D} \to \mathcal{E}] \mapsto \mathcal{E}_{\mathcal{F}^l} \).

- For any category \( \mathcal{C} \), the join functor
\[
\text{Cat} \to \text{Cat}_{\mathcal{C}^l} / \quad \mathcal{D} \mapsto \mathcal{C} \times \mathcal{D}
\]
admits a right adjoint, given on objects by the coslice construction \([F : \mathcal{C} \to \mathcal{E}] \mapsto \mathcal{E}_{\mathcal{F}^l} \).

Remark 4.3.2.18. Let \( G : \mathcal{D} \to \mathcal{E} \) be a functor between categories. According to Remark 4.3.1.10 the slice category \( \mathcal{E}_{\mathcal{F}^l} \) can be identified with the iterated fiber product
\[
\text{Fun}([0], \mathcal{E}) \times_{\text{Fun}([0] \times \mathcal{D}, \mathcal{E})} \text{Fun}([1] \times \mathcal{D}, \mathcal{E}) \times_{\text{Fun}([1] \times \mathcal{D}, \mathcal{E})} \{ G \}.
\]
Using Example 4.3.2.15, we can identify the left factor with the functor category \( \text{Fun}(\mathcal{D}^l, \mathcal{E}) \). We therefore obtain a pullback diagram of categories
\[
\begin{array}{ccc}
\mathcal{E}_{\mathcal{F}^l} & \longrightarrow & \text{Fun}(\mathcal{D}^l, \mathcal{E}) \\
\downarrow & & \downarrow \\
\{ F \} & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{E}),
\end{array}
\]
which recovers Example 4.3.2.11 at the level of objects.

Similarly, if \( F : \mathcal{C} \to \mathcal{E} \) is a functor of categories, then the coslice category \( \mathcal{E}_{\mathcal{F}^l} \) fits into a pullback square
\[
\begin{array}{ccc}
\mathcal{E}_{\mathcal{F}^l} & \longrightarrow & \text{Fun}(\mathcal{C}^l, \mathcal{E}) \\
\downarrow & & \downarrow \\
\{ F \} & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{E}).
\end{array}
\]
4.3.3 Joins of Simplicial Sets

Our next goal is to extend the join operation of Definition 4.3.2.1 to the setting of ∞-categories (and more general simplicial sets). We begin with a slightly more general discussion. Let Lin denote the category whose objects are finite linearly ordered sets and whose morphisms are nondecreasing functions. The functor category \( \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \) is equivalent to the category of augmented simplicial sets (see §[?]), and contains a full subcategory which is equivalent to the category of simplicial sets (see Proposition 4.3.3.11 below).

**Notation 4.3.3.1.** Let \( J \) be a linearly ordered set. We say that a subset \( I \subseteq J \) is an initial segment of \( J \) if it is closed downwards: that is, if, for every pair of elements \( i \leq j \) in \( J \), we have \( (j \in I) \Rightarrow (i \in I) \). We will write \( I \sqsubseteq J \) to indicate that \( I \) is an initial segment of \( J \).

**Construction 4.3.3.2 (Joins of Augmented Simplicial Sets).** For every pair of functors \( X, Y : \text{Lin}^{\text{op}} \rightarrow \text{Set} \), we let \( (X \star Y) : \text{Lin}^{\text{op}} \rightarrow \text{Set} \) denote a new functor given objects by the formula

\[
(X \star Y)(J) = \coprod_{I \sqsubseteq J} (X(I) \times Y(J \setminus I)).
\]

Here the coproduct indexed by the collection of all initial segments \( I \sqsubseteq J \).

More formally, the functor \( (X \star Y) : \text{Lin}^{\text{op}} \rightarrow \text{Set} \) can be described as follows:

- For every finite linearly ordered set \( J \), \( (X \star Y)(J) \) is the collection of all triples \( (I, x, y) \), where \( I \) is an initial segment of \( J \), \( x \) is an element of \( X(I) \), and \( y \) is an element of \( Y(J \setminus I) \).
- If \( \alpha : J' \rightarrow J \) is a nondecreasing function, then the induced map \( (X \star Y)(\alpha) : (X \star Y)(J) \rightarrow (X \star Y)(J') \) is given by the constructoin

\[
(I, x, y) \mapsto (\alpha^{-1}(I), X(\alpha|_{\alpha^{-1}(I)})(x), Y(\alpha|_{\alpha^{-1}(J\setminus I)})(y)).
\]

We will refer to \( X \star Y \) as the join of \( X \) and \( Y \).

**Example 4.3.3.3.** Let \( E : \text{Lin}^{\text{op}} \rightarrow \text{Set} \) denote the functor given by

\[
E(I) = \begin{cases} * & \text{if } I = \emptyset \\ \emptyset & \text{otherwise}. \end{cases}
\]

For every functor \( X : \text{Lin}^{\text{op}} \rightarrow \text{Set} \), we have canonical bijections

\[
(X \star E)(J) = \coprod_{I \sqsubseteq J} (X(I) \times E(J \setminus I)) \cong X(J) \times E(\emptyset) \cong X(J)
\]
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\[ (E \star X)(J) = \prod_{I \subseteq J} (E(I) \times X(J \setminus I)) \simeq E(\emptyset) \times X(J) \simeq X(J). \]

These bijections depend functorially on \( J \), and therefore determine isomorphisms of functors

\[ X \star E \simeq X \simeq E \star X. \]

**Remark 4.3.3.4** (Functoriality). Construction 4.3.3.2 determines a functor

\[ \star : \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \times \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \to \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \quad (X, Y) \mapsto X \star Y. \]

Note that this functor preserves colimits separately in each variable.

**Remark 4.3.3.5** (Associativity). Let \( X, Y, \) and \( Z \) be functors from \( \text{Lin}^{\text{op}} \) to the category of sets. For every finite linearly ordered set \( K \), we have a canonical bijection

\[
(X \star (Y \star Z))(K) = \prod_{I \subseteq K} (X(I) \times (Y \star Z)(K \setminus I))
\]

\[
= \prod_{I \subseteq K} \left( \prod_{J \subseteq K \setminus I} (Y(J) \times Z(K \setminus (I \cup J))) \right)
\]

\[
\simeq \prod_{I \subseteq K} \left( \prod_{J \subseteq K \setminus I} (X(I) \times Y(J) \times Z(K \setminus (I \cup J))) \right)
\]

\[
\simeq \prod_{J' \subseteq K} \left( \prod_{I \subseteq J'} (X(I) \times Y(J' \setminus I) \times Z(K \setminus J')) \right)
\]

\[
\simeq \prod_{J' \subseteq K} ((X \star Y)(J') \star Z(K \setminus J'))
\]

\[
= ((X \star Y) \star Z)(K).
\]

These bijections depend functorially on \( K \in \text{Lin}^{\text{op}} \), and therefore supply an isomorphism of functors \( \alpha_{X,Y,Z} : X \star (Y \star Z) \simeq (X \star Y) \star Z \).

**Remark 4.3.3.6.** The join operation of Construction 4.3.3.2 determines a functor

\[ \star : \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \times \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \to \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}). \]

This functor determines a monoidal structure on the category \( \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \), whose associativity constraints are the isomorphisms \( \alpha_{X,Y,Z} \) of Remark 4.3.3.6 and whose unit object is the functor \( E \) of Example 4.3.3.3.

**Example 4.3.3.7.** For every category \( \mathcal{C} \), let \( h_{\mathcal{C}} : \text{Lin}^{\text{op}} \to \text{Set} \) denote the functor represented by \( \mathcal{C} \), given by the formula

\[ h_{\mathcal{C}}(J) = \{ \text{Functors from } J \text{ to } \mathcal{C} \}. \]
For every pair of categories \( C \) and \( D \) and every finite linearly ordered set \( J \), we have a canonical bijection

\[
(h_C \star h_D)(J) = \prod_{I \subseteq J} h_C(I) \times h_D(J \setminus I)
\]

\[
= \prod_{I \subseteq J} \{\text{Functors } I \to C\} \times \{\text{Functors } (J \setminus I) \to D\}
\]

\[
\simeq \{\text{Functors } J \to C \star D\}
\]

\[
= h_{C \star D}(J).
\]

These bijections depend functorially on \( J \), and therefore determine an isomorphism \( h_C \star h_D \simeq h_{C \star D} \) in the category \( \text{Fun(Lin}^{\text{op}}, \text{Set}) \); here \( C \star D \) denotes the join of the categories \( C \) and \( D \), in the sense of Definition 4.3.2.1.

**Remark 4.3.3.8.** Let \( C \) be a small monoidal category. Then the presheaf category \( \text{Fun}(C^{\text{op}}, \text{Set}) \) inherits a monoidal structure given by *Day convolution* (see \( \S \?[\?] \)), which is characterized up to equivalence by the following properties:

1. The Yoneda embedding
   \[
   \iota : C \to \text{Fun}(C^{\text{op}}, \text{Set}) \quad C \mapsto \text{Hom}_C(\bullet, C)
   \]
   can be promoted to a symmetric monoidal functor.

2. The tensor product on \( \text{Fun}(C^{\text{op}}, \text{Set}) \) preserves small colimits separately in each variable.

Let us specialize to the case where \( C = \text{Lin} \) is the category of finite linearly ordered sets. Note that \( \text{Lin} \) can be identified with a full subcategory of \( \text{Cat} \) which is closed under the formation of joins (and contains the unit object \( \emptyset \in \text{Cat} \)), and therefore inherits the structure of a monoidal category (where the tensor product is given by joins). With respect to this monoidal structure, the Yoneda embedding \( \iota : \text{Lin} \to \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \) satisfies condition (1) (Example 4.3.3.7), and the join functor on \( \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \) satisfies (2) by virtue of Remark 4.3.3.4. It follows that the join operation on \( \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \) is given by Day convolution (with respect to the join operation on the category \( \text{Lin} \)).

We now adapt Construction 4.3.3.2 to the setting of simplicial sets.

**Notation 4.3.3.9.** Let \( \text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) \) denote the full subcategory of \( \text{Fun}(\text{Lin}^{\text{op}}, \text{Set}) \) spanned by those functors \( X : \text{Lin}^{\text{op}} \to \text{Set} \) for which the set \( X(\emptyset) \) is a singleton (that is, the full subcategory spanned by those functors which preserve final objects).

**Remark 4.3.3.10.** For every pair of functors \( X, Y : \text{Lin}^{\text{op}} \to \text{Set} \), we have a canonical bijection \((X \star Y)(\emptyset) = X(\emptyset) \times Y(\emptyset)\). In particular, if \( X \) and \( Y \) belong to the subcategory \( \text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) \) spanned by those functors which preserve final objects, then the join of \( X \) and \( Y \) is again in \( \text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) \).
Fun∗(Linop, Set) ⊆ Fun(Linop, Set), then the join X ⊗ Y also belongs to Fun∗(Linop, Set). Moreover, Fun∗(Linop, Set) contains the unit object E of Example 4.3.3.3. It follows that Fun∗(Linop, Set) inherits the structure of a monoidal category (with respect to the join operation of Construction 4.3.3.2).

Recall that the simplex category ∆ of Definition 1.1.1.2 is the full subcategory of Lin spanned by objects of the form [n] = {0 < 1 < · · · < n} for n ≥ 0.

Proposition 4.3.3.11. The restriction functor

Fun∗(Linop, Set) → Set∆  X ↦ X|∆op

is an equivalence of categories.

Proof. Let S be a one-element set, and let Fun∗′(Linop, Set) denote the full subcategory of Fun(Linop, Set) spanned by those functors X : Linop → Set satisfying X(∅) = S. Since the inclusion functor Fun∗′(Linop, Set) → Fun∗(Linop, Set) is an equivalence of categories, it will suffice to show that the restriction functor Fun∗′(Linop, Set) → Set∆  X ↦ X|∆op

is an equivalence of categories. Let Lin∅ denote the full subcategory of Lin spanned by the nonempty finite linearly ordered sets, so that the category Lin can be identified with the left cone Lin∅/ of Example 4.3.2.5. Using Proposition 4.3.2.13 (and the fact that the forgetful functor Set/∗ → Set is an isomorphism), we deduce that the restriction functor C → Fun(Lin∅, Set) is an isomorphism of categories. We are therefore reduced to showing that the restriction functor Fun(Lin∅, Set) → Fun(∆op, Set) = Set∆ is an equivalence of categories. This is clear, since the inclusion ∆ ↪ Lin∅ is an equivalence (Remark 1.1.1.3).

Remark 4.3.3.12. The inclusion functor ∆ ↪ Lin∅ has a unique left inverse R : Lin∅ → ∆, given on objects by the formula R([n]) = [n] when I has cardinality n + 1. It follows that the equivalence Fun∗(Linop, Set) → Set∆ of Proposition 4.3.3.11 admits an explicit right inverse, which carries a simplicial set X : ∆op → Set to the functor X+ : Linop → Set given by the formula

X+(I) = \begin{cases} X(R([n])) & \text{if } I \text{ is nonempty} \\ * & \text{otherwise.} \end{cases}

Construction 4.3.3.13 (Joins of Simplicial Sets). Let X and Y be simplicial sets. We let X ⊗ Y denote the simplicial set given by the restriction (X+ ⊗ Y+)|∆op. Here X+, Y+ ∈ Fun∗(Linop, Set) are given by Remark 4.3.3.12 and X+ ⊗ Y+ denotes the join of Construction 4.3.3.2. We will refer to X ⊗ Y as the join of X and Y.

The construction X, Y ↦ X ⊗ Y
determines a functor $\star : \text{Set}_\Delta \times \text{Set}_\Delta \to \text{Set}_\Delta$, which we will refer to as the \textit{join functor}. It is characterized (up to isomorphism) by the fact that the diagram

\[
\begin{array}{ccc}
\text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) \times \text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) & \xrightarrow{\star} & \text{Fun}_*(\text{Lin}^{\text{op}}, \text{Set}) \\
\text{Set}_\Delta \times \text{Set}_\Delta & \xrightarrow{\star} & \text{Set}_\Delta
\end{array}
\]

commutes up to isomorphism, where the vertical maps are the equivalences supplied by Proposition 4.3.3.11.

**Remark 4.3.3.14.** For every pair of simplicial sets $X$ and $Y$, we have canonical monomorphisms

\[X \simeq X \star \emptyset \hookrightarrow X \star Y \hookleftarrow \emptyset \star Y \simeq Y.\]

We will often abuse notation by identifying $X$ and $Y$ with the simplicial subsets of $X \star Y$ given by the images of these monomorphisms.

**Remark 4.3.3.15.** Let $X$ and $Y$ be simplicial sets. For each $n$-simplex $\sigma : \Delta^n \to X \star Y$, exactly one of the following conditions holds:

- The morphism $\sigma$ factors through $X$ (where we identify $X$ with a simplicial subset of $X \star Y$ as in Remark 4.3.3.14).

- The morphism $\sigma$ factors through $Y$ (where we identify $Y$ with a simplicial subset of $X \star Y$ as in Remark 4.3.3.14).

- The morphism $\sigma$ factors as a composition

\[\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma_- \star \sigma_+} X \star Y,
\]

for integers $p, q \geq 0$ satisfying $p + 1 + q = n$ and simplices $\sigma_- : \Delta^p \to X$ and $\sigma_+ : \Delta^q \to Y$ of $X$ and $Y$, respectively. Moreover, in this case, the simplices $\sigma_-$ and $\sigma_+$ (and the integers $p, q \geq 0$) are uniquely determined.

**Remark 4.3.3.16.** Let $i : X \hookleftarrow X'$ and $j : Y \hookrightarrow Y'$ be monomorphisms of simplicial sets. From the description of Remark 4.3.3.15, we see that the join $(i \star j) : X \star Y \to X' \star Y'$ is also a monomorphism of simplicial sets.

**Remark 4.3.3.17.** Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. By virtue of Remark 4.3.3.15, the join $(X \star Y)_\bullet$ can be described explicitly by the formula

\[(X \star Y)_n = X_n \amalg \left( \coprod_{p+1+q=n} X_p \times Y_q \right) \amalg Y_n.
\]
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In these terms, the face and degeneracy operators \( \{ d_i : (X \star Y)_n \to (X \star Y)_{n-1} \}_{0 \leq i \leq n} \) and \( \{ s_i : (X \star Y)_n \to (X \star Y)_{n+1} \} \) are given on the first and third summand by the analogous operators for \( X \bullet \) and \( Y \bullet \), and on elements \( (\sigma, \tau) \in X_p \times Y_q \) by the formula

\[
\begin{align*}
    d_i(\sigma, \tau) &= \begin{cases} 
        (d_i(\sigma), \tau) & \text{if } i \leq p \\
        (\sigma, d_{i-1-p}(\tau)) & \text{if } i > p,
    \end{cases} \\
    s_i(\sigma, \tau) &= \begin{cases} 
        (s_i(\sigma), \tau) & \text{if } i \leq p \\
        (\sigma, s_{i-1-p}(\tau)) & \text{if } i > p.
    \end{cases}
\end{align*}
\]

Remark 4.3.3.18. For every pair of simplicial sets \( X \) and \( Y \), we have a canonical isomorphism \( (X \star Y)^{\text{op}} \simeq Y^{\text{op}} \star X^{\text{op}} \).

Example 4.3.3.19. For every simplicial set \( X \), we have canonical isomorphisms \( X \star \emptyset \simeq X \simeq \emptyset \star X \) (compare with Example 4.3.3.3).

Example 4.3.3.20. Let \( C \) and \( D \) be categories. Using Example 4.3.3.7, we obtain a canonical isomorphism of simplicial sets \( N_{\bullet}(C) \star N_{\bullet}(D) \simeq N_{\bullet}(C \star D) \), where \( C \star D \) denotes the join of the categories \( C \) and \( D \).

In particular, for integers \( p, q \geq 0 \), there is a unique isomorphism of simplicial sets

\[ \Delta^p \star \Delta^q \simeq \Delta^{p+1+q}, \]

which is given on vertices of \( \Delta^p \) by the construction \( i \mapsto i \) and on vertices of \( \Delta^q \) by \( j \mapsto p + 1 + j \).

Proposition 4.3.3.21. Let \( X \) and \( Y \) be \( \infty \)-categories. Then the join \( X \star Y \) is an \( \infty \)-category.

Proof. Fix integers \( 0 < i < n \) and a map of simplicial sets \( \sigma_0 : \Lambda^n_i \to X \star Y \); we wish to show that \( \sigma_0 \) can be extended to an \( n \)-simplex of \( X \star Y \). Let \( J \subseteq [n] \) denote the collection of vertices \( j \) of \( \Lambda^n_i \subseteq \Delta^n \) for which \( \sigma_0(j) \) belongs to \( X \). We first claim that \( J \) is an initial segment of \( [n] \). If not, then there exists an integer \( 0 < j \leq n \) such that \( j \in V \) and \( j - 1 \notin V \). Let \( e \) be the edge of \( \Lambda^n_i \) having source \( j - 1 \) and target \( j \). Then \( \sigma_0(e) \) is an edge of \( X \star Y \) having source \( \sigma_0(j - 1) \in Y \) and target \( \sigma_0(j) \in X \), which is impossible.

We now consider three cases:

- Suppose that \( J = [n] \): that is, the morphism \( \sigma_0 \) carries each vertex of \( \Lambda^n_i \) to a vertex of the simplicial subset \( X \subseteq X \star Y \). Then \( \sigma_0 \) factors through \( X \), so we can extend \( \sigma_0 \) to a morphism of simplicial sets \( \sigma : \Delta^n \to X \) by virtue of our assumption that \( X \) is an \( \infty \)-category.

- Suppose that \( J = \emptyset \): that is, the morphism \( \sigma_0 \) carries each vertex of \( \Lambda^n_i \) to a vertex of the simplicial subset \( Y \subseteq X \star Y \). Then \( \sigma_0 \) factors through \( Y \), so we can extend \( \sigma_0 \) to a morphism of simplicial sets \( \sigma : \Delta^n \to Y \) by virtue of our assumption that \( Y \) is an \( \infty \)-category.
• Suppose that $\emptyset \subset J \subset [n]$. Let us identify the nerve $N_\bullet(J)$ with its image in $N_\bullet([n]) \simeq \Delta^n$, so that $N_\bullet(J)$ is contained in the horn $\Lambda_i^n$. Then the composite map

$$N_\bullet(J) \hookrightarrow \Lambda_i^n \xrightarrow{\sigma_0} X \star Y$$

factors through $X$, and therefore determines a map of simplicial sets $\sigma_- : N_\bullet(J) \to X$. Similarly, the composite map

$$N_\bullet([n] \setminus J) \hookrightarrow \Lambda_i^n \xrightarrow{\sigma_0} X \star Y$$

determines a map of simplicial sets $\sigma_+ : N_\bullet([n] \setminus J) \to Y$. The map $\sigma_0$ then admits a unique extension $\sigma : \Delta^n \to X \star Y$, given concretely by the composition

$$\Delta^n = N_\bullet([n]) \simeq N_\bullet(J) \star N_\bullet([n] \setminus J) \xrightarrow{\sigma_- \star \sigma_+} X \star Y.$$

\[\square\]

**Construction 4.3.3.22.** Let $X$ be a simplicial set. We will denote the join $\Delta^0 \star X$ by $X^d$ and refer to it as the **left cone** of $X$. Similarly, we denote the join $X \star \Delta^0$ by $X^o$ and refer to it as the **right cone** of $X$. We will often abuse notation by using Remark 4.3.3.14 to identify $X$ with its image in the cones $X^d$ and $X^o$. Moreover, Remark 4.3.3.14 also supplies morphisms of simplicial sets $X^d \leftarrow \Delta^0 \leftarrow X^o$, which we can identify with vertices which we refer to as the **cone points** of $X^d$ and $X^o$, respectively.

**Example 4.3.3.23.** Let $\mathcal{C}$ be a category. Then Example 4.3.3.20 supplies canonical isomorphisms

$$N_\bullet(\mathcal{C}^d) \simeq N_\bullet(\mathcal{C}^o) \quad N_\bullet(\mathcal{C})^o \simeq N_\bullet(\mathcal{C}^o),$$

where $\mathcal{C}^d$ and $\mathcal{C}^o$ denote the left and right cones of $\mathcal{C}$ (see Example 4.3.2.5).

**Example 4.3.3.24.** Let $n \geq 0$, and let $\Delta^n$ denote the standard $n$-simplex. Using Example 4.3.3.23, we see that there is a unique isomorphism of simplicial sets $(\Delta^n)^o \simeq \Delta^{n+1}$, which carries each vertex $i \in \{0,1,\ldots,n\}$ to itself and the cone point of $(\Delta^n)^o$ to the final vertex $n + 1$. This isomorphism carries the simplicial subset $(\partial \Delta^n)^o \subseteq (\Delta^n)^o$ to the horn $\Lambda_{n+1}^{n+1} \subseteq \Delta^{n+1}$. Similarly, the left cone $(\partial \Delta^n)^d$ is isomorphic to the horn $\Lambda_0^{n+1}$.

**Remark 4.3.3.25.** For every simplicial set $X$, Remark 4.3.3.18 supplies a canonical isomorphism $(X^d)^{\text{op}} \simeq (X^{\text{op}})^{o}$, carrying the cone point of $X^d$ to the cone point of $(X^{\text{op}})^{o}$.

**Remark 4.3.3.26.** Let $X$ be a simplicial set. Then the construction $Y \mapsto X \star Y$ determines a functor

$$\text{Set}_\Delta \to (\text{Set}_\Delta)^X.$$
which preserves small colimits. It follows that the composite functor

$$\text{Set}_\Delta \to (\text{Set}_\Delta)_X \to \text{Set}_\Delta \quad Y \mapsto X \star Y$$

preserves filtered colimits and pushouts. Beware that it does not preserve colimits in general (for example, it carries the initial object $\emptyset \in \text{Set}_\Delta$ to the simplicial set $X$, which need not be initial).

Remark 4.3.3.27 (Associativity). Let $X$, $Y$, and $Z$ be simplicial sets. Then Remark 4.3.3.5 supplies a canonical isomorphism of simplicial sets $\alpha_{X,Y,Z} : X \star (Y \star Z) \simeq (X \star Y) \star Z$. These isomorphisms are associativity constraints for a monoidal structure on the category of simplicial sets, which is characterized (up to isomorphism) by the requirement that the equivalence $\text{Fun}_\ast(\text{Lin}^{\text{op}}, \text{Set}) \to \text{Set}_\Delta$ of Proposition 4.3.3.11 can be promoted to a monoidal functor.

Warning 4.3.3.28. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the join $\mathcal{C} \star \mathcal{D}$ of Definition 4.3.2.1 is characterized (up to isomorphism) by the existence of a pushout diagram

$$
\begin{array}{ccc}
\{(0) \times \mathcal{C} \times \mathcal{D}\} \coprod \{(1) \times \mathcal{C} \times \mathcal{D}\} & \longrightarrow & [1] \times \mathcal{C} \times \mathcal{D} \\
\downarrow & & \downarrow \\
\{(0) \times \mathcal{C}\} \coprod \{(1) \times \mathcal{D}\} & \longrightarrow & \mathcal{C} \star \mathcal{D}
\end{array}
$$

in the category $\text{Cat}$ (see Remark 4.3.2.14). Beware that, in the setting of simplicial sets, the analogous statement is not quite true. To every pair of simplicial sets $X$ and $Y$, one can associate a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\{(0) \times X \times Y\} \coprod \{(1) \times X \times Y\} & \longrightarrow & \Delta^1 \times X \times Y \\
\downarrow & & \downarrow \\
\{(0) \times X\} \coprod \{(1) \times Y\} & \longrightarrow & X \star Y
\end{array}
$$

(see Construction 4.5.5.1). This diagram is almost never a pushout square (Example ?). Nevertheless, the pushout can be regarded as a good approximation to the join $X \star Y$: see Proposition 4.5.5.2 and Theorem 4.5.5.7.

4.3.4 Joins of Topological Spaces

The join operation on simplicial sets admits a topological interpretation.
Construction 4.3.4.1. Let \( X \) and \( Y \) be topological spaces, and let \([0, 1] = |\Delta^1|\) denote the unit interval. We let \( X \star Y \) denote the topological space given by the iterated pushout
\[
X \coprod_{(X \times \{0\} \times Y)} (X \times [0, 1] \times Y) \coprod_{(X \times \{1\} \times Y)} Y.
\]
We will refer to \( X \star Y \) as the **join of \( X \) and \( Y \)**.

Remark 4.3.4.2. Let \( X \) and \( Y \) be topological spaces. Then the join \( X \star Y \) of Construction 4.3.4.1 is equipped with a pair of maps \( \iota_X : X \to X \star Y \) and \( \iota_Y : Y \to X \star Y \). It is not difficult to see that these maps are closed embeddings: that is, they induce homeomorphisms from \( X \) and \( Y \) onto closed subsets of \( X \star Y \). We will generally abuse notation by identifying \( X \) and \( Y \) with their images under \( \iota_X \) and \( \iota_Y \), respectively.

Remark 4.3.4.3. Let \( X \), \( Y \), and \( Z \) be topological spaces. Then the datum of a continuous function \( X \star Y \to Z \) is equivalent to the datum of a triple \( (f_X, f_Y, h) \), where \( f_X : X \to Z \) and \( f_Y : Y \to Z \) are continuous functions and \( h : X \times [0, 1] \times Y \to Z \) is a homotopy from \( f_X \circ \pi_X \) to \( f_Y \circ \pi_Y \); here \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) denote the projection maps.

Remark 4.3.4.4 (Symmetry). Let \( X \) and \( Y \) be topological spaces. Then there is a canonical homeomorphism \( X \star Y \cong Y \star X \), which is induced by the homeomorphism \( X \times [0, 1] \times Y \to Y \times [0, 1] \times X \) given by the formulae
\[
\iota(u_0, \ldots, u_p) = (u_0, \ldots, u_p, 0, \ldots, 0) \quad \iota'(v_0, \ldots, v_q) = (0, \ldots, 0, v_0, \ldots, v_q).
\]

Example 4.3.4.5 (Cones). Let \( \ast \) denote the topological space consisting of a single point. For any topological space \( X \), we write \( X/ \) for the join \( \ast \star X \), and \( X^\circ \) for the join \( X \star \ast \), given more concretely by the formulae
\[
X^\circ = \ast \coprod_{(0) \times X} ([0, 1] \times X) \quad X^\circ = (X \times [0, 1]) \coprod_{(X \times \{1\})} \ast.
\]
We will refer to both \( X^\circ \) and \( X^\circ \) as the **cone on \( X \)** (note that they are canonically homeomorphic, by virtue of Remark 4.3.4.4).

Remark 4.3.4.6. Let \( X \) be a locally compact Hausdorff space. Then the functor
\[
\text{Top} \to \text{Top}_{X/} \quad Y \mapsto X \star Y
\]
preserves colimits. This follows from the fact that the functors \( Y \mapsto X \times Y \) and \( Y \mapsto X \times [0, 1] \times Y \) preserve colimits.

Example 4.3.4.7. For each integer \( n \geq 0 \), let \( |\Delta^n| = \{(u_0, \ldots, u_n) \in \mathbb{R}_{\geq 0} : u_0 + \cdots + u_n = 1\} \) denote the topological \( n \)-simplex. For \( p, q \geq 0 \), we have maps \( |\Delta^p| \to |\Delta^{p+1+q}| \xrightarrow{\iota'} |\Delta^q| \) given by the formulae
\[
\iota(u_0, \ldots, u_p) = (u_0, \ldots, u_p, 0, \ldots, 0) \quad \iota'(v_0, \ldots, v_q) = (0, \ldots, 0, v_0, \ldots, v_q).
\]
Let there be a “straight-line” homotopy $h : |\Delta^p| \times [0, 1] \times |\Delta^q| \to |\Delta^{p+1+q}|$ from $\iota \circ \pi_{|\Delta^p|}$ to $\iota' \circ \pi_{|\Delta^q|}$, given concretely by the formula

$$h((u_0, \ldots, u_p), t, (v_0, \ldots, v_q)) = ((1 - t)u_0, (1 - t)u_1, \ldots, (1 - t)u_p, tv_0, \ldots, tv_q).$$

By virtue of Remark 4.3.4.3, the triple $(\iota, \iota', h)$ can be identified with a continuous function $H_{p,q} : |\Delta^p| \star |\Delta^q| \to |\Delta^{p+1+q}|$.

**Proposition 4.3.4.8.** Let $p$ and $q$ be nonnegative integers. Then the function $H_{p,q} : |\Delta^p| \star |\Delta^q| \to |\Delta^{p+1+q}|$ of Example 4.3.4.7 is a homeomorphism of topological spaces.

**Proof.** Since $|\Delta^p| \star |\Delta^q|$ is compact and $|\Delta^{p+1+q}|$ is Hausdorff, the continuous function $H_{p,q}$ is automatically closed. To complete the proof, it will suffice to show that $H_{p,q}$ is bijective. Fix a point $x$ of $|\Delta^{p+1+q}|$, given by a sequence of nonnegative real numbers $(\varpi_0, \ldots, \varpi_m, \varpi_0, \varpi_1, \ldots, \varpi_n)$ satisfying

$$\varpi_0 + \cdots + \varpi_m + \varpi_0 + \cdots + \varpi_n = 0.$$

Set $t = \varpi_0 + \cdots + \varpi_m$. If $t = 0$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^p|$ (regarded as a subset of $|\Delta^p| \star |\Delta^q|$), given by the sequence $(\varpi_0, \ldots, \varpi_m)$. If $t = 1$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^q|$ (regarded as a subset of $|\Delta^p| \star |\Delta^q|$), given by the sequence $(\varpi_0, \ldots, \varpi_m)$. In the case $0 < t < 1$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^p| \star |\Delta^q|$, given as the image of the triple

$$(\frac{\varpi_0}{1 - t}, \ldots, \frac{\varpi_m}{1 - t}, \frac{\varpi_0}{t}, \ldots, \frac{\varpi_n}{t}) \in |\Delta^p| \times [0, 1] \times |\Delta^n|.$$

□

We now compare the join operation on topological spaces (given by Construction 4.3.4.1) to the join operation on simplicial sets (given by Construction 4.3.3.13).

**Construction 4.3.4.9.** Let $X$ and $Y$ be simplicial sets, and let $\sigma : \Delta^n \to X \star Y$ be a morphism. We define a continuous function $f(\sigma) : |\Delta^n| \to |X| \star |Y|$ as follows (see Remark 4.3.3.15):

- If $\sigma$ factors through $X$, we let $f(\sigma)$ denote the composition

  $$|\Delta^n| \xrightarrow{\sigma} |X| \xrightarrow{i_X} |X| \star |Y|,$$

  where the second map is the inclusion of Remark 4.3.4.2.

- If $\sigma$ factors through $Y$, we let $f(\sigma)$ denote the composition

  $$|\Delta^n| \xrightarrow{\sigma} |Y| \xrightarrow{i_Y} |X| \star |Y|,$$

  where the second map is the inclusion of Remark 4.3.4.2.
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- If \( \sigma \) factors as a composition

\[
\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \ast \Delta^q \xrightarrow{\sigma \ast \sigma_+} X \ast Y,
\]

then we let \( f(\sigma) \) denote the composite map

\[
|\Delta^n| = |\Delta^{p+1+q}| \xrightarrow{H_{p,q}} |\Delta^p| \ast |\Delta^q| \xrightarrow{\sigma_+ \ast |\sigma_+|} X \ast Y,
\]

where \( H_{p,q} \) denotes the homeomorphism of Proposition 4.3.4.8.

The construction \( \sigma \mapsto f(\sigma) \) is compatible with face and degeneracy maps, and therefore determines a morphism of simplicial sets \( f : X \ast Y \to \operatorname{Sing}_\bullet(|X| \ast |Y|) \). We will identify \( f \) with a continuous function \( T_{X,Y} : |X \ast Y| \to |X| \ast |Y| \), which we will refer to as the join comparison map.

**Example 4.3.4.10.** Let \( X = \Delta^p \) and \( Y = \Delta^q \) be standard simplices. Then the join comparison map \( T_{X,Y} : |\Delta^p \ast \Delta^q| \to |\Delta^p| \ast |\Delta^q| \) fits into a commutative diagram

\[
\begin{array}{ccc}
|\Delta^p \ast \Delta^q| & \xrightarrow{T_{X,Y}} & |\Delta^p| \ast |\Delta^q| \\
|\rho| \downarrow & & |\Delta^p| \ast |\Delta^q| \downarrow H_{p,q} \\
|\Delta^{p+1+q}| & \xrightarrow{\rho} & |\Delta^{p+1+q}|,
\end{array}
\]

where \( \rho : \Delta^p \ast \Delta^q \simeq \Delta^{p+1+q} \) denotes the isomorphism of simplicial sets appearing in Example 4.3.3.20 and \( H_{p,q} \) is the homeomorphism of Proposition 4.3.4.8. In particular, \( T_{X,Y} \) is a homeomorphism.

**Proposition 4.3.4.11.** Let \( X \) and \( Y \) be simplicial sets. If either \( X \) or \( Y \) is finite, then the join comparison map \( T_{X,Y} : |X \ast Y| \to |X| \ast |Y| \) of Construction 4.3.4.9 is a homeomorphism.

**Proof.** Without loss of generality, we may assume that \( X \) is finite. Then the geometric realization \( |X| \) is a compact Hausdorff space (Corollary 3.5.1.10). Using Remarks 4.3.4.6 and 4.3.3.26 we see that the functors

\[
\operatorname{Set}_\Delta \to \operatorname{Top}_{|X|/}, \quad Y \mapsto |X \ast Y|, \quad Y \mapsto |X| \ast |Y|
\]

preserve colimits. Consequently, if we regard \( X \) as fixed, then the collection of simplicial sets \( Y \) for which \( T_{X,Y} \) is a homeomorphism is closed under colimits. Since every simplicial set can be realized as a colimit of standard simplices (Corollary 1.1.8.17), it will suffice to prove Proposition 4.3.4.11 in the special case where \( Y = \Delta^q \) is a standard simplex. In this case, \( Y \) is also finite. Repeating the preceding argument (with the roles of \( X \) and \( Y \) reversed), we are reduced to proving that \( T_{X,Y} \) is a homeomorphism in the case where \( X = \Delta^p \) is also a standard simplex. In this case, the desired result follows from Example 4.3.4.10. \( \square \)
Corollary 4.3.4.12. Let $X$ be a simplicial set. Then the join comparison maps $T_{\Delta^0,X}$ and $T_{X,\Delta^0}$ supply homeomorphisms of topological spaces

$$|X^a| \simeq |X|^a \quad |X|^b \simeq |X|^b.$$  

Here $X^a$ and $X^b$ denote the left and right cones on $X$ in the category of simplicial sets (Construction 4.3.3.22), while $|X|^a$ and $|X|^b$ denote the cone $|X|$ in the category of topological spaces (Example 4.3.4.5).

The join comparison map $T_{X,Y} : |X \star Y| \to |X| \star |Y|$ need not be a homeomorphism in general. However, we do have the following:

Corollary 4.3.4.13. Let $X$ and $Y$ be simplicial sets. Then the join comparison map $T_{X,Y} : |X \star Y| \to |X| \star |Y|$ is a bijection.

Proof. As a map of sets, we can realize $T_{X,Y}$ as a filtered colimit of join comparison maps $T_{X',Y}$, where $X'$ ranges over the finite simplicial subsets of $X$ (Remark 3.5.1.8). Each of these maps is bijective (even a homeomorphism), by virtue of Proposition 4.3.4.11. □

Warning 4.3.4.14. Let $X$ and $Y$ be simplicial sets, and let $X \circ Y$ denote the simplicial set given by the iterated coproduct

$$X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y.$$  

Since the formation of geometric realization commutes with the formation of colimits, we have an evident comparison map of topological spaces

$$|X \circ Y| \to |X| \star |Y|.$$  

This map is always bijective, and is a homeomorphism if either $X$ or $Y$ is finite (see Corollary 3.5.2.2). In this case, Corollary 4.3.4.13 supplies a homeomorphism of geometric realizations $|X \circ Y| \simeq |X \star Y|$. Beware that this homeomorphism does not arise from a morphism of simplicial sets. In the case $X = \Delta^p$ and $Y = \Delta^q$, it arises from the homotopy

$$h : |\Delta^p| \times |\Delta^1| \times |\Delta^q| \to |\Delta^{p+1+q}|$$

$$h((u_0, \ldots, u_p), (v_0, \ldots, v_q)) = ((1 - t)u_0, (1 - t)u_1, \ldots, (1 - t)u_p, tv_0, \ldots, tv_q),$$

appearing in Example 4.3.4.7, which is not piecewise-linear with respect to the natural triangulation of the polysimplex $|\Delta^p| \times |\Delta^1| \times |\Delta^q|$.  

4.3.5 Slices of Simplicial Sets

Let \( \mathcal{C} \) be a category. In §4.3.1, we associated to every diagram \( F : \mathcal{K} \to \mathcal{C} \) a slice category \( \mathcal{C}_{/F} \) and a coslice category \( \mathcal{C}_{F/} \) (Construction 4.3.1.7). We now introduce a generalization of this construction, where we replace (the nerves of) \( \mathcal{C} \) and \( \mathcal{K} \) by arbitrary simplicial sets. As our starting point, we recall that the construction \( \mathcal{C} \mapsto \mathcal{C}_{/F} \) can be characterized as the right adjoint of the join functor
\[
\text{Cat} \to \text{Cat} \quad \mathcal{E} \mapsto \mathcal{K} \star \mathcal{E}
\]
(see Corollary 4.3.2.17).

Construction 4.3.5.1 (Slice Simplicial Sets). Let \( f : \mathcal{K} \to X \) be a morphism of simplicial sets. We define a simplicial set \( X_{/f} \) as follows:

- For each \( n \geq 0 \), an \( n \)-simplex of \( X_{/f} \) is a map of simplicial sets \( f : \Delta^n \star \mathcal{K} \to X \) satisfying \( f|_{\mathcal{K}} = f \).
- For every nondecreasing function \( \alpha : [m] \to [n] \) in \( \Delta \), the associated map
  \[
  \alpha^* : \{n\text{-simplices of } X_{/f}\} \to \{m\text{-Simplices of } X_{/f}\}
  \]
  is carries an \( n \)-simplex \( \overline{f} : \Delta^n \star \mathcal{K} \to X \) to the composite map
  \[
  \Delta^m \star \mathcal{K} \xrightarrow{\text{id}_\mathcal{K} \star \alpha} \Delta^n \star \mathcal{K} \xrightarrow{\overline{f}} X.
  \]
We will refer to \( X_{/f} \) as the slice simplicial set of \( X \) over \( f \).

Remark 4.3.5.2. Let \( f : \mathcal{K} \to X \) be a morphism of simplicial sets, and let \( \overline{f} : \Delta^n \star \mathcal{K} \to X \) be an \( n \)-simplex of the slice simplicial set \( X_{/f} \). Then the restriction \( \overline{f}|_{\Delta^n} \) is an \( n \)-simplex of \( X \). The construction \( \overline{f} \mapsto \overline{f}|_{\Delta^n} \) determines a morphism of simplicial sets \( X_{/f} \to X \), which we will refer to as the projection map or the forgetful functor (in the case where \( X \) is an ∞-category). We will often abuse notation by identifying a vertex of \( X_{/f} \) with its image in \( X \).

Variant 4.3.5.3 (Coslice Simplicial Sets). Let \( f : \mathcal{K} \to X \) be a morphism of simplicial sets. We define a simplicial set \( X_{f/} \) as follows:

- For each \( n \geq 0 \), an \( n \)-simplex of \( X_{f/} \) is a map of simplicial sets \( f : \mathcal{K} \star \Delta^n \to X \) satisfying \( f|_{\mathcal{K}} = f \).
- For every nondecreasing function \( \alpha : [m] \to [n] \) in \( \Delta \), the associated map
  \[
  \alpha^* : \{n\text{-simplices of } X_{f/}\} \to \{m\text{-Simplices of } X_{f/}\}
  \]
  is carries an \( n \)-simplex \( \overline{f} : \mathcal{K} \star \Delta^n \to X \) to the composite map
  \[
  K \star \Delta^n \xrightarrow{id_K \star \alpha} K \star \Delta^n \xrightarrow{\overline{f}} X.
  \]
We will refer to $X_{f/}$ as the \textit{coslice simplicial set of $X$ under $f$}. As in Remark 4.3.5.2, it is equipped with a projection map $X_{f/} \rightarrow X$.

**Remark 4.3.5.4.** Construction 4.3.5.1 and Variant 4.3.5.3 are opposite to one another. More precisely, if $f : K \rightarrow X$ is a morphism of simplicial sets and $f^{op} : K^{op} \rightarrow X^{op}$ denotes the induced map of opposite simplicial sets, then we have a canonical isomorphism of simplicial sets $(X_{f/})^{op} \simeq (X^{op})_{f^{op}/}$.

**Remark 4.3.5.5.** Let $f : K \rightarrow X$ be a morphism of simplicial sets. Then vertices of the slice simplicial set $X_{f/}$ are morphisms of simplicial sets $f : K \rightarrow X$ satisfying $f|_{K} = f$. Similarly, vertices of the coslice simplicial set $X_{f/}$ are morphisms of simplicial sets $f : K \rightarrow X$ satisfying $f|_{K} = f$. Here $K^{\Delta}$ and $K^{op}$ denote the left and right cone of $K$ (Construction 4.3.3.22).

**Notation 4.3.5.6 (Slicing over Vertices).** Let $X$ be a simplicial set containing a vertex $x$, and let $f_{x} : \Delta^{0} \rightarrow X$ be the map carrying the unique vertex of $\Delta^{0}$ to $x$. We will generally abuse notation by not distinguishing between the vertex $x$ and the morphism $f_{x}$. For example, we will denote the slice simplicial set $X_{f_{x}/}$ by $X_{x/}$, and the coslice simplicial set $X_{f_{x}/}$ by $X_{x/}$.

**Example 4.3.5.7.** Let $F : \mathcal{K} \rightarrow \mathcal{C}$ be a functor between categories, and let $f = N_{\bullet}(F)$ denote the induced morphism of simplicial sets from $N_{\bullet}(\mathcal{K})$ to $N_{\bullet}(\mathcal{C})$. For each $n \geq 0$, we have canonical bijections

\[
\{n\text{-simplices of } N_{\bullet}(\mathcal{C})_{f/}\} \simeq \{\text{Morphisms } \overline{f} : \Delta^{n} \star N_{\bullet}(\mathcal{K}) \rightarrow N_{\bullet}(\mathcal{C}) \text{ with } \overline{f}|_{N_{\bullet}(\mathcal{K})} = f\}
\]

\[
\simeq \{\text{Morphisms } \overline{f} : N_{\bullet}([n]) \star N_{\bullet}(\mathcal{K}) \rightarrow N_{\bullet}(\mathcal{C}) \text{ with } \overline{f}|_{N_{\bullet}(\mathcal{K})} = f\}
\]

\[
\simeq \{\text{Morphisms } \overline{f} : N_{\bullet}([n] \star \mathcal{K}) \rightarrow N_{\bullet}(\mathcal{C}) \text{ with } \overline{f}|_{N_{\bullet}(\mathcal{K})} = f\}
\]

\[
\simeq \{\text{Functors } \overline{F} : [n] \star \mathcal{K} \rightarrow \mathcal{C} \text{ with } \overline{F}|_{\mathcal{K}} = F\}
\]

\[
\simeq \{\text{Functors } [n] \rightarrow \mathcal{C}_{/F}\}
\]

\[
\simeq \{n\text{-simplices of } N_{\bullet}(\mathcal{C}_{/F})\}.
\]

Here the third bijection comes from Example 4.3.3.20, the fourth from Proposition 1.2.2.1, and the fifth from Proposition 4.3.2.10. These bijections depend functorially on $[n] \in \Delta$, and therefore determine an isomorphism of simplicial sets $N_{\bullet}(\mathcal{C})_{f/} \simeq N_{\bullet}(\mathcal{C}_{/F})$. Similarly, we have a canonical isomorphism $N_{\bullet}(\mathcal{C})_{f/} \simeq N_{\bullet}(\mathcal{C}_{/F})$. For a more general statement, see Corollary 4.3.5.17.

**Example 4.3.5.8.** Let $\mathcal{C}$ be a category containing an object $X$, which we also view as a vertex of the simplicial set $N_{\bullet}(\mathcal{C})$. Specializing Example 4.3.5.7 (and invoking Example 4.3.1.9), we obtain canonical isomorphisms

\[N_{\bullet}(\mathcal{C})_{X/} \simeq N_{\bullet}(\mathcal{C}_{/X}) \quad N_{\bullet}(\mathcal{C})_{X/} \simeq N_{\bullet}(\mathcal{C}_{/X}).\]
Example 4.3.5.9. Let $K$ be a simplicial set, let $Y$ be a topological space, and let $f : K \to \text{Sing}_\bullet(Y)$ be a morphism of simplicial sets, which we will identify with a continuous function $F : |K| \to Y$. For each $n \geq 0$, we have canonical bijections

$$
\{n\text{-simplices of } \text{Sing}_\bullet(Y)/f\} \cong \{\text{Morphisms } \overline{f} : \Delta^n \star K \to \text{Sing}_\bullet(Y) \text{ with } \overline{f}|\Delta^n \star |K| = f\}
$$

$$
\cong \{\text{Continuous maps } \overline{F} : |\Delta^n| \star |K| \to Y \text{ with } \overline{F}|\Delta^n \star |K| = f\}
$$

$$
\cong \{\text{Continuous maps } \overline{F} : |\Delta^n| \star |K| \to Y \text{ with } \overline{F}|\Delta^n \star |K| = F\}
$$

Here the third bijection is provided by Proposition 4.3.4.11. Using the fact that these bijections depend functorially on $n \in \Delta$ and invoking the universal property $|\Delta^n| \star |K|$ (see Remark 4.3.4.3), we obtain an isomorphism of $\text{Sing}_\bullet(Y)/f$ with the iterated fiber product

$$
\text{Sing}_\bullet(Y) \times_{\text{Fun}(\{0\} \times K, \text{Sing}_\bullet(Y))} \text{Fun}(\Delta^1 \times K, \text{Sing}_\bullet(Y)) \times_{\text{Fun}(\{1\} \times K, \text{Sing}_\bullet(Y))} \{f\}.
$$

Example 4.3.5.10. Let $Y$ be a topological space equipped with a base point $y$. Let $P = \{p : [0,1] \to Y\}$ denote the collection of all continuous functions from the unit interval $[0,1]$ to $Y$, and let $P_y = \{p \in P : p(1) = y\}$ denote the subset of $P$ consisting of those continuous paths which end at the point $y$. We regard $P$ as a topological space by equipping it with the compact-open topology, so the singular simplicial set $\text{Sing}_\bullet(P)$ can be identified with $\text{Fun}(\Delta^1, \text{Sing}_\bullet(Y))$ (see Warning 2.4.2.17). Identifying $y$ with a vertex of the singular simplicial set $\text{Sing}_\bullet(Y)$, Example 4.3.5.9 supplies an isomorphism of simplicial sets

$$
\text{Sing}_\bullet(Y)/y \cong \text{Sing}_\bullet(P) \times_{\text{Sing}_\bullet(Y)} \{y\} = \text{Sing}_\bullet(P_y).
$$

In particular, since the topological space $P_y$ is contractible, the simplicial set $\text{Sing}_\bullet(Y)/y$ is a contractible Kan complex (this is a special case of a general phenomenon: see Corollary 4.3.7.13).

Warning 4.3.5.11. If $F : \mathcal{K} \to \mathcal{C}$ is a functor between categories, then the slice category $\mathcal{C}/F$ was defined as the fiber product $\mathcal{C} \times_{\text{Fun}(\mathcal{K},\mathcal{C})} \text{Fun}(\mathcal{K},\mathcal{C})/F$ (where we view $F$ as an object of the functor category $\text{Fun}(\mathcal{K},\mathcal{C})$; see Construction 4.3.1.7). Beware that, in the setting of simplicial sets, our definition is somewhat different. For any morphism of simplicial sets $f : K \to X$, one can associate a comparison map

$$
\rho : X/f \to X \times_{\text{Fun}(\mathcal{K},\mathcal{X})} \text{Fun}(K,X)/f,
$$

where on the right hand side we view $f$ as a vertex of the simplicial set $\text{Fun}(K,X)$ (see Construction [?]). This comparison morphism is an isomorphism when $X$ is the nerve of a category (Example [?]), but is not an isomorphism in general. Nevertheless, we will show later that when $X$ is an $\infty$-category, then $\rho$ is an equivalence of $\infty$-categories (Corollary [?]).
Warning 4.3.5.12. Recall that, if $F: \mathcal{K} \to \mathcal{C}$ is a functor between categories, then the slice category $\mathcal{C}_{/F}$ can be identified with the iterated fiber product

$$\mathcal{C} \times_{\text{Fun}(\{0\} \times \mathcal{K}, \mathcal{C})} \text{Fun}(\{1\} \times \mathcal{K}, \mathcal{C}) \times_{\text{Fun}(\{1\} \times \mathcal{K}, \mathcal{C})} \{F\}$$

(see Remark 4.3.1.10). More generally, for any morphism of simplicial sets $f: \mathcal{K} \to \mathcal{X}$, there is a comparison map

$$\theta: X_f \to X \times_{\text{Fun}(\{0\} \times \mathcal{K}, \mathcal{X})} \text{Fun}(\Delta^1 \times \mathcal{K}, \mathcal{X}) \times_{\text{Fun}(\{1\} \times \mathcal{K}, \mathcal{X})} \{f\}$$

(see Construction [?]) with the following features:

- When $X$ is (the nerve of) a category, the morphism $\theta$ is an isomorphism of simplicial sets (Example [?]).
- When $X$ is an $\infty$-category, the morphism $\theta$ is an equivalence of $\infty$-categories (Proposition [?]).
- When $X = \text{Sing}_\ast(Y)$ is the singular simplicial set of a topological space $Y$, the morphism $\theta$ does not coincide with the isomorphism constructed in Example 4.3.5.9 (however, they are naturally homotopic: see Example [?]).
- The morphism $\theta$ is usually not an isomorphism of simplicial sets (see Warning 4.3.3.28). In fact, the simplicial set $X_f$ need not be abstractly isomorphic to the iterated fiber product

$$X \times_{\text{Fun}(\{0\} \times \mathcal{K}, \mathcal{X})} \text{Fun}(\Delta^1 \times \mathcal{K}, \mathcal{X}) \times_{\text{Fun}(\{1\} \times \mathcal{K}, \mathcal{X})} \{f\},$$

even if $X$ is an $\infty$-category (see Example [?]).

The slice simplicial sets of Construction 4.3.5.1 can be characterized by a universal property.

Construction 4.3.5.13. Let $f: \mathcal{K} \to \mathcal{X}$ be a morphism of simplicial sets. We define a morphism of simplicial sets $c: X_f \star K \to \mathcal{X}$ as follows:

- The restriction of $c$ to the simplicial subset $X_f \subseteq X_f \star K$ is equal to the projection map $X_f \to X$ of Remark 4.3.5.2.
- The restriction of $c$ to the simplicial subset $K \subseteq X_f \star K$ is equal to $f$.
- Let $\sigma: \Delta^n \to X_f \star K$ be an $n$-simplex which does not belong to $X_f$ or $K$, so that $\sigma$ factors (uniquely) as a composition

$$\Delta^n \simeq \Delta^{p+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma \star -} X_f \star K$$
for \( p + 1 + q = n \) (see Remark 4.3.3.15). Using the definition of the simplicial set \( X_{/f} \), we can identify \( \sigma \) with a morphism of simplicial sets \( \overline{f} : \Delta^p \star K \to X \) satisfying \( \overline{f}|_K = f \). We then define \( c(\sigma) \) to be the \( n \)-simplex of \( X \) given by the composite map

\[
\Delta^n \cong \Delta^{p+1+q} \xrightarrow{\Delta^p \star \Delta^q \xrightarrow{\text{id} \star \sigma}} \Delta^p \star K \xrightarrow{\overline{f}} X.
\]

We will refer to \( c \) as the slice contraction morphism. Applying a similar construction to the opposite simplicial sets, we obtain a morphism \( c' : K \star X_{/f} \to X \) which we will refer to as the coslice contraction morphism.

**Proposition 4.3.5.14.** Let \( f : K \to X \) be a morphism of simplicial sets, and let \( c : X_{/f} \star K \to X \) be the slice contraction morphism of Construction 4.3.5.13. Then, for any simplicial set \( Y \), postcomposition with \( c \) induces a bijection

\[
\theta_Y : \text{Hom}_{\Delta}(Y, X_{/f}) \to \{ \text{Morphisms } \overline{f} : Y \star K \to X \text{ satisfying } \overline{f}|_K = f \}
\]

Similarly, postcomposition with the coslice contraction morphism \( c' : K \star X_{/f} \to X \) induces a bijection

\[
\theta'_Y : \text{Hom}_{\Delta}(Y, X_{/f}) \to \{ \text{Morphisms } \overline{f} : K \star Y \to X \text{ satisfying } \overline{f}|_K = f \}.
\]

**Proof.** In the case where \( Y \) is a standard simplex, both assertions follow immediately from the definition of the simplicial sets \( X_{/f} \) and \( X_{/f} \). Since every simplicial set can be realized as a colimit of simplices (Corollary 1.1.8.17), it will suffice to show that the constructions \( Y \mapsto \theta_Y \) and \( Y \mapsto \theta'_Y \) carry colimits of simplicial sets to limits in the arrow category \( \text{Fun}(1, \Delta) \). This follows from the observation that the functors

\[
\text{Set}_{\Delta} \to (\text{Set}_{\Delta})_{/K}, \quad Y \mapsto Y \star K, \quad Y \mapsto K \star Y\]

preserve small colimits (see Remark 4.3.3.26).

**Corollary 4.3.5.15.** Let \( K \) be a simplicial set. Then the join functor

\[
\text{Set}_{\Delta} \to (\text{Set}_{\Delta})_{/K}, \quad Y \mapsto Y \star K
\]

admits a right adjoint, given on objects by the slice construction \( (f : K \to X) \mapsto X_{/f} \). Similarly, the join functor

\[
\text{Set}_{\Delta} \to (\text{Set}_{\Delta})_{/K}, \quad Y \mapsto K \star Y
\]

admits a right adjoint, given on objects by the coslice construction \( (f : K \to X) \mapsto X_{/f} \).
Example 4.3.5.16. Let $X$ be a simplicial set containing a vertex $x$. Let $Y$ be a simplicial set, and let $v$ and $v'$ denote the cone points of $Y^\triangleleft$ and $Y^\triangleright$, respectively. Then Proposition 4.3.5.14 supplies bijections
\[
\text{Hom}_{\Delta}(Y, X_{x/}) \simeq \{\text{Morphisms } f : Y^\triangleleft \to X \text{ with } f(v) = x\}
\]
\[
\text{Hom}_{\Delta}(Y, X_{x/}) \simeq \{\text{Morphisms } f : Y^\triangleright \to X \text{ with } f(v') = x\}.
\]
Example 4.3.5.7 can be adapted to describe any slice or coslice of a simplicial set having the form $N_\bullet(C)$.

Corollary 4.3.5.17. Let $C$ be a category and let $K$ be a simplicial set equipped with a morphism $f : K \to N_\bullet(C)$. Let $u : K \to N_\bullet(K)$ be a morphism of simplicial sets which exhibits $K$ as a homotopy category of $K$ (see Definition 1.2.5.1), so that $f$ factors uniquely as a composition $K \xrightarrow{u} N_\bullet(K) \xrightarrow{N_\bullet(F)} N_\bullet(C)$ for some functor $F : K \to C$. Then $u$ induces isomorphisms of simplicial sets
\[
\theta : N_\bullet(C/F) \simeq N_\bullet(C/F)_{/F} \to N_\bullet(C)_{/F} \quad \theta' : N_\bullet(C/F) \simeq N_\bullet(C)_{N_\bullet(F)/} \to N_\bullet(C)_{/F}.
\]
Proof. We will prove that $\theta$ is an isomorphism; the proof for $\theta'$ is similar. Fix an $n$-simplex $\sigma$ of $N_\bullet(C/F)$, which we identify with a morphism of simplicial sets $\overline{f} : \Delta^n \star K \to N_\bullet(C)$ satisfying $\overline{f}|_K = f$. Let $\overline{f}_0 = \overline{f}|_{\Delta^n}$. Using Proposition 4.3.5.14, we can identify $\overline{f}$ with a morphism of simplicial sets $g : K \to N_\bullet(K)_{/\overline{f}_0}$. We wish to show that $\sigma$ can be lifted uniquely to an $n$-simplex of $N_\bullet(C/F)_{/N_\bullet(F)}$. Equivalently, we wish to show that $g$ admits a unique factorization
\[
K \xrightarrow{u} N_\bullet(K) \xrightarrow{\overline{f}} N_\bullet(C)_{/\overline{f}_0}/\overline{f}_0
\]
for which the composite map $N_\bullet(K) \xrightarrow{\overline{f}} N_\bullet(C)_{/\overline{f}_0}/\overline{f}_0 \to N_\bullet(C)$ is equal to $N_\bullet(F)$. This follows from our assumption that $u$ exhibits $K$ as a homotopy category of $K$, since the simplicial set $N_\bullet(C)_{/\overline{f}_0}/\overline{f}_0$ is isomorphic to the nerve of a category (see Example 4.3.5.7). \qed

Corollary 4.3.5.18. Let $A$ and $B$ be simplicial sets, and let $u : A \to N_\bullet(A)$ and $v : B \to N_\bullet(B)$ be morphisms which exhibit $A$ and $B$ as the homotopy categories of $A$ and $B$, respectively. Then the composite map
\[
A \star B \xrightarrow{u \star v} N_\bullet(A) \star N_\bullet(B) \simeq N_\bullet(A \star B)
\]
exhibits $A \star B$ as the homotopy category of $A \star B$.

Proof. Let $C$ be a category, and suppose we are given a map of simplicial sets $f : A \star B \to C$. Applying Corollary 4.3.5.17 to the morphism $f|_A$, we deduce that $f$ factors uniquely as a composition
\[
A \star B \xrightarrow{u \star \text{id}} N_\bullet(A) \star B \xrightarrow{\overline{f}} N_\bullet(C).
\]
Similarly, \( f' \) factors uniquely as a composition
\[
N_\bullet(\mathcal{A} \star \mathcal{B}) \xrightarrow{\mathcal{I} \cdot \mathcal{V}} N_\bullet(\mathcal{A} \star N_\bullet(\mathcal{B})) \xrightarrow{f''} N_\bullet(\mathcal{C}).
\]
Combining these observations (together with Example 4.3.5.7 and Proposition 1.2.2.1), we conclude that \( f \) factors uniquely as a composition
\[
A \star B \xrightarrow{u \star v} N_\bullet(\mathcal{A}) \star N_\bullet(\mathcal{B}) \simeq N_\bullet(\mathcal{A} \star \mathcal{B}) \xrightarrow{N_\bullet(F)} N_\bullet(\mathcal{C})
\]
for some functor \( F : \mathcal{A} \star \mathcal{B} \to \mathcal{C} \).

\section*{4.3.6 Slices of \( \infty \)-Categories}

We now specialize Construction 4.3.5.1 to the setting of \( \infty \)-categories.

\textbf{Proposition 4.3.6.1 (Joyal [21]).} Let \( K \) be a simplicial set, let \( \mathcal{C} \) be an \( \infty \)-category, and let \( f : K \to \mathcal{C} \) be a diagram. Then the projection map \( \mathcal{C}_{f/} \to \mathcal{C} \) is a left fibration of simplicial sets, and the projection map \( \mathcal{C}_{/f} \to \mathcal{C} \) is a right fibration of simplicial sets. In particular, the simplicial sets \( \mathcal{C}_{f/} \) and \( \mathcal{C}_{/f} \) are \( \infty \)-categories (see Remark 4.2.1.4).

\textbf{Remark 4.3.6.2.} In the special case where \( \mathcal{C} \) is (the nerve of) an ordinary category, Proposition 4.3.6.1 follows from Corollary 4.3.5.17; in fact, both of the simplicial sets \( \mathcal{C}_{f/} \) and \( \mathcal{C}_{/f} \) are (the nerves of) ordinary categories.

We begin with some elementary remarks.

\textbf{Construction 4.3.6.3.} Let \( f : A \leftrightarrow A' \) and \( g : B \leftrightarrow B' \) be monomorphisms of simplicial sets. Using Remark 4.3.3.16, we see that the induced maps
\[
A \star B \xrightarrow{f \star \text{id}_{B'}} A' \star B' \xleftarrow{\text{id}_{A'} \star g} A' \star B
\]
are also monomorphisms. Moreover, the intersection of their images is the image of the monomorphism \( (f \star g) : A \star B \leftrightarrow A' \star B' \). We therefore obtain a monomorphism of simplicial sets
\[
(A \star B') \coprod_{(A \star B)} (A' \star B) \leftrightarrow A' \star B',
\]
which we will refer to as the pushout-join of \( f \) and \( g \).

We will deduce Proposition 4.3.6.1 from the following property of Construction 4.3.6.3.

\textbf{Proposition 4.3.6.4 (Joyal [21]).} Let \( f : A \leftrightarrow A' \) and \( g : B \leftrightarrow B' \) be monomorphisms of simplicial sets. If \( f \) is right anodyne or \( g \) is left anodyne, then the pushout-join
\[
(A \star B') \coprod_{(A \star B)} (A' \star B) \leftrightarrow A' \star B'
\]
is an inner anodyne morphism of simplicial sets.
Proposition 4.3.6.4 implies the following stronger version of Proposition 4.3.6.1:

**Proposition 4.3.6.5.** Let $q : X \to S$ be an inner fibration of simplicial sets, let $f : K \to X$ be any morphism of simplicial sets, let $K_0$ be a simplicial subset of $K$, and set $f_0 = f|_{K_0}$. Then the restriction map

$$X/f \to X_{/f_0} \times S_{/(q \circ f_0)} S_{/(q \circ f)}$$

is a right fibration, and the restriction map

$$X_{f/} \to X_{f_0/} \times S_{/(q \circ f_0)/} S_{/(q \circ f)/}$$

is a left fibration.

**Proof.** We will prove the first assertion; the second follows by a similar argument. By virtue of Proposition 4.2.2.5, it will suffice to show that for every right anodyne morphism $i : A \hookrightarrow A'$, every lifting problem

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X/f \\
\downarrow & & \downarrow \\
A' & \xrightarrow{i} & X_{/f_0} \times S_{/(q \circ f_0)} S_{/(q \circ f)}
\end{array}
\]

admits a solution. Unwinding the definitions, this is equivalent to solving an associated lifting problem

\[
\begin{array}{ccc}
(A * K) \prod_{A \star K_0} (A' * K_0) & \xrightarrow{q} & X \\
\downarrow & & \downarrow q \\
A' * K & \xrightarrow{q} & S
\end{array}
\]

where the left vertical morphism is the pushout-join of Construction 4.3.6.3. Proposition 4.3.6.4 guarantees that this morphism is inner anodyne, so that the desired extension exists by virtue of our assumption that $q$ is an inner fibration (Proposition 4.1.3.1). \qed

**Corollary 4.3.6.6.** Let $q : X \to S$ be an inner fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then the restriction map

$$X/f \to X \times S_{/(q \circ f)}$$

is a right fibration, and the restriction map

$$X_{f/} \to X \times S_{/(q \circ f)/}$$

is a left fibration.
Proof. Apply Proposition 4.3.6.5 in the special case $K_0 = \emptyset$.

**Corollary 4.3.6.7.** Let $\mathcal{C}$ be an $\infty$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $f_0 = f|_{K_0}$ be the restriction of $f$ to a simplicial subset $K_0 \subseteq K$. Then the restriction map $\mathcal{C}/f \to \mathcal{C}/f_0$ is a right fibration, and the restriction map $\mathcal{C}_f/ \to \mathcal{C}_f/0$ is a left fibration.

**Proof.** Apply Proposition 4.3.6.5 to the inner fibration $q : \mathcal{C} \to \Delta^0$.

**Proof of Proposition 4.3.6.1.** Apply Corollary 4.3.6.7 in the special case $K_0 = \emptyset$.

Proposition 4.3.6.4 also yields the following:

**Proposition 4.3.6.8.** Let $q : X \to S$ be an inner fibration of simplicial sets, let $f : K \to X$ be any morphism of simplicial sets, let $K_0$ be a simplicial subset of $K$, and let $f_0 = f|_{K_0}$. If the inclusion $K_0 \hookrightarrow K$ is left anodyne, then the restriction map $X_f/ \to X_f/0 \times S/\left(q_f\circ f_0\right)/ S/\left(q_f\right)$ is a trivial Kan fibration. If the inclusion $K_0 \hookrightarrow K$ is right anodyne, then the restriction map $X_f// \to X_f/0 \times S/\left(q_f\circ f_0\right)/ S/\left(q_f\right)$ is a trivial Kan fibration.

**Proof.** We will prove the first assertion; the second follows by a similar argument. Assume that the inclusion $K_0 \hookrightarrow K$ is right anodyne. We wish to show that, for every monomorphism of simplicial sets $i : A \hookrightarrow A'$, every lifting problem For this, it will suffice to show that if $i : A \hookrightarrow A'$ is any monomorphism of simplicial sets, then any lifting problem

$$
\begin{array}{c}
A \\
\downarrow i \\
A' \\
\downarrow \\
X_f/ \\
\end{array}
$$

admits a solution. Unwinding the definitions, this is equivalent to solving an associated lifting problem

$$
\begin{array}{c}
(A \star K) \coprod_{A \star K_0} (A' \star K_0) \\
\downarrow q \\
X \\
\end{array}
$$

where the left vertical morphism is the pushout-join of Construction 4.3.6.3. Since the left vertical map is inner anodyne (Proposition 4.3.6.4), the desired solution exists by virtue of our assumption that $q$ is an inner fibration (Proposition 4.1.3.1).
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Corollary 4.3.6.9. Let \( C \) be an \( \infty \)-category, let \( f : K \to C \) be a morphism of simplicial sets, and let \( f_0 = f|_{K_0} \) be the restriction of \( f \) to a simplicial subset \( K_0 \subseteq K \). If the inclusion \( K_0 \hookrightarrow K \) is left anodyne, then the restriction map \( C/f \to C/f_0 \) is a trivial Kan fibration. If the inclusion \( K_0 \hookrightarrow K \) is right anodyne, then the restriction map \( C/f \to C/f_0 \) is a trivial Kan fibration.

Proof. Apply Proposition 4.3.6.8 to the inner fibration \( q : C \to \Delta^0 \).

We now turn to the proof of Proposition 4.3.6.4.

Lemma 4.3.6.10 (Joyal [21]). Let \( p, q \geq 0 \) be nonnegative integers. Then:

- Assume \( p > 0 \). Then, for \( 0 \leq i \leq p \), the pushout-join monomorphism
  \[
  (\Lambda^p_i \star \Delta^q) \coprod_{(\Lambda^p_i \star \partial \Delta^q)} (\Delta^p \star \partial \Delta^q) \hookrightarrow \Delta^p \star \Delta^q
  \]
  of Construction 4.3.6.3 is isomorphic to the horn inclusion \( \Lambda_i^{p+1+q} \hookrightarrow \Delta^{p+1+q} \).

- Assume \( q > 0 \). Then, for \( 0 \leq j \leq q \), the pushout-join monomorphism
  \[
  (\partial \Delta^p \star \Delta^q) \coprod_{(\partial \Delta^p \star \Lambda^q_j)} (\Delta^p \star \Lambda^q_j) \hookrightarrow \Delta^p \star \Delta^q
  \]
  of Construction 4.3.6.3 is isomorphic to the horn inclusion \( \Lambda_i^{p+1+q} \hookrightarrow \Delta^{p+1+q} \).

Proof. We will prove the first assertion; the second follows by symmetry. We begin by observing that there is a unique isomorphism of simplicial sets \( u : \Delta^p \star \Delta^q \simeq \Delta^{p+1+q} \) (Example 4.3.3.20). Let \( \sigma \) be an \( n \)-simplex of the join \( \Delta^p \star \Delta^q \); we wish to show that \( u(\sigma) \) belongs to the horn \( \Lambda_i^{p+1+q} \) if and only if \( \sigma \) belongs to the union of the simplicial subsets

\[
\Lambda_i^p \star \Delta^q \subseteq \Delta^p \star \Delta^q \supseteq \Delta^p \star \partial \Delta^q.
\]

We consider three cases (see Remark 4.3.3.15):

- The simplex \( \sigma \) belongs to the simplicial subset \( \Delta^p \subseteq \Delta^p \star \Delta^q \). In this case, \( \sigma \) is contained in \( \Delta^p \star \partial \Delta^q \) and \( u(\sigma) \) is contained in \( \Lambda_i^{p+1+q} \).

- The simplex \( \sigma \) belongs to the simplicial subset \( \Delta^q \subseteq \Delta^p \star \Delta^q \). In this case, \( \sigma \) is contained in \( \Lambda_i^p \star \Delta^q \) and \( u(\sigma) \) is contained in \( \Lambda_i^{p+1+q} \) (since \( p > 0 \)).

- The simplex \( \sigma \) factors as a composition
  \[
  \Delta^n = \Delta^{p+1+q} \simeq \Delta^{p'} \star \Delta^{q'} \xrightarrow{\sigma \times_{\sigma}} \Delta^p \star \Delta^q.
  \]
Let us abuse notation by identifying $\sigma^-$ and $\sigma^+$ with nondecreasing functions $[p'] \to [p]$ and $[q'] \to [q]$, and $u(\sigma)$ with the nondecreasing function $[n] \to [p + 1 + q]$ given by their join. In this case, $\sigma$ fails to belong to the union $(\Lambda^p_i \star \Delta^q) \cup (\Delta^p \star \partial \Delta^q)$ if and only if both of the following conditions are satisfied:

- The image of the nondecreasing function $\sigma^- : [p'] \to [p]$ contains $[p] \setminus \{i\}$.
- The nondecreasing function $\sigma^+ : [q'] \to [q]$ is surjective.

Together, these are equivalent to the assertion that the image of the nondecreasing function $u(\sigma) : [n] \to [p + 1 + q]$ contains $[p + 1 + q] \setminus \{i\}$: that is, it fails to belong to the horn $\Lambda^{p+1+q}_i \subseteq \Delta^{p+1+q}$.

Proof of Proposition 4.3.6.4. For every pair of morphisms of simplicial sets $f : A \to A'$ and $g : B \to B'$, let

$$\theta_{f,g} : (A \star B') \coprod_{(A \star B)} (A' \star B) \to A' \star B'$$

denote their pushout join. We will show that, if $f$ is right anodyne and $g$ is a monomorphism, then $\theta_{f,g}$ is inner anodyne (the analogous assertion for the case where $g$ is left anodyne follows by a similar argument). Let us first regard $f$ as fixed, and let $T$ be the collection of all morphisms $g$ of simplicial sets for which $\theta_{f,g}$ is inner anodyne. Then $T$ weakly saturated (in the sense of Definition 1.4.4.15). We wish to prove that $T$ contains every monomorphism of simplicial sets. By virtue of Proposition 1.4.5.12, we are reduced to proving that the morphism $\theta_{f,g}$ is inner anodyne in the special case where $g$ is the boundary inclusion $\partial \Delta^q \hookrightarrow \Delta^q$ for some $q \geq 0$.

Let us now regard $g : \partial \Delta^q \hookrightarrow \Delta^q$ as fixed, and let $S$ denote the collection of all morphisms of simplicial sets for which $\theta_{f,g}$ is inner anodyne. To complete the proof, we must show that $S$ contains every right anodyne morphism of simplicial sets. As before, we note that $S$ is weakly saturated. It will therefore suffice to show that $S$ contains every horn inclusion $\Lambda^p_i \hookrightarrow \Delta^p$ for $0 < i \leq p$ (see Variant 4.2.2.2). In other words, we are reduced to checking that the pushout-join

$$\theta_{f,g} : (\Lambda^p_i \star \Delta^q) \coprod_{(\Delta^p \star \partial \Delta^q)} (\Delta^p \star \partial \Delta^q) \hookrightarrow \Delta^p \star \Delta^q$$

is inner anodyne. This is clear, since $\theta_{f,g}$ can be identified with the inner horn inclusion $\Lambda^{p+1+q}_i \hookrightarrow \Delta^{p+1+q}$ by virtue of Lemma 4.3.6.10. \qed
4.3.7 Slices of Left and Right Fibrations

In this section, we collect some further applications of Lemma 4.3.6.10.

**Proposition 4.3.7.1.** Let \( f : A \rightarrow A' \) and \( g : B \rightarrow B' \) be monomorphisms of simplicial sets, and let

\[
\theta_{f,g} : (A \star B') \coprod_{(A' \star B)} (A' \star B') \rightarrow A' \star B'
\]

be the pushout-join of Construction 4.3.6.3. If \( f \) is anodyne, then \( \theta_{f,g} \) is left anodyne. If \( g \) is anodyne, then \( \theta_{f,g} \) is right anodyne.

**Proof.** We will prove the first assertion; the proof of the second is similar. We proceed as in the proof of Proposition 4.3.6.4. Let us first regard the anodyne morphism \( f \) as fixed, and let \( T \) be the collection of all morphisms \( g \) of simplicial sets for which \( \theta_{f,g} \) is left anodyne. Then \( T \) weakly saturated (in the sense of Definition 1.4.4.15). We wish to prove that \( T \) contains every monomorphism of simplicial sets. By virtue of Proposition 1.4.5.12, we are reduced to proving that the morphism \( \theta_{f,g} \) is left anodyne in the special case where \( g \) is the boundary inclusion \( \partial \Delta^q \rightarrow \Delta^q \) for some \( q \geq 0 \).

Let us now regard \( g : \partial \Delta^q \rightarrow \Delta^q \) as fixed, and let \( S \) denote the collection of all morphisms of simplicial sets for which \( \theta_{f,g} \) is left anodyne. To complete the proof, we must show that \( S \) contains every horn inclusion \( \Lambda^p_i \rightarrow \Delta^p \) when \( p > 0 \) and \( 0 \leq i \leq p \). In other words, we are reduced to checking that the pushout-join

\[
\theta_{f,g} : (\Lambda^p_i \star \Delta^q) \coprod_{(\Lambda^p_i \star \partial \Delta^q)} (\Delta^p \star \partial \Delta^q) \rightarrow \Delta^p \star \Delta^q
\]

is left anodyne. This is clear, since \( \theta_{f,g} \) can be identified with the horn inclusion \( \Lambda^{p+1+q}_i \rightarrow \Delta^{p+1+q} \) by virtue of Lemma 4.3.6.10. \( \square \)

**Proposition 4.3.7.2.** Let \( f : K \rightarrow X \) and \( q : X \rightarrow S \) be morphisms of simplicial sets, let \( K_0 \subseteq K \) be a simplicial subset, and set \( f_0 = f |_{K_0} \). Then:

- If \( q \) is a left fibration, then the induced map
  \[
  X_{f/f} \rightarrow X_{f/f_0} \times_{S/(q \circ f_0)} S/(q \circ f)
  \]
  is a Kan fibration.

- If \( q \) is a right fibration, then the induced map
  \[
  X_{f/} \rightarrow X_{f_0/} \times_{S/(q \circ f_0)} S/(q \circ f)/
  \]
  is a Kan fibration.
**Proof.** We will prove the first assertion; the proof of the second is similar. Assume that \( q \) is a left fibration; we wish to show that the map \( X_f \to X_{f_0} \times_{S/(q_0f_0)} S/(q_0f) \) is a Kan fibration. Equivalently, we wish to show that every lifting problem

\[
\begin{array}{ccc}
A & \to & X_f \\
\downarrow & & \downarrow \\
A' & \to & X_{f_0} \times_{S/(q_0f_0)} S/(q_0f)
\end{array}
\]

admits a solution, provided that the left vertical map \( A \to A' \) is anodyne. Unwinding the definitions, we see that this can be rephrased as a lifting problem

\[
\begin{array}{ccc}
(A \star K) \coprod_{(A \star K_0)} (A' \star K_0) & \to & X \\
\downarrow & & \downarrow \\
A' \star K & \to & S.
\end{array}
\]

This problem admits a solution, since the vertical map on the left is left anodyne (Proposition 4.3.7.1) and \( q \) is a left fibration.

**Corollary 4.3.7.3.** Let \( f : K \to X \) and \( q : X \to S \) be morphisms of simplicial sets. Then:

- If \( q \) is a left fibration, then the induced map
  \[
  X_f \to X \times_S S/(q_0f)
  \]
  is a Kan fibration.

- If \( q \) is a right fibration, then the induced map
  \[
  X_{f/} \to X \times_S S(q_0f)/
  \]
  is a Kan fibration.

**Proof.** Apply Proposition 4.3.7.2 in the special case \( K_0 = \emptyset \).

**Corollary 4.3.7.4.** Let \( X \) be a Kan complex, let \( f : K \to X \) be a morphism of simplicial sets, let \( K_0 \subseteq K \) be a simplicial subset, and set \( f_0 = f|_{K_0} \). Then the restriction maps

\[
X_f \to X_{f_0}, \quad X_{f/} \to X_{f_0}/
\]

are Kan fibrations.
4.3. THE SLICE AND JOIN CONSTRUCTIONS

Proof. Apply Proposition 4.3.7.2 in the special case \( S = \Delta^0 \).

Corollary 4.3.7.5. Let \( X \) be a Kan complex and let let \( f : K \to X \) be a morphism of simplicial sets. Then the projection maps

\[
X/f \to X \quad X/f/ \to X
\]

are Kan fibrations. In particular, the simplicial sets \( X/f \) and \( X/f/ \) are Kan complexes.

Proof. Apply Corollary 4.3.7.4 in the special case \( K_0 = \emptyset \) (or Corollary 4.3.7.3 in the special case \( S = \Delta^0 \)).

Proposition 4.3.7.6. Let \( f : K \to X \) and \( q : X \to S \) be morphisms of simplicial sets, let \( K_0 \subseteq K \) be a simplicial subset, and set \( f_0 = f|_{K_0} \). Then:

- If \( q \) is a right fibration and the inclusion \( K_0 \hookrightarrow K \) is anodyne, then the induced map
  \[ X/f \to X/f_0 \times_{S/(q\circ f_0)} S/(q\circ f) \]
  is a trivial Kan fibration.

- If \( q \) is a left fibration and the inclusion \( K_0 \hookrightarrow K \) is anodyne, then the induced map
  \[ X/f/ \to X/f_0/ \times_{S/(q\circ f_0)} S/(q\circ f)/ \]
  is a trivial Kan fibration.

Proof. We will prove the first assertion; the proof of the second is similar. Assume that \( q \) is a right fibration and that the inclusion \( K_0 \hookrightarrow K \) is anodyne. We wish to show that the map

\[
X/f \to X/f_0 \times_{S/(q\circ f_0)} S/(q\circ f)
\]

is a trivial Kan fibration. Equivalently, we wish to show that every lifting problem

\[
\begin{array}{ccc}
A & \to & X/f \\
\downarrow & & \downarrow \\
A' & \to & X/f_0 \times_{S/(q\circ f_0)} S/(q\circ f)
\end{array}
\]

admits a solution, provided that the left vertical map \( A \to A' \) is a monomorphism. Unwinding the definitions, we see that this can be rephrased as a lifting problem

\[
\begin{array}{ccc}
(A \star K) \coprod_{(A \star K_0)} (A' \star K_0) & \to & X \\
\downarrow & & \downarrow q \\
A' \star K & \to & S.
\end{array}
\]
CHAPTER 4. THE HOMOTOPY THEORY OF ∞-CATEGORIES

This problem admits a solution, since the vertical map on the left is right anodyne (Proposition 4.3.7.1) and \( q \) is a right fibration.

**Corollary 4.3.7.7.** Let \( X \) be a Kan complex, let \( f : K \to X \) be a morphism of simplicial sets, and let \( K_0 \subseteq K \) be a simplicial subset for which the inclusion \( K_0 \hookrightarrow K \) is anodyne, and set \( f_0 = f|_{K_0} \). Then the restriction maps

\[
X/f \to X \quad X_{f/} \to X
\]

are trivial Kan fibrations.

**Proof.** Apply Proposition 4.3.7.6 in the special case \( S = \Delta^0 \).

We now record some variants of the preceding results.

**Lemma 4.3.7.8.** Let \( f : A \hookrightarrow B \) be a monomorphism of simplicial sets. Then the inclusion \( f^\circ : A^\circ \hookrightarrow B^\circ \) is right anodyne, and the inclusion \( A^\circ \hookrightarrow B^\circ \) is left anodyne.

**Proof.** We will prove the first assertion (the second follows by a similar argument). Let \( T \) be the collection of all morphisms \( f \) of simplicial sets for which \( f^\circ \) is right anodyne. We wish to show that every monomorphism belongs to \( T \). Since the collection \( T \) is weakly saturated, it will suffice to show that every horn inclusion \( f : \partial \Delta^n \hookrightarrow \Delta^n \) belongs to \( T \) (Proposition 1.4.5.12). In this case, we can identify \( f^\circ \) with the with the horn inclusion \( \Lambda^{n+1}_{n+1} \hookrightarrow \Delta^n \) (see Example 4.3.3.24).

**Lemma 4.3.7.8** immediately implies the following stronger assertion:

**Proposition 4.3.7.9.** Let \( X \) and \( Y \) be simplicial sets. If \( X \) is weakly contractible and \( Y' \) is a simplicial subset of \( Y \), then the inclusion \( X \star Y' \hookrightarrow X \star Y \) is left anodyne. If \( Y \) is weakly contractible and \( X' \) is a simplicial subset of \( X \), then the inclusion \( X' \star Y \hookrightarrow X \star Y \) is right anodyne.

**Proof.** We will prove the first assertion; the second follows by a similar argument. Fix a vertex \( x \in X \), so that the inclusion morphism \( \iota : X \star Y' \hookrightarrow X \star Y \) factors as a composition

\[
X \star Y' \xrightarrow{\iota'} (X \star Y') \bigcup (\{x\} \star Y) \xrightarrow{\iota''} X \star Y.
\]

The morphism \( \iota' \) is a pushout of the inclusion \( Y'^\circ \hookrightarrow Y^\circ \), and is left anodyne by virtue of Lemma 4.3.7.8. It will therefore suffice to show that \( \iota'' \) is left anodyne. This is a special case of Proposition 4.3.7.1 since the inclusion map \( \{x\} \hookrightarrow X \) is a weak homotopy equivalence (by virtue of our assumption that \( X \) is weakly contractible) and therefore anodyne (by virtue of Corollary 3.3.7.5). \( \square \)
Example 4.3.7.10. Let $X$ and $Y$ be simplicial sets. If $X$ is weakly contractible, then Proposition 4.3.7.9 guarantees that the inclusion $ι_X : X \hookrightarrow X \ast Y$ is left anodyne. If $Y$ is weakly contractible, then Proposition 4.3.7.9 guarantees that the inclusion $ι_Y : Y \hookrightarrow X \ast Y$ is right anodyne.

Proposition 4.3.7.11. Let $q : X \to S$ and $f : K \to X$ be morphisms of simplicial sets. Then:

- If $q$ is a right fibration and $K$ is weakly contractible, then the induced map $X/f \to S/(q \circ f)$ is a trivial Kan fibration.
- If $q$ is a left fibration and $K$ is weakly contractible, then the induced map $X_{f/} \to S_{(q \circ f)/}$ is a trivial Kan fibration.

Proof. We will prove the first assertion; the second follows by a similar argument. To show that the morphism $X/f \to S/(q \circ f)$ is a trivial Kan fibration, we must prove that every lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \to & X/q \\
\downarrow & & \downarrow \\
\Delta^n & \to & S_{q \circ f}
\end{array}
\]

admits a solution. Unwinding the definitions, we can rephrase this as a lifting problem

\[
\begin{array}{ccc}
(\partial \Delta^n) \ast K & \to & X \\
\downarrow & \searrow & \downarrow q \\
(\Delta^n) \ast K & \to & S.
\end{array}
\]

This lifting problem admits a solution, since $q$ is assumed to be a right fibration and the left vertical map is right anodyne (Proposition 4.3.7.9).

Corollary 4.3.7.12. Let $q : X \to S$ be a morphism of simplicial sets, and let $x \in X$ be a vertex having image $s = q(x)$ in $S$. Then:

- If $q$ is a right fibration, then the induced map $X/_{x} \to S/_s$ is a trivial Kan fibration.
- If $q$ is a left fibration, then the induced map $X_{x/} \to S_{s/}$ is a trivial Kan fibration.

Corollary 4.3.7.13. Let $X$ be a Kan complex containing a vertex $x$. Then the simplicial sets $X/_{x}$ and $X_{x/}$ are contractible Kan complexes.
Proof. Apply Corollary 4.3.7.12 in the special case $S = \Delta^0$. \hfill \Box

**Proposition 4.3.7.14.** Let $f : K \to X$ and $q : X \to S$ be morphisms of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset, and set $f_0 = f|_{K_0}$. If $q$ is a trivial Kan fibration, then the induced maps

$$X/f \to X/f_0 \times_{S/(qf_0)} S/(qf) \quad X/f \to X/f_0 \times_{S/(qf_0)} S/(qf)/$$

are also trivial Kan fibrations.

**Proof.** To show that the map $X/f \to X/f_0 \times_{S/(qf_0)} S/(qf)$ is a trivial Kan fibration, we must show that every lifting problem admits a solution, provided that the left vertical map $A \to A'$ is a monomorphism. Unwinding the definitions, we see that this can be rephrased as a lifting problem

$$
\begin{array}{ccc}
A & \to & X/f \\
\downarrow & & \downarrow \\
A' & \to & X/f_0 \times_{S/(qf_0)} S/(qf)
\end{array}
$$

This problem admits a solution, since the vertical map on the left is a monomorphism (Proposition 4.3.7.1) and $q$ is a trivial Kan fibration. \hfill \Box

**Corollary 4.3.7.15.** Let $q : X \to S$ be a trivial Kan fibrations of simplicial sets, and let $f : K \to X$ be any morphism of simplicial sets. Then the induced maps

$$X/f \to X \times_{S} S/(qf) \quad X/f_0 \to X \times_{S} S/(qf)/$$

are trivial Kan fibrations.

**Proof.** Apply Proposition 4.3.7.14 in the special case $K_0 = \emptyset$. \hfill \Box

**Corollary 4.3.7.16.** Let $X$ be a contractible Kan complex, let $f : K \to X$ be a morphism of simplicial sets, let $K_0$ be a simplicial subset of $K$, and set $f_0 = f|_{K_0}$. Then the restriction maps

$$X/f \to X/f_0 \quad X/f_0 \to X_0/$$

are trivial Kan fibrations.
4.4. ISOMORPHISMS AND ISOFIBRATIONS

Proof. Apply Proposition 4.3.7.14 in the special case $S = \Delta^0$.

Corollary 4.3.7.17. Let $X$ be a contractible Kan complex and let $f : K \to X$ be a morphism of simplicial sets. Then the projection maps

$$X_f \to X \quad X_{f/} \to X$$

are trivial Kan fibrations. In particular, $X_f$ and $X_{f/}$ are also contractible Kan complexes.

Proof. Apply Corollary 4.3.7.15 in the special case $S = \Delta^0$ (or Corollary 4.3.7.16 in the special case $K_0 = \emptyset$).

4.4 ISOMORPHISMS AND ISOFIBRATIONS

Let $C$ be an $\infty$-category. Recall that a morphism $u : X \to Y$ in $C$ is an isomorphism if the homotopy class $[u]$ is an isomorphism in the homotopy category $hC$ (Definition 1.3.6.1). Our goal in this section is to study the notion of isomorphism in more detail.

Our first goal is to show that the class of isomorphisms can be characterized by a lifting property. Let $u : X \to Y$ be an isomorphism in an $\infty$-category $C$, and let $f : X \to Z$ be any other morphism in $C$. Then the composition $[f] \circ [u]^{-1} \in \operatorname{Hom}_{hC}(Y, Z)$ can be written as the homotopy class of some morphism $g : Y \to Z$ in $C$. The equality of homotopy classes $[f] = [g] \circ [u]$ is witnessed by some 2-simplex $\sigma$ which we depict as a diagram

```
  Y
 /\   \
/     \
\     /\    \
   /   \  \\
 /     \ \\
X---f---Z.
```

Phrased differently, $u$ and $f$ determine a morphism of simplicial sets $\sigma_0 : \Lambda_0^2 \to C$, and the preceding argument shows that $\sigma_0$ can be extended to a 2-simplex of $C$. In §4.4.2 we extend this argument to simplices of higher dimension. Suppose that we are given an integer $n \geq 2$ and a morphism of simplicial sets $\sigma_0 : \Lambda_0^n \to C$. If $0 < i < n$, then $\sigma_0$ can be extended to an $n$-simplex of $C$ by virtue of our assumption that $C$ is an $\infty$-category. In the extreme cases $i = 0$ and $i = n$, such an extension need not exist. However, we will show that it exists in the case $i = 0$ when $\sigma_0$ carries the initial edge $N_0(\{0 < 1\}) \subseteq \Lambda_0^n$ to an isomorphism in $C$, or in the case $i = n$ when $\sigma_0$ carries the final edge $N_0(\{n - 1 < n\}) \subseteq \Lambda_0^n$ to an isomorphism in $C$ (Theorem 4.4.2.5).

Theorem 4.4.2.5 has a number of useful consequences. For example, it implies that an $\infty$-category $\mathcal{C}$ is a Kan complex if and only if every morphism of $\mathcal{C}$ is an isomorphism (Proposition 4.4.2.1). More generally, it implies that any $\infty$-category $\mathcal{C}$ contains a largest
Kan complex, which we will denote by $C^\simeq$ and refer to as the core of $C$ (Construction 1.2.4.4). The construction $C \mapsto C^\simeq$ provides a link between the theory of $\infty$-categories and the classical homotopy theory of Kan complexes, which will play an important role throughout this book.

Let $F : C \to D$ be an inner fibration of $\infty$-categories. Then, for every object $D \in D$, the fiber $C_D = \{D\} \times_D C$ is an $\infty$-category (Remark 4.1.1.6). Beware that, in general, this construction behaves poorly with respect to isomorphisms. For example, if the fiber $C_D$ is nonempty and $D' \in D$ is an object which is isomorphic to $D$, then the fiber $C_{D'}$ could be empty. One can rule out this sort of behavior by imposing an additional assumption on the functor $F$. In §4.4.1, we introduce the notion of an isofibration of $\infty$-categories (Definition 4.4.1.4). Roughly speaking, an isofibration between $\infty$-categories is an inner fibration which also satisfies a path lifting property for isomorphisms. This condition guarantees that passage to the fiber is a homotopy invariant operation. For example, if $F : C \to D$ is an isofibration of $\infty$-categories, then it restricts to a Kan fibration of cores $F^\simeq : C^\simeq \to D^\simeq$ (Proposition 4.4.3.6).

Let $B$ be a simplicial set containing a simplicial subset $A$. Recall that, for every $\infty$-category $C$, the restriction functor $\theta : \text{Fun}(B, C) \to \text{Fun}(A, C)$ is an inner fibration (Corollary 4.1.4.2). In §4.4.5, we prove that $\theta$ is an isofibration (Corollary 4.4.5.3; see Proposition 4.4.5.1 for a stronger relative statement). The proof is based on the following recognition principle, which we establish in §4.4.4: if $C$ is an $\infty$-category and $u : F \to G$ is a morphism in an $\infty$-category of the form $\text{Fun}(X, C)$, then $u$ is an isomorphism in $\text{Fun}(X, C)$ if and only if, for every vertex $x \in X$, the induced map $u_x : F(x) \to G(x)$ is an isomorphism in the $\infty$-category $C$ (Theorem 4.4.4.4). In other words, if each $u_x$ admits a homotopy inverse $v_x : G(x) \to F(x)$, then we can choose the morphisms $\{v_x\}_{x \in X}$ (and homotopies witnessing the identifications $v_x \circ u_x \simeq \text{id}_{F(x)}$ and $u_x \circ v_x \simeq \text{id}_{G(x)}$) to depend functorially on $x \in X$.

### 4.4.1 Isofibrations of $\infty$-Categories

Let us begin by reviewing a bit of classical category theory.

**Definition 4.4.1.1.** Let $F : C \to D$ be a functor between categories. We say that $F$ is an isofibration if it satisfies the following condition:

(*) For every object $C \in C$ and every isomorphism $u : D \to F(C)$ in the category $D$, there exists an isomorphism $\overline{u} : D \to C$ in the category $C$ satisfying $F(\overline{u}) = C$.

**Example 4.4.1.2.** Let $F : C \to D$ be a functor between categories. If $F$ is a fibration in groupoids (or an opfibration in groupoids), then $F$ is an isofibration. For a more general statement, see Example 4.4.1.10.

The notion of isofibration is self-dual:
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Proposition 4.4.1.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then $F$ is an isofibration if and only if the opposite functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is an isofibration.

Proof. Assume that $F$ is an isofibration; we will show that $F^{\text{op}}$ is also an isofibration (the reverse implication follows by the same argument). Fix an object $C \in \mathcal{C}$ and an isomorphism $u : D \to F(C)$ in the category $\mathcal{D}$. Since $F$ is an isofibration, the inverse isomorphism $u^{-1} : D \to F(C)$ can be lifted to an isomorphism $v : D \to C$ in the category $\mathcal{C}$. Then $v^{-1} : C \to D$ satisfies $F(v^{-1}) = v$. \hfill \Box

We now introduce an $\infty$-categorical counterpart of Definition 4.4.1.1.

Definition 4.4.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. We say that $F$ is an isofibration if it is an inner fibration (Definition 4.1.1.1) which satisfies the following additional condition:

(*) For every object $C \in \mathcal{C}$ and every isomorphism $u : D \to F(C)$ in the category $\mathcal{D}$, there exists an isomorphism $\overline{u} : D \to C$ in the category $\mathcal{C}$ satisfying $F(\overline{u}) = C$.

Example 4.4.1.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then $F$ is an isofibration (in the sense of Definition 4.4.1.1) if and only if the induced map of simplicial sets $\text{N} \mathcal{C}(F) : \text{N} \mathcal{C} \to \text{N} \mathcal{D}$ is isofibration of $\infty$-categories. This follows from the observation that $\text{N} \mathcal{C}(F)$ is automatically an inner fibration (see Proposition 4.1.1.10).

Example 4.4.1.6. Let $\mathcal{C}$ be an $\infty$-category. Then the projection map $\mathcal{C} \to \Delta^0$ is an isofibration.

Proposition 4.4.1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between $\infty$-categories. Then $F$ is an isofibration of $\infty$-categories (in the sense of Definition 4.4.1.4) if and only if the induced functor of homotopy categories $f : \text{h}\mathcal{C} \to \text{h}\mathcal{D}$ is an isofibration of ordinary categories (in the sense of Definition 4.4.1.1).
satisfying \( f([v]) = [u] \). Then \([v]\) can be realized as the homotopy class of some morphism \( v : D \to C \) in the \( \infty \)-category \( C \), which is automatically an isomorphism. The equation \( f([v]) = [u] \) guarantees that there exists a homotopy from \( F(v) \) to \( u \) in the \( \infty \)-category \( D \), given by a 2-simplex \( \sigma : \\
\begin{array}{ccc}
F(C) & \xrightarrow{id_{F(C)}} & C \\
\downarrow F(v) & & \downarrow u \\
D & \xrightarrow{\pi} & C.
\end{array}
\\n\end{array}
\)

Since \( F \) is an inner fibration, it has the right lifting property with respect to the inclusion \( \Lambda^2_1 \to \Delta^2 \). We can therefore lift \( \sigma \) to a 2-simplex \( \sigma : \\
\begin{array}{ccc}
C & \xrightarrow{id_C} & C \\
\downarrow v & & \downarrow \pi \\
D & \xrightarrow{\pi} & C.
\end{array}
\\n\end{array}
\)

in the \( \infty \)-category \( C \). Since \( v \) and \( id_C \) are isomorphisms, it follows that \( \pi \) is an isomorphism (Remark 1.3.6.3). Allowing \( C \) and \( u \) to vary, we conclude that \( F \) is an isofibration of \( \infty \)-categories. 

**Corollary 4.4.1.8.** Let \( F : C \to D \) be a functor between \( \infty \)-categories. Then \( F \) is an isofibration if and only if the opposite functor \( F^{\text{op}} : C^{\text{op}} \to D^{\text{op}} \) is an isofibration.

**Proof.** Combine Proposition 4.4.1.7, Proposition 4.4.1.3, and Remark 4.1.1.3. 

**Remark 4.4.1.9.** Let \( F : C \to D \) and \( G : D \to E \) be isofibrations of \( \infty \)-categories. Then the composition \( G \circ F \) is also an isofibration of \( \infty \)-categories (for a more general statement, see Remark 4.5.7.9).

**Example 4.4.1.10.** Let \( F : C \to D \) be a right fibration between \( \infty \)-categories. Then \( F \) is an inner fibration (Remark 4.2.1.4), and any isomorphism \( u : D \to F(C) \) can be lifted to a morphism \( \pi : D \to C \) in \( C \), which is automatically an isomorphism by virtue of Proposition 4.4.2.9. It follows that \( F \) is an isofibration. Similarly, any left fibration of \( \infty \)-categories is an isofibration. For a more general statement, see Theorem [?].

**Example 4.4.1.11 (Replete Subcategories).** Let \( C \) be an \( \infty \)-category and let \( C' \subseteq C \) be a subcategory (Definition 4.1.2.2). The following conditions are equivalent:

1) The inclusion functor \( C' \hookrightarrow C \) is an isofibration.
4.4. **ISOMORPHISMS AND ISOFIBRATIONS**

(2) If \( u : X \rightarrow Y \) is an isomorphism in \( C \) and the object \( Y \) belongs to the subcategory \( C' \), then the isomorphism \( u \) also belongs to the subcategory \( C' \) (and, in particular, the object \( X \) also belongs to \( C' \)).

(3) If \( u : X \rightarrow Y \) is an isomorphism in \( C \) and the object \( X \) belongs to the subcategory \( C' \), then the isomorphism \( u \) also belongs to the subcategory \( C' \) (and, in particular, the object \( Y \) also belongs to \( C' \)).

If these conditions are satisfied, then we say that the subcategory \( C' \subseteq C \) is replete.

**Exercise 4.4.1.12.** Let \( X \) be a Kan complex, and let \( Y \subseteq X \) be a simplicial subset. Show that \( Y \) is a summand of \( X \) (Definition 1.1.6.1) if and only if it is a replete full subcategory of \( X \).

**Example 4.4.1.13.** Let \( C \) be an \( \infty \)-category, and let \( \text{Isom}(C) \) denote the full subcategory of \( \text{Fun}(\Delta^1, C) \) spanned by the isomorphisms in \( C \). Then the subcategory \( \text{Isom}(C) \subseteq \text{Fun}(\Delta^1, C) \) is replete. Unwinding the definitions, this amounts to the observation that for every commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{v} & & \downarrow{v'} \\
X' & \xrightarrow{u'} & Y'
\end{array}
\]

in the \( \infty \)-category \( C \) where \( u, v, \) and \( v' \) are isomorphisms, the morphism \( u' \) is also an isomorphism. This follows immediately from the two-out-of-three property of Remark 1.3.6.3.

### 4.4.2 ISOMORPHISMS AND LIFTING PROPERTIES

Recall that a morphism of simplicial sets \( X \rightarrow S \) is a Kan fibration if and only if it is both a left fibration and a right fibration (Example 4.2.1.5). In the special case \( S = \Delta^0 \), either one of these conditions is individually sufficient.

**Proposition 4.4.2.1** (Joyal [21]). Let \( X \) be a simplicial set. The following conditions are equivalent:

(a) The projection map \( X \rightarrow \Delta^0 \) is a Kan fibration.

(b) The simplicial set \( X \) is a Kan complex.

(c) The simplicial set \( X \) is an \( \infty \)-category and the homotopy category \( \text{h}X \) is a groupoid.

(d) The simplicial set \( X \) is an \( \infty \)-category and every morphism in \( X \) is an isomorphism.
(e) The projection map \( X \rightarrow \Delta^0 \) is a left fibration.

(f) The projection map \( X \rightarrow \Delta^0 \) is a right fibration.

**Corollary 4.4.2.2.** Let \( q : X \rightarrow S \) be morphism of simplicial sets which is either a left or a right fibration. Then, for every vertex \( s \in S \), the fiber \( X_s = \{ s \} \times_S X \) is a Kan complex.

**Proof.** Combine Proposition 4.4.2.1 with Remark 4.2.1.8.

**Corollary 4.4.2.3.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{i} & & \downarrow{q} \\
B & \xrightarrow{g} & S,
\end{array}
\]

where \( i \) is a monomorphism. Then:

- If \( q \) is either a left or right fibration, then the simplicial set \( \text{Fun}_{A/\!/S}(B, X) \) of Construction 3.1.3.7 is a Kan complex.

- If \( q \) is a left fibration and \( i \) is left anodyne, then the Kan complex \( \text{Fun}_{A/\!/S}(B, X) \) is contractible.

- If \( q \) is a right fibration and \( i \) is right anodyne, then the Kan complex \( \text{Fun}_{A/\!/S}(B, X) \) is contractible.

**Proof.** Without loss of generality, we may assume that \( q \) is a left fibration. By virtue of Remark 3.1.3.11, the simplicial set \( \text{Fun}_{A/\!/S}(B, X) \) can be identified with a fiber of the restriction map

\[ \theta : \text{Fun}(B, X) \rightarrow \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, X). \]

Proposition 4.2.3.1 asserts that \( \theta \) is a left fibration of simplicial sets, so its fibers are Kan complexes (Corollary 4.4.2.2). If \( i \) is left anodyne, then \( \theta \) is a trivial Kan fibration (Proposition 4.2.3.4), so its fibers are contractible Kan complexes.

**Corollary 4.4.2.4.** Let \( q : X \rightarrow S \) and \( g : B \rightarrow S \) be morphisms of simplicial sets. If \( q \) is either a left fibration or a right fibration, then the simplicial set \( \text{Fun}_{/S}(B, X) \) is a Kan complex.

**Proof.** Apply Corollary 4.4.2.3 in the special case \( A = \emptyset \).
Our proof of Proposition 4.4.2.1 is based on the following characterization of isomorphisms in an ∞-category C:

**Theorem 4.4.2.5** (Joyal). Let C be an ∞-category and let u : X → Y be a morphism of C. The following conditions are equivalent:

1. The morphism u is an isomorphism.
2. Let n ≥ 2 and let σ₀ : Λ^n_0 → C be a morphism of simplicial sets for which the initial edge
   \[ \Delta^1 \simeq N_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} C \]
   is equal to u. Then σ₀ can be extended to an n-simplex σ : Δ^n → C.
3. Let n ≥ 2 and let σ₀ : Λ^n_n → C be a morphism of simplicial sets for which the final edge
   \[ \Delta^1 \simeq N_\bullet(\{n - 1 < n\}) \hookrightarrow \Lambda^n_n \xrightarrow{\sigma_0} C \]
   is equal to u. Then σ₀ can be extended to an n-simplex σ : Δ^n → C.

**Proof of Proposition 4.4.2.1 from Theorem 4.4.2.5.** Let X be a simplicial set. By definition, the projection map X → Δ^0 is a left fibration if and only if, for every pair of integers 0 < i ≤ n, every morphism of simplicial sets σ₀ : Λ^n_i → X can be extended to an n-simplex σ : Δ^n → X. This condition is automatically satisfied when n = 1 (we can identify σ₀ with a vertex x ∈ X, and take σ to be the degenerate edge id_x), and is satisfied for 0 < i < n if and only if X is an ∞-category. Assuming that X is an ∞-category, it is satisfied for i = n if and only if every morphism in X is an isomorphism (by virtue of Theorem 4.4.2.5). This proves the equivalence (d) ⇔ (e), and the equivalence (d) ⇔ (f) follows by applying the same reasoning to the opposite simplicial set X^{op}. In particular, (e) and (f) are equivalent to one another, and therefore equivalent to (a) (see Example 4.2.1.5). The equivalences (a) ⇔ (b) and (c) ⇔ (d) are immediate from the definitions.

The proof of Theorem 4.4.2.5 will require some preliminaries.

**Definition 4.4.2.6.** Let C and D be ∞-categories. We will say that a functor F : C → D is **conservative** if it satisfies the following condition:

- Let u : X → Y be a morphism in C. If F(u) : F(X) → F(Y) is an isomorphism in the ∞-category D, then u is an isomorphism.

**Example 4.4.2.7.** Let C be an ∞-category. Then the canonical map C → N_•(hC) is conservative.
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Remark 4.4.2.8. Let \(F : C \to D\) and \(G : D \to E\) be functors between \(\infty\)-categories, where \(G\) is conservative. Then \(F\) is conservative if and only if the composition \((G \circ F) : C \to E\) is conservative.

Proposition 4.4.2.9. Let \(F : C \to D\) be a functor between \(\infty\)-categories. If \(F\) is a left or a right fibration, then \(F\) is conservative.

Proof. Without loss of generality, we may assume that \(F\) is a left fibration. Let \(u : X \to Y\) be a morphism in \(C\), and suppose that \(F(u)\) is an isomorphism in \(D\). Let \(\bar{v} : F(Y) \to F(X)\) is a homotopy inverse to \(F(u)\), so that there exists a 2-simplex \(\sigma\) of \(D\) as depicted in the following diagram:

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{\bar{v}} & F(X) \\
\downarrow^{F(u)} & & \downarrow^{F(id_X)} \\
F(X) & \xrightarrow{F(id_X)} & F(X).
\end{array}
\]

Invoking our assumption that \(F\) is a left fibration, we can lift \(\sigma\) to a diagram in the \(\infty\)-category \(C\). This lift supplies a morphism \(v : Y \to X\) and witnesses that \(id_X\) as a composition of \(v\) with \(u\), so that \(v\) is a left homotopy inverse to \(u\). Moreover, the image \(F(v) = \bar{v}\) is an isomorphism in \(D\). Repeating the preceding argument (with \(u : X \to Y\) replaced by \(v : Y \to X\)), we deduce that there exists a morphism \(w : X \to Y\) which is left homotopy inverse to \(v\). It follows that \(u\) and \(w\) are homotopic, so that \(v\) is a homotopy inverse to \(u\) (Remark 1.3.6.6). In particular, \(u\) is an isomorphism. \(\square\)

Proposition 4.4.2.10. Let \(q : C \to D\) be an inner fibration of \(\infty\)-categories, let \(u\) be an isomorphism in \(C\), let \(n \geq 2\) be an integer, and suppose we are given a lifting problem

\[
\begin{array}{c}
\Delta^n \xrightarrow{\sigma_0} C \\
\Delta^n \xrightarrow{\sigma} \Delta^n \\
\end{array}
\]
If the composite map
\[
\Delta^1 \simeq N_*\{n - 1 < n\} \hookrightarrow \Lambda^n_n \xrightarrow{\sigma_0} C
\]
is equal to \(u\), then there exists an \(n\)-simplex \(\sigma : \Delta^n \to C\) rendering the diagram commutative.

**Proof.** Using Lemma 4.3.6.10, we can identify the horn \(\Lambda^n_n\) with the pushout
\[
(\Delta^n - \{1\}) \coprod_{(\partial\Delta^n - \{1\})} (\partial\Delta^n - \{1\}) \subseteq \Delta^n - \{1\} \simeq \Delta^n.
\]
Set \(f = \sigma_0|_{\Delta^n - \{1\}}\) and \(f_0 = \sigma_0|_{\partial\Delta^n - \{1\}}\), and let \(E\) denote the fiber product \(C_{f_0/} \times_{D(q\circ f_0)/} D(qf)/\).

Note that there is an evident projection map \(\theta : E \to C\), given by the composition
\[
E \xrightarrow{\theta'} C_{f_0/} \xrightarrow{\theta''} C.
\]
The morphism \(\theta''\) is a left fibration (Proposition 4.3.6.1), and the morphism \(\theta'\) is a pullback of the restriction map \(D(q\circ f)/ \to D(q\circ f_0)/\) and is therefore also a left fibration (Corollary 4.3.6.9). It follows that \(\theta : E \to C\) is a left fibration (Remark 4.2.1.11), and in particular \(E\) is an \(\infty\)-category (Remark 4.1.1.9).

Note that the restriction of \(\sigma_0\) to \(\Delta^n - \{1\}\) can be identified with an object \(Y\) of the coslice \(\infty\)-category \(C_{f/}\). Let
\[
\rho : C_{f/} \to C_{f_0/} \times_{D(q\circ f_0)/} D(qf)/ = E
\]
be the left fibration of Proposition 4.3.6.5 and set \(Y = \rho(Y) \in E\). Then the restriction \(\sigma_0|_{\partial\Delta^n - \{1\}}\) and \(\tau\) determine a morphism \(\tau : X \to Y\) in the \(\infty\)-category \(E\). Unwinding the definitions, we see that choosing an \(n\)-simplex \(\sigma : \Delta^n \to C\) satisfying the requirements of Proposition 4.4.2.10 is equivalent to choosing a morphism \(v : X \to Y\) in \(C_{f/}\) satisfying \(\rho(v) = \tau\). Since \(\rho\) is a left fibration, it is an isofibration (Example 4.4.1.10). Consequently, to prove the existence of \(v\), it will suffice to show that \(\tau\) is an isomorphism in the \(\infty\)-category \(E\). Since \(\theta\) is a left fibration, this follows from our assumption that \(u = \theta(\tau)\) is an isomorphism in the \(\infty\)-category \(C\) (Proposition 4.4.2.9).

**Proof of Theorem 4.4.2.5.** The implication (1) \(\Rightarrow\) (3) is a special case of Proposition 4.4.2.10. We will complete the proof by showing that (3) \(\Rightarrow\) (1) (a similar argument shows that (1) and (2) are equivalent). Let \(u : X \to Y\) be a morphism in an \(\infty\)-category \(C\), and consider the map \(\sigma_0 : \Lambda^2_2 \to C\) depicted in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{v} & & \downarrow{id_Y} \\
Y & \xrightarrow{id_Y} & Y
\end{array}
\]
If \( u \) satisfies condition (3), then we can complete \( \sigma_0 \) to a 2-simplex of \( \mathcal{C} \), which witnesses the morphism \( v = d_2(\sigma) \) as a right homotopy inverse of \( u \). The tuple \((\sigma, s_0(u), s_1(u), \bullet)\) then determines a morphism of simplicial sets \( \tau_0 : \Lambda^n_3 \to \mathcal{C} \) (see Exercise 1.1.2.14). Invoking assumption (3) again, we can extend \( \tau_0 \) to a 3-simplex \( \tau \) of \( \mathcal{C} \). The face \( d_3(\tau) \) then witnesses that \( v \) is also a left homotopy inverse to \( u \), so that \( u \) is an isomorphism as desired. \( \square \)

### 4.4.3 The Core of an \( \infty \)-Category

Let \( \mathcal{C} \) be a category. Recall that the core of \( \mathcal{C} \) is the subcategory \( \mathcal{C}^\simeq \subseteq \mathcal{C} \) comprised of all objects of \( \mathcal{C} \) and all isomorphisms between them (Construction 1.2.4.4). In this section, we generalize this construction to the setting of \( \infty \)-categories.

**Construction 4.4.3.1.** Let \( \mathcal{C} \) be an \( \infty \)-category. We let \( \mathcal{C}^\simeq \) denote the simplicial subset of \( \mathcal{C} \) comprised of those simplices \( \sigma : \Delta^n \to \mathcal{C} \) which carry each edge of \( \Delta^n \) to an isomorphism in \( \mathcal{C} \). We will refer to \( \mathcal{C}^\simeq \) as the core of \( \mathcal{C} \).

**Remark 4.4.3.2.** Let \( \mathcal{C} \) be an \( \infty \)-category, let \( h\mathcal{C} \) be its homotopy category, and let \( h\mathcal{C}^\simeq \) denote the core of \( h\mathcal{C} \). Then the core \( \mathcal{C}^\simeq \subseteq \mathcal{C} \) fits into a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
\mathcal{C}^\simeq & \to & \mathcal{C} \\
\downarrow & & \downarrow \\
N_\bullet(h\mathcal{C}^\simeq) & \to & N_\bullet(h\mathcal{C}). \\
\end{array}
\]

**Example 4.4.3.3.** Let \( \mathcal{C} \) be an ordinary category, and let \( \mathcal{C}^\simeq \) denote its core (in the sense of Construction 1.2.4.4). Then the core of the \( \infty \)-category \( N_\bullet(\mathcal{C}) \) (in the sense of Construction 4.4.3.1) can be identified with the nerve of \( \mathcal{C}^\simeq \). That is, we have a canonical isomorphism \( N_\bullet(\mathcal{C})^\simeq \simeq N_\bullet(\mathcal{C}^\simeq) \).

**Remark 4.4.3.4 (Functoriality).** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor of \( \infty \)-categories. Then \( F \) carries the core \( \mathcal{C}^\simeq \) into the core \( \mathcal{D}^\simeq \) (see Remark 1.4.1.6), and therefore restricts to a morphism of Kan complexes \( F^\simeq : \mathcal{C}^\simeq \to \mathcal{D}^\simeq \).

**Proposition 4.4.3.5.** Let \( \mathcal{C} \) be an \( \infty \)-category. Then the core \( \mathcal{C}^\simeq \) is a subcategory of \( \mathcal{C} \) (Definition 4.1.2.2): that is, the inclusion \( \mathcal{C}^\simeq \hookrightarrow \mathcal{C} \) is an inner fibration.

**Proof.** Combine Example 4.1.2.4, Remark 4.1.2.6 and Remark 4.4.3.2. \( \square \)

**Proposition 4.4.3.6.** Let \( F : \mathcal{C} \to \mathcal{D} \) be an isofibration of \( \infty \)-categories. Then the induced map \( F^\simeq : \mathcal{C}^\simeq \to \mathcal{D}^\simeq \) is a Kan fibration.
Proof. Fix integers $n > 0$ and $0 \leq i \leq n$; we wish to show that every lifting problem

\[
\begin{array}{c}
\Lambda^n_i \xrightarrow{\sigma_0} C \xrightarrow{\sigma} C' \xrightarrow{\sigma} C'' \\
\Delta^n \xrightarrow{\sigma} D \xrightarrow{\sigma} D'
\end{array}
\]

admits a solution. In the case $n = 1$, this follows either from our definition of isofibration (in the case $i = 1$) or from Corollary 4.4.1.8 (in the case $i = 0$). We may therefore assume that $n \geq 2$. We claim that $\sigma_0$ can be extended to an $n$-simplex $\sigma : \Delta^n \to C$ satisfying $F(\sigma) = \sigma$. If $0 < i < n$, this follows from the fact that $F$ is an inner fibration. The extremal cases $i = 0$ and $i = n$ follow from Proposition 4.4.2.10 (applied to the inner fibration $F : C \to D$ and its opposite $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$). To complete the proof, it will suffice to show that $\sigma$ carries each edge of $\Delta^n$ to an isomorphism in $C$. For $n > 2$, this is automatic (since the horn $\Lambda^n_i$ contains every edge of $\Delta^n$). In the case $n = 2$ it follows from the two-out-of-three property for isomorphisms in $C$ (Remark 1.3.6.3).

\[\square\]

**Corollary 4.4.3.7.** Let $C$ be an $\infty$-category. Then the core $C^\simeq$ is a Kan complex.

**Proof.** Apply Proposition 4.4.3.6 to the inner fibration $C \to \Delta^0$.

\[\square\]

**Exercise 4.4.3.8.** Deduce Corollary 4.4.3.7 directly from the criterion of Proposition 4.4.2.1.

**Corollary 4.4.3.9.** Let $C$ be an $\infty$-category and let $u : X \to Y$ be a morphism of $C$. The following conditions are equivalent:

1. The morphism $u$ is an isomorphism.

2. There exists a Kan complex $E$, a morphism $\overline{u} : \overline{X} \to \overline{Y}$ in $E$, and a functor $F : E \to C$ satisfying $F(\overline{u}) = u$.

3. There exists a contractible Kan complex $E$, a morphism $\overline{u} : \overline{X} \to \overline{Y}$ in $E$, and a functor $F : E \to C$ satisfying $F(\overline{u}) = u$.

**Proof.** If $u$ is an isomorphism, then it belongs to the image of the inclusion functor $C^\simeq \hookrightarrow C$. Since the core $C^\simeq$ is a Kan complex, this proves that (1) \implies (2). Conversely, if we can write $u = F(\overline{u})$ for some functor $F : E \to C$ where $E$ is a Kan complex, then Remark 1.4.1.6 guarantees that $u$ is an isomorphism in $C$ (since $\overline{u}$ is automatically an isomorphism in $E$, by virtue of Proposition 1.3.6.11). This proves that (2) \implies (1).

The implication (3) \implies (2) is immediate. We will complete the proof by showing that (2) implies (3). Let $E$ be a Kan complex, let $F : E \to C$ be a functor, and let $\overline{u}$ be an edge of
\[ E \\] satisfying \( F(\overline{u}) = u \). Let us identify \( u \) with a morphism of simplicial sets \( \Delta^1 \rightarrow E \). By virtue of Proposition 3.1.6.1, this morphism factors as a composition \( \Delta^1 \xrightarrow{\nu} \mathcal{E}' \xrightarrow{q} E \), where \( \nu \) is anodyne and \( q \) is a Kan fibration. Since \( E \) is a Kan complex and \( q \) is a Kan fibration, the simplicial set \( \mathcal{E}' \) is a Kan complex (Remark 3.1.1.11). Because \( \Delta^1 \) is weakly contractible and \( \nu \) is a weak homotopy equivalence, the Kan complex \( \mathcal{E}' \) is contractible. We can then write \( u = F'(v) \) where \( F' = F \circ q \).

**Corollary 4.4.3.10.** Let \( \mathcal{C} \) be an \( \infty \)-category containing objects \( X \) and \( Y \). The following conditions are equivalent:

1. The objects \( X \) and \( Y \) are isomorphic.
2. There exists a connected Kan complex \( \mathcal{E} \), a pair of vertices \( \overline{X}, \overline{Y} \in \mathcal{E} \), and a morphism \( f : \mathcal{E} \rightarrow \mathcal{C} \) satisfying \( f(\overline{X}) = X \) and \( f(\overline{Y}) = Y \).
3. There exists a connected Kan complex \( \mathcal{E} \), a pair of vertices \( \overline{X}, \overline{Y} \in \mathcal{E} \), and a morphism \( f : \mathcal{E} \rightarrow \mathcal{C} \) satisfying \( f(\overline{X}) = X \) and \( f(\overline{Y}) = Y \).

**Notation 4.4.3.11.** Let \( \mathcal{C} \) be an \( \infty \)-category. We let \( \pi_0(\mathcal{C}^\simeq) \) denote the set of connected components of the Kan complex \( \mathcal{C}^\simeq \). Note that \( \pi_0(\mathcal{C}^\simeq) \) can be identified with the set of isomorphism classes of objects of \( \mathcal{C} \) (that is, the quotient of the set of objects of \( \mathcal{C} \) by the equivalence relation of isomorphism).

If \( \mathcal{C} \) is an \( \infty \)-category, then the Kan complex \( \mathcal{C}^\simeq \) can be characterized by a universal property:

**Proposition 4.4.3.12.** Let \( \mathcal{C} \) be an \( \infty \)-category and let \( X \) be a Kan complex. Then composition with the inclusion \( \mathcal{C}^\simeq \hookrightarrow \mathcal{C} \) induces a bijection \( \text{Hom}_{\text{Set}_\Delta}(X, \mathcal{C}^\simeq) \rightarrow \text{Hom}_{\text{Set}_\Delta}(X, \mathcal{C}) \).

**Proof.** Let \( F : X \rightarrow \mathcal{C} \) be a morphism of simplicial sets. To show that \( F \) factors through the core \( \mathcal{C}^\simeq \subseteq \mathcal{C} \), we must show that for every edge \( u : x \rightarrow y \) of the Kan complex \( X \), the image \( F(u) \) is an isomorphism in \( \mathcal{C} \). This follows from Remark 1.4.1.6 since \( u \) is automatically an isomorphism in the \( \infty \)-category \( X \) (Proposition 1.3.6.11).

**Corollary 4.4.3.13.** Let \( \mathcal{C} \) be an \( \infty \)-category. Then the core \( \mathcal{C}^\simeq \) is the largest Kan complex which is contained in \( \mathcal{C} \).

**Proof.** Combine Corollary 4.4.3.7 with Proposition 4.4.3.12.

\[ \square \]
Corollary 4.4.3.14 (Pullbacks of Isofibrations). Suppose we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
C' & \xrightarrow{F} & C \\
\downarrow{q'} & & \downarrow{q} \\
D' & \xrightarrow{F'} & D,
\end{array}
\]

where \( q \) is an isofibration of \( \infty \)-categories and \( D' \) is an \( \infty \)-category. Then:

1. The simplicial set \( C' \) is an \( \infty \)-category.
2. The diagram of Kan complexes is a pullback square and a homotopy pullback square.
3. A morphism \( u : X \to Y \) in the \( \infty \)-category \( C' \) is an isomorphism if and only if \( F(u) \) is an isomorphism in the \( \infty \)-category \( C \) and \( q'(u) \) is an isomorphism in the \( \infty \)-category \( D' \).
4. The morphism \( q' \) is an isofibration of \( \infty \)-categories.

Proof. Since \( q \) is an isofibration, it is an inner fibration. It follows that the morphism \( q' \) is also an inner fibration (Remark 4.1.1.5). Since \( D' \) is an \( \infty \)-category, the simplicial set \( C' \) is also an \( \infty \)-category (Remark 4.1.1.9). This proves (1).

Let \( E \) denote the fiber product \( C'^{\simeq} \times_{D'^{\simeq}} D^{\simeq} \), which we regard as a simplicial subset of \( C' = C \times D' \). It follows from Proposition 4.4.3.6 that \( q \) restricts to a Kan fibration \( q^{\simeq} : C^{\simeq} \to D^{\simeq} \). The projection map \( E \to D'^{\simeq} \) is a pullback of \( q^{\simeq} \), and is therefore also a Kan fibration. Since \( D'^{\simeq} \) is a Kan complex (Corollary 4.4.3.7), it follows that \( E \) is a Kan complex (Remark 3.1.1.11). Applying Corollary 4.4.3.13, we deduce that \( E \) is contained in the core \( C'^{\simeq} \subseteq C' \), which proves that the diagram (4.2) is a pullback square. Since \( q^{\simeq} \) is a Kan fibration, it is also a homotopy pullback square (Example 3.4.1.5). This proves assertion (2), and assertion (3) is an immediate consequence.

To complete the proof of (4), it will suffice to show that the morphism \( q' \) satisfies condition \((*)\) of Definition 4.4.1.4. Let \( Y' \) be an object of \( C' \) and let \( \overline{u} : \overline{X}' \to q'(Y') \) be
an isomorphism in the ∞-category \( D' \); we wish to show that \( \overline{m} \) can be written as \( q'(u') \) for some isomorphism \( u' : X' \to Y' \) in the ∞-category \( \mathcal{C}' \). By virtue of (3), this is equivalent to showing that \( F'(\overline{m}) \) can be written as \( q(u) \) for some isomorphism \( u : X \to F(Y) \) in the ∞-category \( \mathcal{C} \), which follows from our assumption that \( q \) is an isofibration. \( \square \)

**Corollary 4.4.3.15.** Let \( q : \mathcal{C} \to \mathcal{D} \) be an isofibration of ∞-categories, and let \( \mathcal{C}_D = \{ D \} \times_D \mathcal{C} \) be the fiber of \( q \) over an object \( D \in \mathcal{D}. \) Then the canonical map \( (\mathcal{C}_D)^\simeq \to \{ D \} \times_D \mathcal{C}^\simeq \) is an isomorphism. In other words, the inclusion functor \( \mathcal{C}_D \to \mathcal{C} \) is conservative.

**Proof.** Apply Corollary 4.4.3.14 in the special case \( D' = \{ D \} \).

We close this section by establishing a relative version of Proposition 4.4.3.12. Let \( \mathcal{C} \) be an ∞-category, and let \( X \) be an arbitrary simplicial set. Then the simplicial set \( \text{Fun}(X, \mathcal{C}) \) is an ∞-category (Theorem 1.4.3.7), and the simplicial set \( \text{Fun}(X, \mathcal{C}^\simeq) \) is a Kan complex (Corollary 3.1.3.4). The inclusion \( \mathcal{C}^\simeq \to \mathcal{C} \) induces a monomorphism of simplicial sets \( \text{Fun}(X, \mathcal{C}^\simeq) \to \text{Fun}(X, \mathcal{C})\simeq \), which automatically factors through the core \( \text{Fun}(X, \mathcal{C})^\simeq \) (Corollary 4.4.3.13).

**Proposition 4.4.3.16.** Let \( \mathcal{C} \) be an ∞-category and let \( X \) be a Kan complex. Then the canonical map

\[
\theta : \text{Fun}(X, \mathcal{C}^\simeq) \to \text{Fun}(X, \mathcal{C})^\simeq
\]

is an isomorphism of simplicial sets.

**Remark 4.4.3.17.** Proposition 4.4.3.12 can be regarded as a special case of Proposition 4.4.3.16; it is equivalent to the assertion that, for every ∞-category \( \mathcal{C} \) and every Kan complex \( X \), the canonical map \( \text{Fun}(X, \mathcal{C}^\simeq) \to \text{Fun}(X, \mathcal{C})^\simeq \) is bijective on vertices.

**Warning 4.4.3.18.** The conclusion of Proposition 4.4.3.16 generally does not hold if \( X \) is not a Kan complex.

**Proof of Proposition 4.4.3.16.** Let \( \sigma : Y \to \text{Fun}(X, \mathcal{C})^\simeq \) be a morphism of simplicial sets, which we identify with a diagram \( F : X \times Y \to \mathcal{C} \). To show that \( \sigma \) factors through the monomorphism \( \theta \), it will suffice to show that \( F \) factors through the core \( \mathcal{C}^\simeq \subseteq \mathcal{C} \). Equivalently, we wish to show that for every edge \( (u, v) : (x, y) \to (x', y') \) in the product simplicial set \( X \times Y \), the morphism \( F(u, v) : F(x, y) \to F(x', y') \) is an isomorphism in the ∞-category \( \mathcal{C} \). Note that \( F(u, v) \) can be identified with a composition of morphisms

\[
F(x, y) \xrightarrow{F(u, \text{id}_y)} F(x', y) \xrightarrow{F(\text{id}_x', v)} F(x', y')
\]

in the ∞-category \( \mathcal{C} \). Since the collection of isomorphisms in \( \mathcal{C} \) is closed under composition (Remark 1.3.6.3), it will suffice to show that \( F(u, \text{id}_y) \) and \( F(\text{id}_x', v) \) are isomorphisms in \( \mathcal{C} \). In the first case, this follows from Proposition 4.4.3.12 (applied to the morphism \( F|_{X \times \{y\}} \)),
since \( X \) is a Kan complex. In the second case, it follows from our assumption that \( \sigma \) factors through the core \( \text{Fun}(X, \mathcal{C})^\simeq \subseteq \text{Fun}(X, \mathcal{C}) \) (and therefore carries the edge \( v : y \to y' \) to an isomorphism in the diagram \( \infty \)-category \( \text{Fun}(X, \mathcal{C}) \)).

### 4.4.4 Natural Isomorphisms

Recall that, if \( X \) is an arbitrary simplicial set and \( \mathcal{C} \) is an \( \infty \)-category, then the simplicial set \( \text{Fun}(X, \mathcal{C}) \) is also an \( \infty \)-category (Theorem 1.4.3.7). In this section, we study isomorphisms in \( \infty \)-categories of the form \( \text{Fun}(X, \mathcal{C}) \).

**Definition 4.4.4.1.** Let \( \mathcal{C} \) be an \( \infty \)-category, let \( X \) be a simplicial set, and suppose we are given a pair of diagrams \( f, f' : X \to \mathcal{C} \). A natural transformation from \( f \) to \( f' \) is a morphism \( u : f \to f' \) in the \( \infty \)-category \( \text{Fun}(X, \mathcal{C}) \). A natural isomorphism from \( f \) to \( f' \) is a natural transformation \( u : f \to f' \) which is an isomorphism in the \( \infty \)-category \( \text{Fun}(X, \mathcal{C}) \) (Definition 1.3.6.1). We say that \( f \) and \( f' \) are naturally isomorphic if there exists a natural isomorphism from \( f \) to \( f' \).

**Remark 4.4.4.2.** In the situation of Definition 4.4.4.1 a natural transformation from \( f \) to \( f' \) is simply a homotopy from \( f \) to \( f' \), in the sense of Definition 3.1.4.2 that is, a map of simplicial sets \( h : \Delta^1 \times X \to \mathcal{C} \) satisfying \( h|_{\{0\} \times X} = f \) and \( h|_{\{1\} \times X} = f' \). However, the terminology of Definition 4.4.4.1 is intended to signal a shift in emphasis. We will generally reserve use of the term homotopy between diagrams \( f, f' : X \to \mathcal{C} \) for the case where \( \mathcal{C} \) is a Kan complex, and use the term natural transformation when \( \mathcal{C} \) is a more general \( \infty \)-category.

**Example 4.4.4.3.** Let \( \mathcal{C} \) be an ordinary category, and suppose we are given a pair of diagrams \( f, f' : X \to N_\bullet(\mathcal{C}) \). Then a natural transformation from \( f \) to \( f' \) can be identified with a collection of morphisms \( \{u_x : f(x) \to f'(x)\}_{x \in X} \) with the following property: for every edge \( e : x \to y \) of the simplicial set \( X \), the diagram

\[
\begin{array}{ccc}
  f(x) & \xrightarrow{u_x} & f'(x) \\
  \downarrow f(e) & & \downarrow f'(e) \\
  f(y) & \xrightarrow{u_y} & f'(y)
\end{array}
\]

commutes (in the category \( \mathcal{C} \)).

In particular, if \( \mathcal{C} \) and \( \mathcal{D} \) are ordinary categories and we are given a pair of functors \( f, f' : \mathcal{D} \to \mathcal{C} \), then giving a natural transformation from \( f \) to \( f' \) (in the sense of classical category theory) is equivalent to giving a natural transformation from \( N_\bullet(f) : N_\bullet(\mathcal{D}) \to N_\bullet(\mathcal{C}) \) to \( N_\bullet(f') : N_\bullet(\mathcal{D}) \to N_\bullet(\mathcal{C}) \).
Let $\mathcal{C}$ be an $\infty$-category and let $X$ be an arbitrary simplicial set. For every vertex $x \in X$, evaluation at $x$ determines a functor

$$\text{ev}_x : \text{Fun}(X, \mathcal{C}) \to \text{Fun}(\{x\}, \mathcal{C}) \simeq \mathcal{C}.$$ 

In particular, if $u : f \to f'$ is an isomorphism in the $\infty$-category $\text{Fun}(X, \mathcal{C})$, then $\text{ev}_x(u) : f(x) \to f'(x)$ is an isomorphism in the $\infty$-category $\mathcal{C}$. Our goal in this section is to prove the converse:

**Theorem 4.4.4.4.** Let $\mathcal{C}$ be an $\infty$-category, let $f, f' : X \to \mathcal{C}$ be diagrams in $\mathcal{C}$ indexed by a simplicial set $X$, and let $u : f \to f'$ be a natural transformation. Then $u$ is a natural isomorphism if and only if, for every vertex $x \in X$, the induced map $\text{ev}_x(u) : f(x) \to f'(x)$ is an isomorphism in the $\infty$-category $\mathcal{C}$.

**Remark 4.4.4.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories and suppose we are given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$, which restrict to functors between their cores $F^\simeq, G^\simeq : \mathcal{C}^\simeq \to \mathcal{D}^\simeq$ (see Remark 4.4.3.4). Let $u$ be a natural transformation from $F$ to $G$, which we identify with a map of simplicial sets $u : \Delta^1 \times \mathcal{C} \to \mathcal{D}$. If $u$ is a natural isomorphism, then it restricts to a map of simplicial sets $u_0 : \Delta^1 \times \mathcal{C}^\simeq \to \mathcal{D}^\simeq$, which we can regard as a homotopy from $F^\simeq$ to $G^\simeq$. In particular, if the functors $F$ and $G$ are naturally isomorphic, then the morphisms $F^\simeq$ and $G^\simeq$ are homotopic.

**Corollary 4.4.4.6.** Let $\mathcal{C}$ be an $\infty$-category. Then the functor

$$(\text{Set}_\Delta)^{\text{op}} \to \text{Set}_\Delta \quad X \mapsto \text{Fun}(X, \mathcal{C})^\simeq$$

preserves limits (that is, it carries colimits in the category of simplicial sets to limits of Kan complexes).

The proof of Theorem 4.4.4.4 will use the following combinatorial assertion:

**Lemma 4.4.4.7.** Let $m \geq 0$ and $n \geq 2$ be integers. Then there exists a sequence of simplicial subsets

$$X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(t) = \Delta^m \times \Delta^n$$

with the following properties:

1. The simplicial subset $X(0) \subseteq \Delta^m \times \Delta^n$ is the union of $\Delta^m \times \Lambda_0^0$ and $\partial \Delta^m \times \Delta^n$.
2. For each $0 < s \leq t$, there exist integers $q \geq 2$ and $p < q$ and a pushout diagram of simplicial sets

$$\begin{array}{ccc}
\Delta^q & \overset{\sigma}{\longrightarrow} & X(s) \\
\downarrow & & \downarrow \\
\Lambda^q_p & \longrightarrow & X(s-1)
\end{array}$$
Moreover, if \( p = 0 \), then the map \( \sigma : \Delta^q \to X(s) \subseteq \Delta^m \times \Delta^n \) satisfies \( \sigma(0) = (0,0) \) and \( \sigma(1) = (0,1) \).

**Proof.** Let \( \sigma \) be a nondegenerate \( q \)-simplex of the product \( \Delta^m \times \Delta^n \), given by a chain

\[
(i_0, j_0) < (i_1, j_1) < \cdots < (i_q, j_q).
\]

We will say that \( \sigma \) is **free** if the composite maps

\[
\Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \xrightarrow{\pi_m} \Delta^m \\
\Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \xrightarrow{\pi_n} \Delta^n
\]

are surjective and there exists an integer \( 0 \leq p < q \) such that \( (i_p, j_p) = (p, 0) \) and \( (i_{p+1}, j_{p+1}) = (p, 1) \). If this condition is satisfied, then the integer \( p \) is uniquely determined; we will refer to \( p \) as the **index** of \( \sigma \) and denote it by \( p(\sigma) \). We also denote the dimension \( q \) of \( \sigma \) by \( q(\sigma) \).

Let \( \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \) be an enumeration of the collection of all free simplices of the product \( \Delta^m \times \Delta^n \). Without loss of generality, we may assume that this enumeration satisfies the following pair of conditions:

- For \( 1 \leq s \leq s' \leq t \), we have \( q(\sigma_s) \leq q(\sigma_{s'}) \).
- If \( 1 \leq s \leq s' \leq t \) are integers satisfying \( q(\sigma_s) \leq q(\sigma_{s'}) \), then \( p(\sigma_s) \geq p(\sigma_{s'}) \).

Let \( X(0) \) denote the union \((\Delta^m \times \Lambda^0_0) \cup (\partial \Delta^m \times \Delta^n) \subseteq \Delta^m \times \Delta^n\). For \( 0 < s \leq t \), we let \( X(s) \) denote the smallest simplicial subset of \( \Delta^m \times \Delta^n \) which contains \( X(0) \) together with the simplices \( \{\sigma_1, \sigma_2, \ldots, \sigma_s\} \). We will show that the sequence

\[
X(0) \subset X(1) \subset \cdots \subset X(t)
\]

satisfies the requirements of Lemma 4.4.4.7.

We first claim that \( X(t) = \Delta^m \times \Delta^n \). Let \( \sigma \) be an arbitrary nondegenerate \( q \)-simplex of \( \Delta^m \times \Delta^n \), which we will identify with a sequence

\[
(i_0, j_0) < (i_1, j_1) < \cdots < (i_q, j_q)
\]

of elements of the partially ordered set \([m] \times [n]\). We wish to show that \( \sigma \) is contained in \( X(t) \). Without loss of generality, we may assume that the sequence \((i_0, i_1, \ldots, i_q)\) contains every element of the set \([m] = \{0 < 1 < \cdots < m\}\). (otherwise, \( \sigma \) is contained in the simplicial subset \( \partial \Delta^m \times \Delta^n \subseteq X(0) \subseteq X(t) \)). Similarly, we may assume that the sequence \((j_0, j_1, \ldots, j_q)\) contains every element of the set \([1 < 2 < \cdots < n]\) (otherwise, \( \sigma \) is contained in the simplicial subset \( \Delta^m \times \Lambda^n_0 \subseteq X(0) \subseteq X(t) \)). In particular, the sequence \( \sigma \) contains \((p, 1)\), for some integer \( 0 \leq p \leq n \). Let us assume that \( p \) is chosen as small as possible. In this case, there are two possibilities:
• The sequence $\sigma$ also contains the pair $(p, 0)$. In this case, $\sigma$ is a free simplex of $\Delta^m \times \Delta^n$, and therefore belongs to $X(t)$.

• The sequence $\sigma$ does not contain $(p, 0)$, and therefore has the form

$$(0, 0) < (1, 0) < \cdots < (p - 1, 0) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

We can then identify $\sigma$ with the $p$th face of the $(q+1)$-simplex $\sigma'$ given by the sequence

$$(0, 0) < (1, 0) < \cdots < (p - 1, 0) < (p, 0) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

The simplex $\sigma'$ is free and therefore belongs to $X(t)$, so that $\sigma$ belongs to $X(t)$ as well.

We now complete the proof by verifying requirement (2) of Lemma 4.4.4.7. Fix an integer $0 < s \leq t$ and let $\sigma = \sigma_s$ be the corresponding free simplex of $\Delta^m \times \Delta^n$. Let $q = q(\sigma)$ be the dimension of $\sigma$ and let $p = p(\sigma)$ be the index of $\sigma$, so that $0 \leq p < q$ and $\sigma$ has the form

$$(0, 0) < (1, 0) < \cdots < (p, 0) < (p, 1) < (i_{p+2}, j_{p+2}) < \cdots < (i_q, j_q).$$

By construction, the simplicial subset $X(s) \subseteq \Delta^m \times \Delta^n$ is the union of $X(s - 1)$ with the image of $\sigma$. We therefore have a pushout diagram of simplicial sets

$$\begin{array}{ccc}
K & \longrightarrow & X(s - 1) \\
\downarrow & & \downarrow \\
\Delta^q & \xrightarrow{\sigma} & X(s),
\end{array}$$

where $K \subseteq \Delta^q$ denotes the inverse image of $X(s - 1)$. We will complete the proof by showing that $K$ is equal to the horn $\Lambda^q_p \subseteq \Delta^q$.

We first show that the horn $\Lambda^q_p$ is contained in $K$. For this, it will suffice to show that for every integer $0 \leq p' \leq q$ satisfying $p' \neq p$, the face $\tau = d_{p'}(\sigma)$ is contained in $X(s - 1)$. We consider three cases:

• For $p' < p$, the simplex $\tau$ is given by the sequence

$$(0, 0) < \cdots < (p' - 1, 0) < (p' + 1, 0) < \cdots < (p, 0) < (p, 1) < \cdots < (i_q, j_q),$$

which is contained in the simplicial subset $\partial \Delta^m \times \Delta^n \subseteq X(0) \subseteq X(s - 1)$.

• For $p' = p + 1$, the simplex $\tau$ is given by the sequence

$$(0, 0) < (1, 0) < \cdots < (p, 0) < (i_{p+2}, j_{p+2}) < \cdots < (i_q, j_q).$$
If \( j_{p+2} \geq 2 \), then \( \tau \) belongs to the simplicial subset \( \Delta^m \times \Lambda^n_0 \subseteq X(0) \subseteq X(s-1) \). Otherwise, we must have \( (i_{p+2}, j_{p+2}) = (p+1, 1) \), so that \( \tau \) occurs as a face of the free simplex \( \sigma' \) given by the sequence

\[
(0, 0) < (1, 0) < \cdots < (p, 0) < (p+1, 0) < (p+1, 1) < \cdots < (i_q, j_q),
\]

which has dimension \( q \) and index \( p + 1 \). By construction, \( \sigma' \) belongs to the set \( \{\sigma_1, \sigma_2, \ldots, \sigma_{s-1}\} \), and is therefore contained in the simplicial subset \( X(s-1) \subseteq \Delta^m \times \Delta^n \).

- For \( p' > p + 1 \), the simplex \( \tau \) is given by the sequence

\[
(0, 0) < \cdots < (p, 0) < (p, 1) < \cdots < (i_{p'-1}, j_{p'-1}) < (i_{p'+1}, j_{p'+1}) < \cdots < (i_q, j_q).
\]

It follows that \( \tau \) is either contained in the simplicial subset \( X(0) = (\Delta^m \times \Lambda^n_0) \cup (\partial \Delta^m \times \Delta^n) \) or that it is a free simplex of \( \Delta^m \times \Delta^n \) having dimension \( q - 1 \). In the latter case, \( \tau \) must belong to the set \( \{\sigma_1, \ldots, \sigma_{s-1}\} \), and is therefore contained in the simplicial subset \( X(s-1) \subseteq \Delta^m \times \Delta^n \).

To show that the inclusion \( \Lambda^q_p \subseteq K \) is an equality, it will suffice to show that \( K \) does not contain the \( p \)-th face of \( \Delta^q \). Let \( \tau = d_p(\sigma) \) be the \( p \)-th face of \( \sigma \), given by the sequence

\[
(0, 0) < (1, 0) < \cdots < (p-1, 0) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).
\]

We wish to show that \( \tau \) is not contained in \( X(s-1) \). Assume otherwise. Since \( \tau \) is not contained in \( X(0) \), we conclude that \( \tau \) is contained in some free simplex \( \sigma' \in \{\sigma_1, \sigma_2, \ldots, \sigma_{s-1}\} \).

Note that \( \tau \neq \sigma' \) (since \( \tau \) is not free), so we have inequalities

\[
q - 1 = q(\tau) < q(\sigma') \leq q(\sigma) = q.
\]

It follows that \( \sigma' \) is a free \( q \)-simplex of \( \Delta^m \times \Delta^n \) which contains \( \tau \) and is not equal to \( \sigma \), and is therefore necessarily given by the sequence

\[
(0, 0) < (1, 0) < \cdots < (p-1, 0) < (p-1, 1) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).
\]

We therefore have \( p(\sigma') = p-1 < p = p(\sigma) \), which contradicts our assumption regarding the choice of enumeration \( \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \). \( \square \)

**Lemma 4.4.4.8.** Let \( \mathcal{C} \) be an \( \infty \)-category, let \( A \subseteq B \) be simplicial sets, let \( n \geq 2 \) be an integer, and suppose we are given a morphism of simplicial sets

\[
F_0 : (A \times \Delta^n) \coprod_{(A \times \Lambda^n_0)} (B \times \Lambda^n_0) \to \mathcal{C}.
\]
Assume that, for every vertex \( x \in B \), the composite map
\[
\Delta^1 \cong \{x\} \times \text{N}_\bullet(\{0,1\}) \hookrightarrow B \times \Lambda^n_0 \xrightarrow{F_0} C
\]
corresponds to an isomorphism in \( C \). Then \( F_0 \) can be extended to a morphism of simplicial sets \( F : B \times \Delta^n \to C \).

**Proof.** Let \( P \) denote the collection of all pairs \((K, F_K)\), where \( K \subseteq B \) is a simplicial subset containing \( A \) and \( F_K : K \times \Delta^n \to C \) is a morphism of simplicial sets satisfying
\[
F_K|_{A \times \Delta^n} = F_0|_{A \times \Delta^n} \quad F_K|_{K \times \Lambda^n_0} = F_0|_{K \times \Lambda^n_0}.
\]
We regard \( P \) as partially ordered set, where \((K, F_K) \leq (K', F_K')\) if \( K \subseteq K' \) and \( F_K = F_K'|_{K' \times \Delta^n} \). The partially ordered set \( P \) satisfies the hypotheses of Zorn’s lemma, and therefore has a maximal element \((K_{\text{max}}, F_{K_{\text{max}}})\). We will complete the proof by showing that \( K_{\text{max}} = B \). Assume otherwise. Then there exists some nondegenerate \( m \)-simplex \( \tau : \Delta^m \to B \) whose image is not contained in \( K_{\text{max}} \). Choosing \( m \) as small as possible, we can assume that \( \tau \) carries the boundary \( \partial \Delta^m \) into \( B \). Let \( K' \subseteq B \) be the union of \( K_{\text{max}} \) with the image of \( \tau \), so that we have a pushout diagram of simplicial sets
\[
\begin{array}{ccc}
\partial \Delta^m & \to & K_{\text{max}} \\
\downarrow & & \downarrow \\
\Delta^m & \to & K'.
\end{array}
\]

We will complete the proof by showing that the lifting problem
\[
(K_{\text{max}} \times \Delta^n) \coprod_{(K_{\text{max}} \times \Lambda^n_0)} (K' \times \Lambda^n_0) \xrightarrow{(F_K, F_0|_{K' \times \Lambda^n_0})} C
\]

admit a solution (contradicting the maximality of the pair \((K_{\text{max}}, F_{K_{\text{max}}})\)). To prove this, we can replace the inclusion \( K_{\text{max}} \hookrightarrow K' \) by \( \partial \Delta^m \hookrightarrow \Delta^m \). We are therefore reduced to proving Lemma 4.4.4.8 in the special case where \( B = \Delta^m \) is a simplex and \( A = \partial \Delta^m \) is its boundary.

Choose a sequence of simplicial subsets
\[
X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(t) = \Delta^m \times \Delta^n
\]
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satisfying the requirements of Lemma 4.4.4.7 so that $F_0$ can be identified with a morphism $X(0) \to \mathcal{C}$. We will show that, for $0 \leq s \leq t$, there exists a morphism of simplicial sets $F_s : X(s) \to \mathcal{C}$ extending $F_0$ (taking $s = t$, this will complete the proof of Lemma 4.4.4.8). We proceed by induction on $s$, the case $s = 0$ being vacuous. Assume that $s > 0$ and that we have already constructed a morphism $F_{s-1} : X(s-1) \to \mathcal{C}$ satisfying $F_{s-1}|_{X(0)} = F_0$. By construction, there exists integers $q \geq 2$, $p > 0$, and a pushout diagram of simplicial sets

Moreover, in the special case $p = 0$, we can assume that $\sigma(0) = (0, 0)$ and $\sigma(1) = (0, 1)$, so that the composite map

$$ \Delta^1 \simeq N_\bullet(\{0 \leq 1\}) \hookrightarrow \Lambda_q^p \xrightarrow{\sigma_0} X(s-1) \xrightarrow{F_{s-1}} \mathcal{C} $$

corresponds to an isomorphism in $\mathcal{C}$. To construct the desired extension $F_s : X(s) \to \mathcal{C}$, it will suffice to show that the composite map

$$ \Lambda_q^p \xrightarrow{\sigma_0} X(s-1) \xrightarrow{F_{s-1}} \mathcal{C} $$

can be extended to a $q$-simplex of $\mathcal{C}$. In the case $0 < p < q$, the existence of the desired extension follows from our assumption that $\mathcal{C}$ is an $\infty$-category. In the special case $p = 0$, it follows instead from the characterization of isomorphisms given by Theorem 4.4.2.5.

Proof of Theorem 4.4.4.4. Let $\mathcal{C}$ be an $\infty$-category, let $X$ be a simplicial set, and let $u : f \to f'$ be a natural transformation between diagrams $f, f' : X \to \mathcal{C}$ having the property that, for every vertex $x \in X$, the induced map $u_x : f(x) \to f'(x)$ is an isomorphism in the $\infty$-category $\mathcal{C}$. We wish to show that $u$ is an isomorphism in the functor $\infty$-category $\text{Fun}(X, \mathcal{C})$. We will prove this by verifying the criterion of Theorem 4.4.2.5. Let $n \geq 2$ be an integer, and suppose we are given a morphism of simplicial sets $\sigma_0 : \Lambda^n_0 \to \text{Fun}(X, \mathcal{C})$ for which the composite map

$$ \Delta^1 \simeq N_\bullet(\{0, 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} \text{Fun}(X, \mathcal{C}) $$

is equal to $u$. We wish to show that $\sigma_0$ can be extended to an $n$-simplex $\sigma$ of the diagram $\infty$-category $\text{Fun}(X, \mathcal{C})$. Let us identify $\sigma_0$ with a morphism of simplicial sets $\rho_0 : X \times \Lambda^n_0 \to \mathcal{C}$; we wish to show that $\rho_0$ can be extended to a morphism $\rho : X \times \Delta^n \to \mathcal{C}$. This follows by applying Lemma 4.4.4.8 in the special case $B = X$ and $A = \emptyset$. \qed
4.4.5 Exponentiation for Isofibrations

We now show that the formation of ∞-categories of functors behaves well with respect to isofibrations.

Proposition 4.4.5.1. Let \( F : C \to D \) be an isofibration of ∞-categories, let \( B \) be a simplicial set, and let \( A \subseteq B \) be a simplicial subset. Then the restriction map

\[
F' : \text{Fun}(B, C) \to \text{Fun}(A, C) \times_{\text{Fun}(A, D)} \text{Fun}(B, D)
\]

is an isofibration of ∞-categories.

Remark 4.4.5.2. Proposition 4.4.5.1 generalizes to isofibrations between arbitrary simplicial sets: see Proposition 4.5.7.10.

We will give the proof of Proposition 4.4.5.1 at the end of this section.

Corollary 4.4.5.3. Let \( C \) be an ∞-category, let \( B \) be a simplicial set, and let \( A \subseteq B \) be a simplicial subset. Then the restriction map \( \text{Fun}(B, C) \to \text{Fun}(A, C) \) is an isofibration.

Proof. Apply Proposition 4.4.5.1 in the special case \( D = \Delta^0 \).

Corollary 4.4.5.4. Let \( C \) be an ∞-category, let \( B \) be a simplicial set, and let \( A \subseteq B \) be a simplicial subset. Then the restriction functor \( \text{Fun}(B, C) \to \text{Fun}(A, C) \) induces a Kan fibration of simplicial sets \( \text{Fun}(B, C)^\simeq \to \text{Fun}(A, C)^\simeq \).

Proof. Combine Corollary 4.4.5.3 with Proposition 4.4.3.6.

Corollary 4.4.5.5. Let \( q : C \to D \) be an isofibration of ∞-categories. For every simplicial set \( B \), the induced map \( \text{Fun}(B, C) \to \text{Fun}(B, D) \) is also an isofibration of ∞-categories.

Proof. Apply Proposition 4.4.5.1 in the special case \( A = \emptyset \).

The main ingredient needed in our proof of Proposition 4.4.5.1 is the following isomorphism extension result:

Proposition 4.4.5.6. Let \( F : C \to D \) be an isofibration of ∞-categories, let \( B \) be a simplicial set, let \( A \subseteq B \) be a simplicial subset, and suppose we are given a lifting problem

\[
(\Delta^1 \times A) \coprod_{(1) \times A} \{1 \} \times B \quad \xrightarrow{h_0} \quad C \quad \xrightarrow{h} \quad D
\]

with the following properties:

\[
\Delta^1 \times B \quad \xrightarrow{\overline{h}} \quad D
\]
For every vertex \( x \in A \), the edge \( \Delta^1 \simeq \Delta^1 \times \{x\} \xrightarrow{h_0} C \) is an isomorphism in \( C \).

For every vertex \( b \in B \), the edge \( \Delta^1 \simeq \Delta^1 \times \{x\} \xrightarrow{h} D \) is an isomorphism in \( D \).

Then \( h_0 \) can be extended to a diagram \( h : \Delta^1 \times B \to C \) satisfying \( \overline{h} = F \circ h \). Moreover, we can arrange that for every vertex \( x \in B \), the edge \( \Delta^1 \simeq \Delta^1 \times \{x\} \xrightarrow{h} C \) is an isomorphism in the \( \infty \)-category \( C \) (so that \( h \) can be regarded as an isomorphism in the diagram \( \infty \)-category \( \text{Fun}(B, C) \), by virtue of Theorem 4.4.4.4).

**Proof.** We proceed as in the proof of Lemma 4.4.4.8 with some minor modifications. Let \( P \) denote the collection of all pairs \((K, h_K)\), where \( K \subseteq B \) is a simplicial subset containing \( A \) and \( h_K : \Delta^1 \times K \to C \) is a morphism of simplicial sets satisfying

\[
h_K|_{\Delta^1 \times A} = h_0|_{\Delta^1 \times A} \quad h_K|_{\{1\} \times K} = h_0|_{\{1\} \times K}
\]

and having the property that, for each vertex \( x \in K \), the edge \( h_K|_{\Delta^1 \times \{x\}} \) is an isomorphism in the \( \infty \)-category \( C \). We regard \( P \) as partially ordered set, where \((K, h_K) \leq (K', h_{K'})\) if \( K \subseteq K' \) and \( h_K = h_{K'}|_{\Delta^1 \times K'} \). The partially ordered set \( P \) satisfies the hypotheses of Zorn’s lemma, and therefore has a maximal element \((K_{\text{max}}, h_{K_{\text{max}}})\). We will complete the proof by showing that \( K_{\text{max}} = B \). Assume otherwise. Then there exists some nondegenerate \( n \)-simplex \( \tau : \Delta^n \to B \) whose image is not contained in \( K_{\text{max}} \). Choosing \( n \) as small as possible, we can assume that \( \tau \) carries the boundary \( \partial \Delta^n \) into \( B \). Let \( K' \subseteq B \) be the union of \( K_{\text{max}} \) with the image of \( \tau \), so that we have a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\partial \Delta^n & \to & K_{\text{max}} \\
\downarrow & & \downarrow \\
\Delta^n & \to & K'.
\end{array}
\]

We will complete the proof by showing that the lifting problem

\[
(\Delta^1 \times K_{\text{max}}) \coprod_{\{1\} \times K_{\text{max}}} (\{1\} \times K') \xrightarrow{(h_K, h_0|_{\{1\} \times K'})} C
\]

\[
\Delta^1 \times K' \xrightarrow{} \Delta^0
\]
admit a solution (contradicting the maximality of the pair \((K_{\text{max}}, h_{K_{\text{max}}})\)). To prove this, we can replace the inclusion \(K_{\text{max}} \hookrightarrow K\) by \(\partial \Delta^n \hookrightarrow \Delta^n\). We are therefore reduced to proving Lemma 4.4.4.8 in the special case where \(B = \Delta^n\) is a simplex and \(A = \partial \Delta^n\) is its boundary.

If \(n = 0\), then the existence of the desired extension follows from our assumption that \(F\) is an isofibration. We may therefore assume without loss of generality that \(n > 0\). Let

\[
(\Delta^1 \times \partial \Delta^n) \cup (\{1\} \times \Delta^n) = X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(n+1) = \Delta^1 \times \Delta^n
\]

be the sequence of simplicial subsets appearing in the proof of Lemma 3.1.2.9 so that \(h_0\) can be identified with morphism of simplicial sets from \(X(0)\) to \(C\). We will show that, for \(0 \leq i \leq n+1\), there exists a morphism of simplicial sets \(h_i : X(i) \to C\) satisfying \(h_i|_{X(0)} = h_0\) and \(F \circ h_i = h_i|_{X(i)}\) (taking \(i = n+1\), this will complete the proof of Proposition 4.4.5.6).

We proceed by induction on \(i\), the case \(i = 0\) being vacuous. Assume that \(i > 0\) and that we have already constructed a morphism \(h_{i-1} : X(i-1) \to C\) satisfying \(h_{i-1}|_{X(0)} = h_0\) and \(F \circ h_{i-1} = h_{i-1}|_{X(i-1)}\). By virtue of Lemma 3.1.2.9, we have a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\Lambda_i^{n+1} & \xrightarrow{\sigma_0} & X(i-1) \\
\downarrow & & \downarrow \\
\Delta^{n+1} & \xrightarrow{\sigma} & X(i).
\end{array}
\]

Consequently, to prove the existence of \(h_i\), it suffices to solve the lifting problem

\[
\begin{array}{ccc}
\Lambda_i^{n+1} & \xrightarrow{h_{i-1}\circ \sigma_0} & C \\
\downarrow & & \downarrow F \\
\Delta^{n+1} & \xrightarrow{h_0 \circ \sigma} & D.
\end{array}
\]

For \(0 < i < n+1\), the existence of the desired solution follows from our assumption that \(F\) is an inner fibration. In the case \(i = n+1\), the existence follows from Proposition 4.4.2.10 since the map \(\sigma : \Delta^{n+1} \to \Delta^1 \times \Delta^n\) carries the final edge \(N_{\bullet}(\{n < n+1\}) \subseteq \Delta^{n+1}\) to the edge \(\Delta^1 \times \{n\} \subseteq \Delta^1 \times \Delta^n\), which \(h_0\) carries to an isomorphism in the \(\infty\)-category \(C\).

**Proof of Proposition 4.4.5.7** Let \(F : C \to D\) be an isofibration of \(\infty\)-categories, let \(B\) be a simplicial set, and let \(A \subseteq B\) be a simplicial subset. We wish to show that the restriction map

\[
F' : \text{Fun}(B,C) \to \text{Fun}(A,C) \times_{\text{Fun}(A,D)} \text{Fun}(B,D)
\]
is an isofibration of \(\infty\)-categories. We first note that the projection map

\[
\text{Fun}(A, C) \times_{\text{Fun}(A,D)} \text{Fun}(B, D) \to \text{Fun}(A, C)
\]

is a pullback of the inner fibration \(\text{Fun}(B, D) \to \text{Fun}(A, D)\) (see Corollary 4.1.4.2). Since \(\text{Fun}(A, C)\) is an \(\infty\)-category (Theorem 1.4.3.7), it follows that \(\text{Fun}(A, C) \times_{\text{Fun}(A,D)} \text{Fun}(B, D)\) is also an \(\infty\)-category (Remark 4.1.1.9). It follows from Proposition 4.1.4.1 that \(F'\) is an inner fibration. It will therefore suffice to show that, for every object \(Y \in \text{Fun}(B, C)\) and every isomorphism \(u : X \to F'(Y)\) in the \(\infty\)-category \(\text{Fun}(A, C) \times_{\text{Fun}(A,D)} \text{Fun}(B, D)\) can be lifted to an isomorphism \(\pi : X \to Y\) in the \(\infty\)-category \(\text{Fun}(B, C)\). This follows immediately from Proposition 4.4.5.6.

\[\square\]

## 4.5 Equivalence

Let \(C\) and \(D\) be categories. We say that a functor \(F : C \to D\) is an \textit{isomorphism of categories} if there exists a functor \(G : D \to C\) satisfying the identities \(G \circ F = \text{id}_C\) and \(F \circ G = \text{id}_D\). This condition is somewhat unnatural, since it refers to equalities between \textit{objects} of the functor categories \(\text{Fun}(C, C)\) and \(\text{Fun}(D, D)\). For most purposes, it is better to adopt a looser definition. We say that a functor \(F : C \to D\) is an \textit{equivalence of categories} if there exists a functor \(G : D \to C\) for which the composite functors \(G \circ F\) and \(F \circ G\) are isomorphic to the identity functors \(\text{id}_C\) and \(\text{id}_D\), respectively. In category theory, the notion of equivalence between categories plays a much more central role than the notion of isomorphism between categories, and virtually all important concepts are invariant under equivalence.

In §4.5.1, we extend the notion of equivalence to the \(\infty\)-categorical setting. If \(C\) and \(D\) are \(\infty\)-categories, we will say that a functor \(F : C \to D\) is an \textit{equivalence of \(\infty\)-categories} if there exists a functor \(G : D \to C\) for which the composite maps \(G \circ F\) and \(F \circ G\) are isomorphic to \(\text{id}_C\) and \(\text{id}_D\), when viewed as objects of the \(\infty\)-categories \(\text{Fun}(C,C)\) and \(\text{Fun}(D,D)\), respectively (Definition 4.5.1.9). Phrased differently, a functor \(F\) is an equivalence of \(\infty\)-categories if it is an isomorphism when viewed as an object of the category \(\text{hCat}_{\infty}\), whose objects are \(\infty\)-categories and whose morphisms are isomorphism classes of functors (Construction 4.5.1.1).

In the study of \(\infty\)-categories, it can be technically convenient to work with simplicial sets which do \textit{not} satisfy the weak Kan extension condition. For example, it is often harmless to replace the standard \(n\)-simplex \(\Delta^n\) by its spine \(\text{Spine}[n] \subseteq \Delta^n\): for any \(\infty\)-category \(C\), the restriction map \(\text{Fun}(\Delta^n, C) \to \text{Fun}(\text{Spine}[n], C)\) is a trivial Kan fibration (see Example 1.4.7.7). In §4.5.2 we formalize this observation by introducing the notion of \textit{categorical equivalence} between simplicial sets. By definition, a morphism of simplicial sets \(f : X \to Y\) is a categorical equivalence if, for every \(\infty\)-category \(C\), the induced functor of \(\infty\)-categories
Fun(Y, C) \to Fun(X, C) is bijective on isomorphism classes of objects (Definition 4.5.2.1). If X and Y are \infty-categories, this reduces to the condition that f is an equivalence of \infty-categories in the sense of \ref{sec:1.0} (Example 4.5.2.3). However, we will encounter many other examples of categorical equivalences between simplicial sets which are not \infty-categories: for example, every inner anodyne morphism of simplicial sets is a categorical equivalence (Corollary 4.5.2.11).

Recall that every \infty-category C contains a largest Kan complex, which we denote by C^\simeq and refer to as the core of C (Construction 4.4.3.1). The construction C \mapsto C^\simeq can often be used to reformulate questions about \infty-categories in terms of the classical homotopy theory of Kan complexes. In \ref{sec:6.0} we illustrate this principle by showing that a functor of \infty-categories F : C \to D is an equivalence if and only if, for every simplicial set K, the induced map Fun(K, C)^\simeq \xrightarrow{F^\simeq} Fun(K, D)^\simeq is a homotopy equivalence of Kan complexes. Moreover, it suffices this condition in the special case K = \Delta^1 (Theorem 4.5.4.1). As an application, we show that the collection of categorical equivalences is stable under the formation of filtered colimits (Corollary 4.5.4.2), and under pullback along isofibrations of \infty-categories (Corollary 4.5.4.3).

In \ref{sec:0.0}, we study an important class of categorical equivalences the theory of joins developed in \ref{sec:0.0}. Recall that, if C and D are categories, then the join C \star D is isomorphic to the iterated pushout
\[
\begin{array}{cccc}
C \coprod_{(C \times \{0\} \times D)} (C \times [1] \times D) \coprod_{(C \times \{1\} \times D)} D,
\end{array}
\]
formed in the category Cat of (small) categories (Remark 4.3.2.14). In the setting of \infty-categories, the situation is more subtle (Warning 4.3.3.28). For any simplicial sets X and Y, there is a natural comparison map
\[
c_{X,Y} : X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y \to X \star Y
\]
(Notation 4.5.5.3), which is almost never an isomorphism. Nevertheless, we show in \ref{sec:0.0} that c_{X,Y} is always a categorical equivalence of simplicial sets (Theorem 4.5.5.7). To carry out the proof, it will be useful to employ to use the language of categorical pushout diagrams, which we explain in \ref{sec:0.0}.

Let F : C \to D be a functor of \infty-categories. Recall that F is an inner fibration if and only if every lifting problem
\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & & \downarrow \\
B & \rightarrow & D
\end{array}
\]
(4.3)
admits a solution, provided that the morphism \( i : A \hookrightarrow B \) is inner anodyne (Proposition 4.1.3.1). In §4.5.6, we show that \( F \) is an isofibration if and only if the following stronger condition: the lifting problem (4.3) admits a solution whenever the map \( i : A \hookrightarrow B \) is both a monomorphism and a categorical equivalence (Proposition 4.5.6.1). In §4.5.7, we apply this result to extend the notion of isofibration to simplicial sets which are not necessarily \( \infty \)-categories (Definition 4.5.7.1).

### 4.5.1 Equivalences of \( \infty \)-Categories

The collection of \( \infty \)-categories can be organized into a category, in which the morphisms are given by isomorphism classes of functors.

**Construction 4.5.1.1** (The Homotopy Category of \( \infty \)-Categories). We define a category \( \text{hCat}_\infty \) as follows:

- The objects of \( \text{hCat}_\infty \) are \( \infty \)-categories.

- If \( \mathcal{C} \) and \( \mathcal{D} \) are \( \infty \)-categories, then \( \text{Hom}_{\text{hCat}_\infty}(\mathcal{C}, \mathcal{D}) = \pi_0(\text{Fun}(\mathcal{C}, \mathcal{D})) \) is the set of isomorphism classes of objects of the \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) (or, equivalently, of the homotopy category \( \text{hFun}(\mathcal{C}, \mathcal{D}) \)). If \( F : \mathcal{C} \to \mathcal{D} \) is a functor, we denote its isomorphism class by \([F]\) \( \in \text{Hom}_{\text{hCat}_\infty}(\mathcal{C}, \mathcal{D}) \).

- If \( \mathcal{C}, \mathcal{D}, \) and \( \mathcal{E} \) are \( \infty \)-categories, then the composition law

\[
\circ : \text{Hom}_{\text{hCat}_\infty}(\mathcal{D}, \mathcal{E}) \times \text{Hom}_{\text{hCat}_\infty}(\mathcal{C}, \mathcal{D}) \to \text{Hom}_{\text{hCat}_\infty}(\mathcal{C}, \mathcal{E})
\]

is characterized by the formula \([G] \circ [F] = [G \circ F]\).

We will refer to \( \text{hCat}_\infty \) as the *homotopy category of \( \infty \)-categories*.

**Remark 4.5.1.2.** We will later study a refinement of Construction 4.5.1.1. The collection of (small) \( \infty \)-categories can itself be organized into a (large) \( \infty \)-category \( \text{Cat}_\infty \), whose homotopy category can be identified with the ordinary category \( \text{hCat}_\infty \) of Construction 4.5.1.1. We refer the reader to Chapter [?] for more details.

**Remark 4.5.1.3.** Let \( \text{hKan} \) denote the homotopy category of Kan complexes (Construction 3.1.4.10). Then we can regard \( \text{hKan} \) as a full subcategory of the \( \infty \)-category \( \text{hCat}_\infty \) (Construction 4.5.1.1), spanned by those \( \infty \)-categories which are Kan complexes. This follows from the observation that if \( Y \) is a Kan complex, then a pair of morphisms \( f, g : X \to Y \) are isomorphic as objects of the \( \infty \)-category \( \text{Fun}(X, Y) \) if and only if they are homotopic (Proposition 3.1.4.4).

The inclusion functor \( \text{hKan} \hookrightarrow \text{hCat}_\infty \) has both left and right adjoints.
Proposition 4.5.1.4. Let $C$ be an $\infty$-category and let $C^\simeq$ denote its core (Construction 4.4.3.1). For every Kan complex $X$, composition with the inclusion map $\iota : C^\simeq \hookrightarrow C$ induces a bijection
\[
\text{Hom}_{h\text{Kan}}(X, C^\simeq) = \text{Hom}_{h\text{Cat}_\infty}(X, C^\simeq) \to \text{Hom}_{h\text{Cat}_\infty}(X, C).
\]

Proof. By virtue of Proposition 4.4.3.16, postcomposition with $\iota$ induces an isomorphism of Kan complexes $\text{Fun}(X, C^\simeq) \to \text{Fun}(X, C)^\simeq$. Proposition 4.5.1.4 follows by passing to connected components. \qed

Corollary 4.5.1.5. The inclusion functor $h\text{Kan} \hookrightarrow h\text{Cat}_\infty$ of Remark 4.5.1.3 admits a right adjoint, given on objects by the construction $C \mapsto C^\simeq$.

Remark 4.5.1.6. The right adjoint $h\text{Cat}_\infty \to h\text{Kan}$ of Corollary 4.5.1.5 can be described more explicitly as follows:

- To each $\infty$-category $C$, it associates the Kan complex $C^\simeq$ of Construction 4.4.3.1.
- To each morphism $[F] : C \to D$ in the homotopy category $h\text{Cat}_\infty$ (given by the isomorphism class of a functor $F : C \to D$), it associates the homotopy class $[F^\simeq]$ of the underlying map of cores $F^\simeq = F|_{C^\simeq}$ (note that the homotopy class of $F^\simeq$ depends only on the isomorphism class of $F$, by virtue of Remark 4.4.4.5).

Proposition 4.5.1.7. The inclusion functor $h\text{Kan} \hookrightarrow h\text{Cat}_\infty$ of Remark 4.5.1.3 admits a left adjoint.

Proof. Let $C$ be an $\infty$-category. We wish to show that there exists a Kan complex $X$ and a morphism $u : C \to X$ with the following property: for every Kan complex $Y$, precomposition with $u$ induces a bijection
\[
\text{Hom}_{h\text{Kan}}(X, Y) = \text{Hom}_{h\text{Cat}_\infty}(X, Y) \to \text{Hom}_{h\text{Cat}_\infty}(C, Y).
\]

Unwinding the definitions, we see that this is a reformulation of the requirement that $u$ is a weak homotopy equivalence of simplicial sets. The existence of $u$ now follows from Corollary 3.1.6.2. \qed

Remark 4.5.1.8. The left adjoint $h\text{Cat}_\infty \to h\text{Kan}$ of Proposition 4.5.1.7 admits a category-theoretic interpretation: it carries an $\infty$-category $C$ to the localization $C[W^{-1}]$ obtained by formally inverting the collection $W$ of all morphisms in $C$ (see Proposition 5.3.1.18).

Definition 4.5.1.9. Let $C$ and $D$ be $\infty$-categories. We will say that a functor $F : C \to D$ is an equivalence of $\infty$-categories if the homotopy class $[F]$ is an isomorphism in the homotopy category $h\text{Cat}_\infty$ of Construction 4.5.1.1.
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**Remark 4.5.1.10.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor between \( \infty \)-categories. Then \( F \) is an equivalence of \( \infty \)-categories if and only if there exists a functor \( G : \mathcal{D} \to \mathcal{C} \) such that \( G \circ F \) is isomorphic to the identity functor \( \text{id}_\mathcal{C} \) (as an object of the functor \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{C}) \)) and \( F \circ G \) is isomorphic to the identity functor \( \text{id}_\mathcal{D} \) (as an object of the functor \( \infty \)-category \( \text{Fun}(\mathcal{D}, \mathcal{D}) \)). If these conditions are satisfied, then the functor \( G \) is also an equivalence of \( \infty \)-categories, and its isomorphism class (as an object of the \( \infty \)-category \( \text{Fun}(\mathcal{D}, \mathcal{C}) \)) depends only on the isomorphism class of \( F \) (as an object of the \( \infty \)-category \( \text{Fun}(\mathcal{C}, \mathcal{D}) \)). We will emphasize this dependence by referring to \( G \) as an inverse to the functor \( F \).

**Example 4.5.1.11.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be an isomorphism of simplicial sets. Then \( F \) is an equivalence of \( \infty \)-categories. In particular, for every \( \infty \)-category \( \mathcal{C} \), the identity functor \( \text{id}_\mathcal{C} \) is an equivalence of \( \infty \)-categories.

**Example 4.5.1.12.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor between categories. Then the induced map \( N_\bullet(F) : N_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{D}) \) is an equivalence of \( \infty \)-categories if and only if \( F \) is an equivalence of categories.

**Example 4.5.1.13.** Let \( f : X \to Y \) be a morphism of Kan complexes. Then \( f \) is a homotopy equivalence if and only if it is an equivalence of \( \infty \)-categories (see Remark 4.5.1.3). In this case, a morphism \( g : Y \to X \) is an inverse to \( f \) (in the sense of Remark 4.5.1.10) if and only if it is a homotopy inverse to \( f \) (in the sense of Definition 3.1.5.1).

**Warning 4.5.1.14.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \infty \)-categories, and let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. If \( F \) is an equivalence of \( \infty \)-categories (in the sense of Definition 4.5.1.9), then it is a homotopy equivalence of simplicial sets (in the sense of Definition 3.1.5.1). More precisely, if \( G : \mathcal{D} \to \mathcal{C} \) is an inverse to the functor \( F \) (in the sense of Remark 4.5.1.10), then \( G \) is also a homotopy inverse to \( F \) (in the sense of Definition 3.1.5.1). Beware that the converse assertion is false in general. For example, the projection map \( \Delta^1 \to \Delta^0 \) is a homotopy equivalence of simplicial sets (with homotopy inverse given by the inclusion \( \Delta^0 \simeq \{0\} \hookrightarrow \Delta^1 \)), but not an equivalence of \( \infty \)-categories.

**Remark 4.5.1.15.** Let \( X \) be an arbitrary simplicial set. Then the construction \( \mathcal{C} \mapsto \text{Fun}(X, \mathcal{C}) \) determines a functor from the homotopy category \( \text{hCat}_\infty \) to itself. In particular, if \( F : \mathcal{C} \to \mathcal{D} \) be an equivalence of \( \infty \)-categories, then the induced map \( \text{Fun}(X, \mathcal{C}) \to \text{Fun}(X, \mathcal{D}) \) is also an equivalence of \( \infty \)-categories.

**Remark 4.5.1.16 (Two-out-of-Six).** Let \( F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}, \) and \( H : \mathcal{C} \to \mathcal{D} \) be functors between \( \infty \)-categories. If \( G \circ F \) and \( H \circ G \) are equivalences of \( \infty \)-categories, then \( F, G, \) and \( H \) are equivalences of \( \infty \)-categories.

**Remark 4.5.1.17 (Two-out-of-Three).** Let \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{E} \) be functors between \( \infty \)-categories. If any two of the functors \( F, G, \) and \( G \circ F \) is an equivalence of \( \infty \)-categories, then so is the third. In particular, the collection of equivalences is closed under composition.
Remark 4.5.1.18. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. If $F$ is an equivalence of $\infty$-categories, then the induced map of cores $F^\simeq: \mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is a homotopy equivalence of Kan complexes. This follows from Corollary 4.5.1.5 (and Remark 4.5.1.6): if the isomorphism class $[F]$ is an invertible morphism in the homotopy category $\text{hCat}_\infty$, then the homotopy class $[F^\simeq]$ is an invertible morphism in the homotopy category $\text{hKan}$.

Remark 4.5.1.19. Let $F: \mathcal{C} \to \mathcal{D}$ be an equivalence of $\infty$-categories. Then the induced functor $\text{h}F: \text{hC} \to \text{hD}$ is an equivalence of ordinary categories. In particular, the functor $F$ carries isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$.

Remark 4.5.1.20. Let $F: \mathcal{C} \to \mathcal{D}$ be an equivalence of $\infty$-categories. If $\mathcal{D}$ is a Kan complex, then $\mathcal{C}$ is a Kan complex. To prove this, it suffices to show that every morphism $u: X \to Y$ in $\mathcal{C}$ is an isomorphism (Proposition 4.4.2.1). By virtue of Remark 4.5.1.19, this is equivalent to the assertion that $F(u): F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$, which is automatic when $\mathcal{D}$ is a Kan complex (Proposition 1.3.6.11). Similarly, if $\mathcal{C}$ is a Kan complex, then $\mathcal{D}$ is a Kan complex (this follows by applying the same argument to an inverse equivalence $\mathcal{D} \to \mathcal{C}$).

4.5.2 Categorical Equivalences of Simplicial Sets

Recall that a morphism of simplicial sets $f: X \to Y$ is a weak homotopy equivalence if, for every Kan complex $Z$, precomposition with $f$ induces a bijection $\pi_0(\text{Fun}(Y, Z)) \to \pi_0(\text{Fun}(X, Z))$ (Definition 3.1.5.11). If this condition is satisfied, then one should regard $X$ and $Y$ as indistinguishable from the perspective of classical homotopy theory. However, from the $\infty$-categorical perspective, the relation of weak homotopy equivalence is somewhat too coarse: it is possible for a functor of $\infty$-categories $F: \mathcal{C} \to \mathcal{D}$ to be a weak homotopy equivalence (or even a homotopy equivalence) without being an equivalence of $\infty$-categories (Warning 4.5.1.14). For this reason, it will be convenient to introduce a finer notion of equivalence.

Definition 4.5.2.1. Let $f: X \to Y$ be a morphism of simplicial sets. We say that $f$ is a categorical equivalence if, for every $\infty$-category $\mathcal{C}$, the induced functor $\text{Fun}(Y, \mathcal{C}) \to \text{Fun}(X, \mathcal{C})$ induces a bijection on isomorphism classes $\pi_0(\text{Fun}(Y, \mathcal{C})^\simeq) \to \pi_0(\text{Fun}(X, \mathcal{C})^\simeq)$.

Example 4.5.2.2. Every isomorphism of simplicial sets is a categorical equivalence.

Example 4.5.2.3. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor of $\infty$-categories. Then $F$ is a categorical equivalence (in the sense of Definition 4.5.2.1) if and only if it is an equivalence of $\infty$-categories (in the sense of Definition 4.5.1.9). Both conditions are equivalent to the assertion that for every $\infty$-category $\mathcal{E}$, precomposition with $F$ induces a bijection $\text{Hom}_{\text{hCat}_\infty}(\mathcal{D}, \mathcal{E}) \to \text{Hom}_{\text{hCat}_\infty}(\mathcal{C}, \mathcal{E})$. 
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**Remark 4.5.2.4.** Let \( f : X \to Y \) be a categorical equivalence of simplicial sets. Then \( f \) is a weak homotopy equivalence (since every Kan complex is an \( \infty \)-category). Beware that the converse is generally false.

**Remark 4.5.2.5 (Two-out-of-Six).** Let \( f : W \to X, \ g : X \to Y, \) and \( h : Y \to Z \) be morphisms of simplicial sets. If \( g \circ f \) and \( h \circ g \) are categorical equivalences, then \( f, g, \) and \( h \) are categorical equivalences.

**Remark 4.5.2.6 (Two-out-of-Three).** Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of simplicial sets. If any two of the morphisms \( f, g, \) and \( g \circ f \) is a categorical equivalence, then so is the third. In particular, the collection of categorical equivalences is closed under composition.

**Remark 4.5.2.7.** Let \( f : X \to Y \) be a categorical equivalence of simplicial sets. Then, for any simplicial set \( K \), the induced map \( f^K : X \times K \to Y \times K \) is also a categorical equivalence of simplicial sets. To prove this, we must show that for every \( \infty \)-category \( C \), the restriction map \( \theta : \text{Fun}(Y \times K, C) \to \text{Fun}(X \times K, C) \) induces a bijection on isomorphism classes of objects. This follows from our assumption that \( f \) is a categorical equivalence, since \( \theta \) can be identified with the map \( \text{Fun}(Y, \text{Fun}(K, C)) \to \text{Fun}(X, \text{Fun}(K, C)) \) given by precomposition with \( f \).

**Proposition 4.5.2.8.** Let \( f : X \to Y \) be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism \( f : X \to Y \) is a categorical equivalence. That is, for every \( \infty \)-category \( C \), precomposition with \( f \) induces a bijection
   \[
   \pi_0(\text{Fun}(Y, C)^\simeq) \to \pi_0(\text{Fun}(X, C)^\simeq).
   \]

2. For every \( \infty \)-category \( C \), precomposition with \( f \) induces a homotopy equivalence of Kan complexes \( \text{Fun}(Y, C)^\simeq \to \text{Fun}(X, C)^\simeq \).

3. For every \( \infty \)-category \( C \), precomposition with \( f \) induces an equivalence of \( \infty \)-categories \( \text{Fun}(Y, C) \to \text{Fun}(X, C) \).

**Proof.** The implication (2) \( \Rightarrow \) (1) follows from Remark 3.1.5.4 and the implication (3) \( \Rightarrow \) (2) follows from Remark 4.5.1.18. We will complete the proof by showing that (1) implies (3). Assume that \( f \) is a categorical equivalence of simplicial sets, let \( C \) be an \( \infty \)-category, and let \( f^* : \text{Fun}(Y, C) \to \text{Fun}(X, C) \) denote the functor given by precomposition with \( f \). We wish to show that \([f^*]\) is an isomorphism in the homotopy category \( \text{hCat}_\infty \). For this, it will suffice to show that for any \( \infty \)-category \( D \), the induced map

\[
\theta : \pi_0(\text{Fun}(D, \text{Fun}(Y, C))^\simeq) \to \pi_0(\text{Fun}(D, \text{Fun}(X, C))^\simeq)
\]
is bijective. We conclude by observing that $\theta$ can be identified with the map

$$\pi_0(\text{Fun}(Y, \text{Fun}(\mathcal{D}, \mathcal{C}))^\simeq) \to \pi_0(\text{Fun}(X, \text{Fun}(\mathcal{D}, \mathcal{C}))^\simeq)$$

given by precomposition with $f$.

**Proposition 4.5.2.9.** Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then $q$ is a categorical equivalence.

**Proof.** Let $\mathcal{C}$ be an $\infty$-category. We wish to show that precomposition with $f$ induces a bijection

$$f^* : \pi_0(\text{Fun}(Y, \mathcal{C})^\simeq) \to \pi_0(\text{Fun}(X, \mathcal{C})^\simeq).$$

Let $s : Y \to X$ be a section of $f$ (so that $f \circ s = \text{id}_Y$). Then precomposition with $s$ induces a function $s^* : \pi_0(\text{Fun}(X, \mathcal{C})^\simeq) \to \pi_0(\text{Fun}(Y, \mathcal{C})^\simeq)$ for which the composition $s^* \circ f^*$ is equal to the identity on the set $\pi_0(\text{Fun}(Y, \mathcal{C})^\simeq)$. We will complete the proof by showing that the composition $f^* \circ s^*$ is isomorphic to the identity on $\pi_0(\text{Fun}(X, \mathcal{C})^\simeq)$. Fix a map of simplicial sets $g : X \to \mathcal{C}$; we wish to show that $g$ is isomorphic to the composite map

$$X \xrightarrow{f} Y \xrightarrow{s} X \xrightarrow{g} \mathcal{C}$$

as an object of the $\infty$-category $\text{Fun}(X, \mathcal{C})$.

Since $f$ is a trivial Kan fibration, the composition $s \circ f$ is fiberwise homotopic to the identity map $\text{id}_X$: that is, we can choose a morphism of simplicial sets $h : \Delta^1 \times X \to X$ which is compatible with the projection to $Y$ and which satisfies $h|_{\{0\} \times X} = s \circ f$ and $h|_{\{1\} \times X} = \text{id}_X$. The composition $g \circ h$ can then be regarded as a natural transformation $u : (g \circ s \circ f) \to g$. We will complete the proof by showing that $u$ is an isomorphism in $\infty$-category $\text{Fun}(X, \mathcal{C})$. By virtue of Theorem 4.4.4.4, it will suffice to prove that for each vertex $x \in X$, the composite map

$$\Delta^1 \simeq \Delta^1 \times \{x\} \hookrightarrow \Delta^1 \times X \xrightarrow{h} X \xrightarrow{g} \mathcal{C}$$

describes an invertible morphism in $\mathcal{C}$. Setting $y = f(x)$, we note that this composite map factors through the (contractible) Kan complex $X_y$, so the desired result follows from Proposition 1.3.6.11. \hfill $\square$

**Corollary 4.5.2.10.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a trivial Kan fibration. Then $F$ is an equivalence of $\infty$-categories.

**Proof.** Combine Proposition 4.5.2.9 with Example 4.5.2.3. \hfill $\square$

**Corollary 4.5.2.11.** Let $f : A \leftrightarrow B$ be an inner anodyne morphism of simplicial sets. Then $f$ is a categorical equivalence.
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Proof. By virtue of Proposition 4.5.2.8, it will suffice to show that for every ∞-category C, the restriction map \( f^* : \text{Fun}(B, C) \to \text{Fun}(A, C) \) is an equivalence of ∞-categories. This follows from Corollary 4.5.2.10 since \( f^* \) is a trivial Kan fibration (Proposition 1.4.7.6).

\[ \text{Warning 4.5.2.12.} \]

Let \( f : A \to B \) be a morphism of simplicial sets. By virtue of Corollary 3.3.7.5, the morphism \( f \) is anodyne if and only if it is both a monomorphism and a weak homotopy equivalence. Beware that the analogous assertion for inner anodyne morphisms is false. If \( f \) is inner anodyne, then it is both a monomorphism (Remark 1.4.6.5) and a categorical equivalence (Corollary 4.5.2.11). However, the converse fails: a monomorphism \( A \hookrightarrow B \) which is a categorical equivalence need not be inner anodyne. For example, an inner anodyne morphism of simplicial sets is automatically bijective on vertices (Exercise 1.4.6.6). However, there can be other obstructions as well: see Example 4.5.2.13.

\[ \text{Example 4.5.2.13 (II).} \]

Let \( X = \Delta^2 \amalg_{\mathbb{N} \setminus \{1, 2\}} \Delta^0 \) be the simplicial set obtained from the standard 2-simplex by collapsing the final edge to a point, which we represent by the diagram

\[
\begin{array}{ccc}
\cdot & \xrightarrow{e'} & \cdot \\
\downarrow{e} & & \downarrow{e}
\end{array}
\]

Then \( X \) has exactly two nondegenerate edges \( e, e' : \Delta^1 \to X \), as indicated in the diagram. We now make the following observations:

- There is a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{} & \Delta^2 \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{e'} & X.
\end{array}
\]

Consequently, the morphism \( e' : \Delta^1 \to X \) is inner anodyne, and therefore a categorical equivalence (Corollary 4.5.2.11).

- There is a unique morphism of simplicial sets \( r : X \to \Delta^1 \) satisfying \( r \circ e' = \text{id}_{\Delta^1} \); the composite map \( \Delta^2 \to X \xrightarrow{r} \Delta^1 \) is given on vertices by \( 0 \mapsto 0, 1 \mapsto 1, \) and \( 2 \mapsto 1 \). Since \( e' \) is a categorical equivalence, it follows that \( r \) is also a categorical equivalence (Remark 4.5.2.6).
The composite map $\Delta^1 \xrightarrow{e} X \xrightarrow{r} \Delta^1$ is equal to the identity map $\text{id}_{\Delta^1}$. Since $r$ is a categorical equivalence, it follows that $e$ is also a categorical equivalence. Moreover, $e$ is also a monomorphism of simplicial sets which is bijective vertices.

The morphism $e : \Delta^1 \hookrightarrow X$ is an inner fibration. This follows from Remark 4.1.1.5 since we have a pullback diagram of simplicial sets

$$
\begin{array}{ccc}
\Lambda^2_2 & \longrightarrow & \Delta^1 \\
\downarrow & & \downarrow e \\
\Delta^2 & \longrightarrow & X,
\end{array}
$$

where the horizontal maps are surjective and the inclusion $\Lambda^2_2 \hookrightarrow \Delta^2$ is an inner fibration (since can be realized as the nerve of a morphism between partially ordered sets).

The morphism $e$ is not inner anodyne, since the lifting problem

$$
\begin{array}{ccc}
\Lambda^1 & \longrightarrow & \Delta^1 \\
\downarrow e & \searrow \text{id} & \downarrow e \\
\Delta^1 & \downarrow e & \Delta^1 \\
\downarrow & \searrow \text{id} & \downarrow \\
X & \longrightarrow & X
\end{array}
$$

has no solution.

**Remark 4.5.2.14 (Axioms for Categorical Equivalence).** The collection of categorical equivalences of simplicial sets has the following properties:

(A) If $\mathcal{C}$ and $\mathcal{D}$ are $\infty$-categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is a categorical equivalence if and only if it is an equivalence of $\infty$-categories (Example 4.5.2.3).

(B) Every inner anodyne morphism of simplicial sets is a categorical equivalence (Corollary 4.5.2.11).

(C) If $f : X \to Y$ and $g : Y \to Z$ have the property that two of the morphisms $f$, $g$, and $g \circ f$ are categorical equivalences, then so is the third (Remark 4.5.2.6).

In fact, the collection of categorical equivalences is characterized by assertions (A), (B) and (C). Let $f : X \to Y$ be a morphism of simplicial sets. Invoking Proposition 4.1.3.1 twice, we
can construct a commutative diagram of simplicial sets

\[ \begin{array}{ccc} X & \xrightarrow{u} & C \\ \downarrow f & & \downarrow F \\ Y & \xrightarrow{v} & D. \end{array} \]

It follows from (A), (B) and (C) that the morphism \( f \) is a categorical equivalence if and only if the functor \( F \) is an equivalence of \( \infty \)-categories.

### 4.5.3 Categorical Pushout Diagrams

Recall that a commutative diagram of simplicial sets

\[ \begin{array}{ccc} A & \xrightarrow{} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{} & D \end{array} \]

is a homotopy pushout diagram if, for every Kan complex \( X \), the diagram of Kan complexes

\[ \begin{array}{ccc} \text{Fun}(A, X) & \xleftarrow{} & \text{Fun}(B, X) \\ \uparrow & & \uparrow \\ \text{Fun}(C, X) & \xleftarrow{} & \text{Fun}(D, X) \end{array} \]

is homotopy Cartesian (Definition 3.4.2.1). In this section, we study a stronger version of this condition.

**Definition 4.5.3.1.** A commutative diagram of simplicial sets

\[ \begin{array}{ccc} A & \xrightarrow{} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{} & D \end{array} \]
is a categorical pushout diagram if, for every $\infty$-category $\mathcal{C}$, the diagram of Kan complexes

$$\begin{array}{ccc}
\text{Fun}(A, \mathcal{C}) \cong & \longleftarrow & \text{Fun}(B, \mathcal{C}) \cong \\
\uparrow & & \uparrow \\
\text{Fun}(C, \mathcal{C}) \cong & \longleftarrow & \text{Fun}(D, \mathcal{C}) \cong
\end{array}$$

is homotopy Cartesian.

**Remark 4.5.3.2.** Every categorical pushout diagram of simplicial sets is also a homotopy pushout diagram of simplicial sets (since every Kan complex $X$ is an $\infty$-category which satisfies $\text{Fun}(K, X)^\simeq = \text{Fun}(K, X)$ for every simplicial set $X$).

We now summarize some of the basic formal properties of Definition 4.5.3.1 (which are counterparts of the analogous assertions for the class of homotopy pushout diagrams, established in §3.4.2).

**Proposition 4.5.3.3 (Symmetry).** A commutative diagram of simplicial sets

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}$$

is a categorical pushout diagram if and only if the transposed diagram

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & D
\end{array}$$

is a categorical pushout diagram.

**Proof.** Apply Proposition 3.4.1.7.

**Proposition 4.5.3.4 (Transitivity).** Suppose we are given a commutative diagram of simplicial sets

$$\begin{array}{ccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}$$

is a categorical pushout diagram.
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where the left square is a categorical pushout diagram. Then the right square is a categorical pushout diagram if and only if the outer rectangle is a categorical pushout diagram.

Proof. Apply Proposition 3.4.1.9.

Proposition 4.5.3.5 (Homotopy Invariance). Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{w_A} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{w_B} & B' \\
\downarrow & & \downarrow \\
C & \xrightarrow{w_C} & D \\
\downarrow & & \downarrow \\
C' & \xrightarrow{w_D} & D',
\end{array}
\]

where the morphisms $w_A$, $w_B$, and $w_C$ are categorical equivalences. Then any two of the following three conditions imply the third:

1. The commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{} & D
\end{array}
\]

is a categorical pushout square.

2. The commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{} & B' \\
\downarrow & & \downarrow \\
C' & \xrightarrow{} & D'
\end{array}
\]

is a categorical pushout square.

3. The morphism $w_D$ is a categorical equivalence.

Proof. Combine Corollary 3.4.1.10 with Proposition 4.5.2.8.
Proposition 4.5.3.6. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{f'} & D
\end{array}
\]  \hfill (4.4)

where \(f\) is a categorical equivalence. Then (4.4) is a categorical pushout diagram if and only if \(f'\) is a categorical equivalence.

Proof. For every \(\infty\)-category \(C\), we obtain a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Fun}(A, C) & \xleftarrow{u} & \text{Fun}(B, C) \\
\uparrow & & \uparrow \\
\text{Fun}(C, C) & \xleftarrow{u'} & \text{Fun}(D, C)
\end{array}
\]  \hfill (4.5)

where \(u\) is a homotopy equivalence of Kan complexes (Proposition 4.5.2.8). Applying Corollary 3.4.1.3, we conclude that (4.5) is homotopy Cartesian if and only if \(u\) is a homotopy equivalence of Kan complexes. Consequently, (4.4) is a categorical pushout square if and only if, for every \(\infty\)-category \(C\), the composition with \(f'\) induces a homotopy equivalence \(\text{Fun}(D, C) \xrightarrow{\sim} \text{Fun}(C, C)\). By virtue of Proposition 4.5.2.8, this is equivalent to the requirement that \(f'\) is a weak homotopy equivalence. \(\Box\)

Proposition 4.5.3.7. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \to & D,
\end{array}
\]  \hfill (4.6)

where \(f\) is a monomorphism. Then (3.11) is a categorical pushout square if and only if the induced map \(\rho : C \amalg_A B \to D\) is a categorical equivalence of simplicial sets.
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Proof. For every $\infty$-category $C$, we obtain a commutative diagram of Kan complexes

$$
\begin{array}{c}
\text{Fun}(A, C)^\simeq \leftarrow \leftarrow \\
\text{Fun}(B, C)^\simeq \leftarrow \leftarrow \\
\text{Fun}(C, C)^\simeq \leftarrow \leftarrow \\
\text{Fun}(D, C)^\simeq
\end{array}
$$

(4.7)

where $u$ is a Kan fibration (Corollary 4.4.5.4). It follows that the diagram (4.7) is homotopy Cartesian if and only if the induced map

$$
\theta : \text{Fun}(D, C)^\simeq \rightarrow \text{Fun}(C, C)^\simeq \times_{\text{Fun}(A, C)^\simeq} \text{Fun}(B, C)^\simeq
$$

is a weak homotopy equivalence (Example 3.4.1.5). By virtue of Corollary 4.4.4.6 we can identify $\theta$ with the map $\text{Fun}(D, C)^\simeq \rightarrow \text{Fun}(B \coprod_A C, C)^\simeq$ given by precomposition with $\rho$. It follows that the diagram (3.11) is a categorical pushout square if and only if, for every $\infty$-category $C$, the induced map $\text{Fun}(D, C)^\simeq \rightarrow \text{Fun}(C \coprod_A B, C)^\simeq$ is a homotopy equivalence of Kan complexes. By virtue of Proposition 4.5.2.8 this is equivalent to the requirement that $\rho$ is a categorical equivalence.

Example 4.5.3.8. Suppose we are given a pushout diagram of simplicial sets

$$
\begin{array}{c}
A \xrightarrow{f} & B \\
\downarrow & \downarrow \\
C \xrightarrow{g} & D
\end{array}
$$

(4.8)

If $f$ is a monomorphism, then (4.8) is also a homotopy pushout diagram.

Remark 4.5.3.9. Suppose we are given a pushout diagram of simplicial sets

$$
\begin{array}{c}
A \xrightarrow{f} & B \\
\downarrow & \downarrow \\
C \xrightarrow{g} & D
\end{array}
$$

where $f$ is a monomorphism. If $g$ is a categorical equivalence, then $g'$ is also a categorical equivalence. This follows by combining Example 4.5.3.8 with Proposition 4.5.3.6.
Corollary 4.5.3.10. Let \( i : A \to B \) and \( i' : A' \to B' \) be morphisms of simplicial sets. Assume that \( i \) is monomorphism, and that either \( i \) or \( i' \) is a categorical equivalence. Then the induced map

\[
(A \times B') \coprod_{(A \times A')} (B \times A') \to B \times B'
\]
is a categorical equivalence.

Proof. By virtue of Proposition 4.5.3.7, it will suffice to show that the diagram

\[
\begin{array}{ccc}
A \times A' & \longrightarrow & B \times A' \\
\downarrow & & \downarrow \\
A \times B' & \longrightarrow & B \times B'
\end{array}
\]

is a categorical pushout square. This follows from the criterion of Proposition 4.5.3.6: if \( i \) is a categorical equivalence, then the horizontal maps in the diagram (4.9) are categorical equivalences (Remark 4.5.2.7). Similarly, if \( i' \) is a categorical equivalence, then the vertical maps in the diagram (4.9) are categorical equivalences.

Remark 4.5.3.11. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D.
\end{array}
\]

Using Exercise 3.1.6.11, we can factor \( f \) as a composition \( A \xrightarrow{f'} B' \xrightarrow{w} B \), where \( f' \) is a monomorphism and \( w \) is a trivial Kan fibration (and therefore also a categorical equivalence, by virtue of Proposition 4.5.2.9). Combining Propositions 4.5.3.7 and 4.5.3.5, we conclude that diagram (4.10) is a categorical pushout square if and only if the induced map \( u : C \coprod_A B' \to D \) is a categorical equivalence In particular, the condition that \( u \) is a weak homotopy equivalence does not depend on the choice of factorization \( f = w \circ f' \).

4.5.4 Detecting Equivalences of \( \infty \)-Categories

Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor between \( \infty \)-categories. If \( F \) is an equivalence of \( \infty \)-categories, then the induced map \( F^\simeq : \mathcal{C}^\simeq \to \mathcal{D}^\simeq \) is a homotopy equivalence of Kan complexes (Remark 4.5.1.18). The converse assertion is not true in general. For example, the inclusion map
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$C^\simeq \to C$ induces an isomorphism on cores, but is never an equivalence of $\infty$-categories unless $C$ is a Kan complex. However, we have the following result:

**Theorem 4.5.4.1.** Let $F : C \to D$ be a functor of $\infty$-categories. The following conditions are equivalent:

1. The functor $F$ is an equivalence of $\infty$-categories.
2. For every simplicial set $X$, composition with $F$ induces an equivalence of $\infty$-categories $\operatorname{Fun}(X, C) \to \operatorname{Fun}(X, D)$.
3. For every simplicial set $X$, composition with $F$ induces a homotopy equivalence of Kan complexes $\operatorname{Fun}(X, C)^\simeq \to \operatorname{Fun}(X, D)^\simeq$.
4. Composition with $F$ induces a homotopy equivalence of Kan complexes $\operatorname{Fun}(\Delta^1, C)^\simeq \to \operatorname{Fun}(\Delta^1, D)^\simeq$.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from Remark 4.5.1.15, the implication (2) $\Rightarrow$ (3) from Remark 4.5.1.18, and the implication (3) $\Rightarrow$ (4) is immediate. If (3) is satisfied, then for every $\infty$-category $\mathcal{E}$, composition with the isomorphism class $[F] \in \operatorname{Hom}_{\mathcal{h}\operatorname{Cat}_\infty}(C, D)$ induces a bijection

$$\operatorname{Hom}_{\mathcal{h}\operatorname{Cat}_\infty}(\mathcal{E}, C) = \pi_0(\operatorname{Fun}(\mathcal{E}, C)^\simeq) \simeq \pi_0(\operatorname{Fun}(\mathcal{E}, D)^\simeq) = \operatorname{Hom}_{\mathcal{h}\operatorname{Cat}_\infty}(\mathcal{E}, D),$$

so that $[F]$ is an isomorphism in the category $\mathcal{h}\operatorname{Cat}_\infty$ and (1) is satisfied. We will complete the proof by showing that (4) implies (3).

For every simplicial set $X$, let $\theta_X : \operatorname{Fun}(X, C)^\simeq \to \operatorname{Fun}(X, D)^\simeq$ denote the map given by postcomposition with the functor $F$. Let us say that $X$ is *good* if the morphism $\theta_X$ is a homotopy equivalence. We now make the following observations:

(a) Let $X$ be the colimit of a diagram of monomorphisms

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

We then obtain a commutative diagram of Kan complexes

$$\begin{array}{cccc}
\operatorname{Fun}(X(0), C)^\simeq & \leftarrow & \operatorname{Fun}(X(1), C)^\simeq & \leftarrow & \operatorname{Fun}(X(2), C)^\simeq & \cdots \\
\operatorname{Fun}(X(0), D)^\simeq & \leftarrow & \operatorname{Fun}(X(1), D)^\simeq & \leftarrow & \operatorname{Fun}(X(2), D)^\simeq & \cdots ,
\end{array}$$

where the horizontal maps are Kan fibrations (Corollary 4.4.5.4). Moreover, the induced map of inverse limits can be identified with the map $\theta_X : \operatorname{Fun}(X, C)^\simeq \to \operatorname{Fun}(X, D)^\simeq$. 


Fun(\mathcal{X}, \mathcal{D}) \simeq \text{(Corollary 4.4.4.6). If each } \mathcal{X}(n) \text{ is good, then the vertical maps appearing in the diagram are homotopy equivalences, so that } \theta_X \text{ is also a homotopy equivalence (Proposition 3.3.9.6). It follows that } \mathcal{X} \text{ is also good.}

(b) Let } \mathcal{X} \text{ be a simplicial set which is given as a coproduct } \coprod \mathcal{X}(\alpha) \text{ of a collection of simplicial sets } \mathcal{X}(\alpha). \text{ Then } \theta_\mathcal{X} \text{ can be identified with the product of the maps } \theta_{\mathcal{X}(\alpha)} \text{ (Corollary 4.4.4.6). Consequently, if each of the summands } \mathcal{X}(\alpha) \text{ is good, then } \mathcal{X} \text{ is also good (Remark 3.1.5.8).}

(c) Let } u : \mathcal{X} \to \mathcal{Y} \text{ be an inner anodyne morphism of simplicial sets. Then we have a commutative diagram of Kan complexes}

\[
\begin{array}{ccc}
\text{Fun}(\mathcal{X}, \mathcal{C}) \simeq & \longrightarrow & \text{Fun}(\mathcal{Y}, \mathcal{C}) \simeq \\
\downarrow & & \downarrow \\
\text{Fun}(\mathcal{X}, \mathcal{D}) \simeq & \longrightarrow & \text{Fun}(\mathcal{Y}, \mathcal{D}) \simeq,
\end{array}
\]

where the horizontal maps are homotopy equivalences (Proposition 4.5.2.8). It follows that } \mathcal{X} \text{ is good if and only if } \mathcal{Y} \text{ is good.}

(d) Suppose are given a categorical pushout square of simplicial sets

\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{Y}'.
\end{array}
\]

If } \mathcal{X}, \mathcal{X}', \text{ and } \mathcal{Y} \text{ are good, then } \mathcal{Y}' \text{ is also good (see Corollary 3.4.1.10).}

(e) Let } \mathcal{X} \text{ be a retract of a simplicial set } \mathcal{Y}. \text{ If } \mathcal{Y} \text{ is good, then } \mathcal{X} \text{ is also good.}

Now suppose that condition (4) is satisfied, so that the simplex } \Delta^1 \text{ is good. We wish to show that every simplicial set } \mathcal{X} \text{ is good. Writing } \mathcal{X} \text{ as the direct limit of its skeleta } \{\text{sk}_n(\mathcal{X})\}_{n \geq 0} \text{ and using (a), we can reduce to the case where } \mathcal{X} \text{ is has dimension } \leq n \text{ for some integer } n. \text{ We proceed by induction on } n. \text{ The case } n = -1 \text{ is trivial (in this case, the simplicial set } \mathcal{X} \text{ is empty and the morphism } \theta_\mathcal{X} \text{ is an isomorphism). We may therefore assume that } n \geq 0. \text{ Let } S \text{ be the collection of nondegenerate } n\text{-simplices of } \mathcal{X}, \text{ so that}
Proposition 4.1.3.13 supplies a pushout diagram

$$\coprod_{\sigma \in S} \partial \Delta^n \rightarrow \coprod_{\sigma \in S} \Delta^n \rightarrow sk_{n-1}(X) \rightarrow X.$$ 

Since the horizontal maps in this diagram are monomorphisms, it is also a categorical pushout diagram (Example 4.5.3.8). Moreover, our inductive hypothesis guarantees that the simplicial sets $sk_{n-1}(X)$ and $\coprod_{\sigma \in S} \partial \Delta^n$ are good. Applying (d), we are reduced to showing that the coproduct $\coprod_{\sigma \in S} \Delta^n$ is good. Using (b), we are reduced to showing that the standard simplex $\Delta^n$ is good. If $n \geq 2$, then the inner horn inclusion $\Lambda^n_1 \rightarrow \Delta^n$ is a categorical equivalence, so that the desired result follows from our inductive hypothesis together with (c). We are therefore reduced to showing that the standard simplices $\Delta^0$ and $\Delta^1$ are good. In the second case this follows from our assumption (4), and in the first case it follows from (e) (since the 0-simplex $\Delta^0$ is a retract of $\Delta^1$).

Corollary 4.5.4.2. Let $\mathcal{W}$ denote the full subcategory of $\text{Fun}([1], \text{Set}_\Delta)$ spanned by those morphisms of simplicial sets $f : X \rightarrow Y$ which are categorical equivalences. Then $\mathcal{W}$ is closed under the formation of filtered colimits in $\text{Fun}([1], \text{Set}_\Delta)$.

Proof. By virtue of Corollary 4.1.3.3, there exists a functor $Q : \text{Set}_\Delta \rightarrow \text{Set}_\Delta$ which commutes with filtered colimits and a natural transformation of functors $u : \text{id}_{\text{Set}_\Delta} \rightarrow Q$ with the property that, for every simplicial set $X$, the simplicial set $Q(X)$ is an $\infty$-category and the morphism $u_X : X \rightarrow Q(X)$ is inner anodyne. For every morphism of simplicial sets $f : X \rightarrow Y$, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{u_X} & & \downarrow^{u_Y} \\
Q(X) & \xrightarrow{Q(f)} & Q(Y)
\end{array}
$$

where the vertical maps are categorical equivalences (Corollary 4.5.2.11). It follows from Remark 4.5.2.6 that $f$ is a categorical equivalence if and only if the functor $Q(f)$ is an equivalence of $\infty$-categories. Using the criterion of Theorem 4.5.4.1 we see that $f$ is a categorical equivalence if and only if the induced map $\text{Fun}(\Delta^1, Q(X))^\simeq \rightarrow \text{Fun}(\Delta^1, Q(Y))^\simeq$ is a homotopy equivalence of Kan complexes. The desired result now follows by observing
that the construction $X \mapsto \text{Fun}(\Delta^1, Q(X))^\simeq$ commutes with filtered colimits, since the collection of homotopy equivalences between Kan complexes is closed under filtered colimits (Proposition 3.2.7.3).

Corollary 4.5.4.3. Suppose we are given a pullback diagram of $\infty$-categories

$$
\begin{array}{c}
\mathcal{C}' \\
\downarrow F' \\
\mathcal{D}' \\
\end{array}
\quad
\begin{array}{c}
\mathcal{C} \\
\downarrow q \\
\mathcal{D}, \\
\end{array}
$$

where $q$ is an isofibration. If $F$ is an equivalence of $\infty$-categories, then $F'$ is also an equivalence of $\infty$-categories.

Proof. Let $X$ be an arbitrary simplicial set. Then we have a pullback diagram of $\infty$-categories

$$
\begin{array}{c}
\text{Fun}(X, \mathcal{C}') \\
\downarrow F_X' \\
\text{Fun}(X, \mathcal{D}') \\
\end{array}
\quad
\begin{array}{c}
\text{Fun}(X, \mathcal{C}) \\
\downarrow q_X \\
\text{Fun}(X, \mathcal{D}) \\
\end{array}
$$

where $q_X$ is also an isofibration (Corollary 4.4.5.5). Invoking Corollary 4.4.3.14 we conclude that the diagram of Kan complexes

$$
\begin{array}{c}
\text{Fun}(X, \mathcal{C}')^\simeq \\
\downarrow F_X^\simeq \\
\text{Fun}(X, \mathcal{D}')^\simeq \\
\end{array}
\quad
\begin{array}{c}
\text{Fun}(X, \mathcal{C})^\simeq \\
\downarrow q_X^\simeq \\
\text{Fun}(X, \mathcal{D})^\simeq \\
\end{array}
$$

is a homotopy pullback square. Since $F$ is an equivalence of $\infty$-categories, the morphism $F_X^\simeq$ is a homotopy equivalence of Kan complexes (Theorem 4.5.4.1). Applying Corollary 3.4.1.3 we conclude that $F_X^\simeq$ is also a homotopy equivalence of Kan complexes. Allowing $X$ to vary, we conclude that $F'$ is an equivalence of $\infty$-categories. □
Corollary 4.5.4.4. Suppose we are given a commutative diagram of $\infty$-categories

\[
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{q'} & \mathcal{C} \\
\downarrow F' & & \downarrow q \\
\mathcal{D}' & \xrightarrow{F} & \mathcal{D},
\end{array}
\]

where $q'$ and $q$ are isofibrations and the functors $F$ and $F'$ are equivalences of $\infty$-categories. Then, for every object $D' \in \mathcal{D}$ having image $D = F(D') \in \mathcal{D}$, the induced functor

\[C'_{D'} = \{D'\} \times_{\mathcal{D}'} \mathcal{C}' \to \{D\} \times_{\mathcal{D}} \mathcal{C} = C_D\]

is an equivalence of $\infty$-categories.

Proof. For every simplicial set $X$, we have a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Fun}(X, \mathcal{C}') & \xrightarrow{\simeq} & \text{Fun}(X, \mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Fun}(X, \mathcal{D}') & \xrightarrow{\simeq} & \text{Fun}(X, \mathcal{D}).
\end{array}
\]

Our assumption that $F$ and $F'$ are equivalences of $\infty$-categories guarantee that the horizontal maps are homotopy equivalences (Theorem 4.5.4.1), and our assumption that $q$ and $q'$ are isofibrations guarantee that the vertical maps are Kan fibrations (Proposition 4.4.3.6). Let $D \in \text{Fun}(X, \mathcal{D})$ and $D' \in \text{Fun}(X, \mathcal{D}')$ be the constant diagrams taking the values $D$ and $D'$, respectively, so that the induced map of fibers

\[\theta : \{D'\} \times_{\text{Fun}(X, \mathcal{D}')} \text{Fun}(X, \mathcal{C}') \to \{D\} \times_{\text{Fun}(X, \mathcal{D})} \text{Fun}(X, \mathcal{C})\]

is also a homotopy equivalence (Proposition 3.2.7.1). Using Corollary 4.4.3.15 (and Corollary 4.4.5.5), we can identify $\theta$ with the natural map $\text{Fun}(X, \mathcal{C}'_{D'}) \simeq \text{Fun}(X, C_{D})$. Allowing $X$ to vary and applying Theorem 4.5.4.1 we conclude that the functor $C'_{D'} \to C_D$ is an equivalence of $\infty$-categories. \qed

Warning 4.5.4.5. Suppose we are given a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow q' & & \downarrow q \\
S' & \xrightarrow{f} & S,
\end{array}
\]
where \( q \) and \( q' \) are Kan fibrations and \( f \) is a homotopy equivalence. By virtue of Proposition 3.2.7.1, the following conditions are equivalent:

1. The morphism \( f' \) is a homotopy equivalence of Kan complexes.
2. For each vertex \( s' \in S' \) having image \( s = f(s') \in S \), the induced map of fibers \( X_{s'} \to X_s \) is a homotopy equivalence of Kan complexes.

Corollary 4.5.4.4 can be regarded as a generalization of the implication (1) \( \Rightarrow \) (2), where we allow \( \infty \)-categories in place of Kan complexes and isofibrations in place of Kan fibrations. Beware that the implication (2) \( \Rightarrow \) (1) does not generalize. For example, we have a commutative diagram of \( \infty \)-categories

\[
\begin{array}{ccc}
\partial \Delta^1 & \to & \Delta^1 \\
\downarrow & & \downarrow id \\
\Delta^1 & \to & \Delta^1,
\end{array}
\]

where the vertical maps are isofibrations, the bottom horizontal map is an isomorphism, and the upper horizontal map restricts to an isomorphism on each fiber, but is nevertheless not an equivalence of \( \infty \)-categories.

### 4.5.5 Application: Universal Property of the Join

Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, and let \( \mathcal{C} \star \mathcal{D} \) denote their join (Definition 4.3.2.1). Proposition 4.3.2.13 (and Remark 4.3.2.14) supplies a pushout diagram of categories.

\[
\begin{array}{ccc}
\mathcal{C} \times \{0\} \times \mathcal{D} & \coprod & \mathcal{C} \times \{1\} \times \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{C} \times \{0\} \coprod \{1\} \times \mathcal{D} & \to & \mathcal{C} \star \mathcal{D}.
\end{array}
\]

Passing to nerves, we obtain a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
N_{\bullet}(\mathcal{C}) \times \{0\} \times N_{\bullet}(\mathcal{D}) & \coprod & N_{\bullet}(\mathcal{C}) \times \{1\} \times N_{\bullet}(\mathcal{D}) \\
\downarrow & & \downarrow \\
N_{\bullet}(\mathcal{C}) \times \Delta^1 \times N_{\bullet}(\mathcal{D}) & \to & N_{\bullet}(\mathcal{C}) \star N_{\bullet}(\mathcal{D}).
\end{array}
\]
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Beware that this diagram is generally not a pushout square. However, we will show in this section that it is nevertheless a categorical pushout square, in the sense of Definition 4.5.3.1. Moreover, an analogous statement holds if we replace $N_\bullet(C)$ and $N_\bullet(D)$ by arbitrary simplicial sets $X$ and $Y$.

**Construction 4.5.5.1.** Let $X$ and $Y$ be simplicial sets, let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projection maps, and let $\iota_X : X \hookrightarrow X \star Y$ and $\iota_Y : Y \hookrightarrow X \star Y$ denote the inclusion maps. Then there is a unique map of simplicial sets $c : X \times \Delta^1 \times Y \to X \star Y$ with the property that $c|_{X \times \{0\} \times Y} = \iota_X \circ \pi_X$ and $c|_{X \times \{1\} \times Y} = \iota_Y \circ \pi_Y$. Concretely, if $\sigma = (\sigma_X, \sigma_{\Delta^1}, \sigma_Y)$ is an $n$-simplex of the product $X \times \Delta^1 \times Y$, then $c(\sigma)$ is the $n$-simplex of $X \star Y$ given by the composition

$$\Delta^n \simeq (\sigma_{\Delta^1}^{-1}\{0\}) \ast (\sigma_{\Delta^1}^{-1}\{1\}) \xrightarrow{\sigma_X \ast \sigma_Y} X \star Y.$$  

We will refer to $c : X \times \Delta^1 \times Y \to X \star Y$ as the collapse map.

**Proposition 4.5.5.2.** Let $X$ and $Y$ be simplicial sets, and let $c : X \times \Delta^1 \times Y \to X \star Y$ denote the collapse map of Construction 4.5.5.1. Then the commutative diagram of simplicial sets

$$
\begin{array}{ccc}
(X \times \{0\} \times Y) \coprod (X \times \{1\} \times Y) & \xrightarrow{\pi_X \coprod \pi_Y} & X \times \Delta^1 \times Y \\
\downarrow c & & \downarrow c \\
(X \times \{0\}) \coprod \{\{1\} \times Y) & \xrightarrow{(\iota_X, \iota_Y)} & X \star Y
\end{array}
$$

is a categorical pushout square.

It will be convenient to state Proposition 4.5.5.2 in a slightly different form.

**Notation 4.5.5.3.** Let $X$ and $Y$ be simplicial sets. We let $X \diamond Y$ denote the simplicial set given by the iterated pushout

$$
\begin{array}{ccc}
X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y
\end{array}
$$

so that we have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
X \times \partial \Delta^1 \times Y & \xrightarrow{\pi_X \coprod \pi_Y} & X \times \Delta^1 \times Y \\
\downarrow & & \downarrow \\
(X \times \{0\}) \coprod \{\{1\} \times Y & \xrightarrow{(\iota_X, \iota_Y)} & X \diamond Y.
\end{array}
$$

It follows that the commutative diagram (4.11) determines a morphism of simplicial sets $c_{X,Y} : X \diamond Y \to X \star Y$, which we will refer to as the comparison map.
Example 4.5.5.4. Let \( X \) and \( Y \) be simplicial sets. If \( X \) is empty, then the simplicial set \( X \diamond Y \) can be identified with \( Y \). If \( Y \) is empty, then the simplicial set \( X \diamond Y \) can be identified with \( X \). In either case, the comparison map \( c_{X,Y} : X \diamond Y \to X \star Y \) is an isomorphism of simplicial sets.

Remark 4.5.5.5 (Functoriality). The construction \((X, Y) \mapsto X \diamond Y\) determines a functor \( \diamond : \text{Set}_\Delta \times \text{Set}_\Delta \to \text{Set}_\Delta \). Moreover:

- For fixed \( X \), the functor

\[
\text{Set}_\Delta \to \text{Set}_\Delta \quad Y \mapsto X \star Y
\]

preserves monomorphisms, filtered colimits and pushout diagrams.

- For fixed \( Y \), the functor

\[
\text{Set}_\Delta \to \text{Set}_\Delta \quad X \mapsto X \star Y
\]

preserves monomorphisms, filtered colimits, and pushout diagrams.

Remark 4.5.6. Let \( f : X \to X' \) and \( g : Y \to Y' \) be categorical equivalences of simplicial sets. Then the induced map \((f \circ g) : X \diamond Y \to X' \diamond Y'\) is also a categorical equivalence. This follows by applying Proposition 4.5.3.7 to the diagram

\[
\begin{array}{ccc}
X \times \partial \Delta^1 \times Y & \xrightarrow{\sim} & X \times \Delta^1 \times Y \\
\downarrow & & \downarrow \\
X' \times \partial \Delta^1 \times Y' & \xrightarrow{\sim} & X' \times \Delta^1 \times Y' \\
(\{0\} \times X) \coprod (\{1\} \times Y) & \to & X \diamond Y \\
\downarrow & & \downarrow \\
(\{0\} \times X') \coprod (\{1\} \times Y') & \to & X' \diamond Y'.
\end{array}
\]

By virtue of Proposition 4.5.3.7, Proposition 4.5.2 can be restated as follows:

Theorem 4.5.5.7. Let \( X \) and \( Y \) be simplicial sets. Then the comparison map \( c_{X,Y} : X \diamond Y \to X \star Y \) of Notation 4.5.5.3 is a categorical equivalence of simplicial sets.
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**Corollary 4.5.5.8.** Let \( f : X \to X' \) and \( g : Y \to Y' \) be categorical equivalences of simplicial sets. Then the induced map \( (f * g) : X * Y \to X' * Y' \) is also a categorical equivalence of simplicial sets.

*Proof.* We have a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X \odot Y & \xrightarrow{f \circ g} & X' \odot Y' \\
\downarrow c_{X,Y} & & \downarrow c_{X',Y'} \\
X * Y & \xrightarrow{f * g} & X' * Y',
\end{array}
\]

where \( f \circ g, c_{X,Y}, \text{ and } c_{X',Y'} \) are categorical equivalences (Remark 4.5.5.6 and Theorem 4.5.5.7). Invoking the two-out-of-three property (Remark 4.5.2.6), we conclude that \( f * g \) is also a categorical equivalence.

The proof of Theorem 4.5.5.7 will require some preliminaries. We begin by reducing to the special case where \( X = \Delta^1 \).

**Lemma 4.5.5.9.** Let \( Y \) be a simplicial set, and suppose that the comparison map \( c_{\Delta^1,Y} : \Delta^1 \odot Y \to \Delta^1 * Y \) is a categorical equivalence. Then, for every simplicial set \( X \), the comparison map \( c_{X,Y} : X \odot Y \to X * Y \) is a categorical equivalence.

*Proof.* Throughout the proof, we regard the simplicial set \( Y \) as fixed. Let us say that a simplicial set \( X \) is *good* if \( c_{X,Y} \) is a categorical equivalence. We begin with some elementary observations:

(a) The collection of good simplicial sets is closed under the formation of filtered colimits (since the collection of categorical equivalences is closed under filtered colimits, by virtue of Corollary 4.5.4.2).

(b) Suppose we are given a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X(0) \\
\downarrow & & \downarrow \\
X(1) & \xrightarrow{} & X(01),
\end{array}
\]
where \( f \) is a monomorphism. If \( X, X(0) \), and \( X(1) \) are good, then \( X(01) \) is good. This follows by applying Proposition 4.5.3.5 to the commutative diagram

\[
\begin{array}{ccc}
X \circ Y & \xrightarrow{c_{X,Y}} & X(0) \circ Y \\
\downarrow & & \downarrow \\
X \ast Y & \xrightarrow{c_{X(0),Y}} & X(0) \ast Y \\
\downarrow & & \downarrow \\
X(1) \circ Y & \xrightarrow{c_{X(1),Y}} & X(01) \circ Y \\
\downarrow & & \downarrow \\
X(1) \ast Y & \xrightarrow{c_{X(01),Y}} & X(01) \ast Y,
\end{array}
\]

noting that the upper and lower squares are categorical pushouts by virtue of Example 4.5.3.8).

(c) Let \( f : X \to X' \) be an inner anodyne morphism of simplicial sets. Then \( X \) is good if and only if \( X' \) is good. To prove this, we observe that there is a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X \circ Y & \xrightarrow{c_{X,Y}} & X \ast Y \\
\downarrow & & \downarrow f \ast \text{id}_Y \\
X' \circ Y & \xrightarrow{c_Y} & X' \ast Y.
\end{array}
\]

By the two-out-of-three property (Remark 4.5.3.6), it will suffice to show that the morphisms \( f \circ \text{id}_Y \) and \( f \ast \text{id}_Y \) are categorical equivalences. In the first case, this follows from Remark 4.5.3.6. For the second, we observe that \( f \ast \text{id}_Y \) is actually inner anodyne, since it factors as a composition

\[
X \ast Y \xrightarrow{u} \prod_X (X \ast Y) \xrightarrow{v} X' \ast Y,
\]

where \( u \) is a pushout of \( f \) (hence inner anodyne because \( f \) is inner anodyne) and \( v \) is left anodyne by virtue of Proposition 4.3.7.1 (hence also inner anodyne).
We wish to show that if the 1-simplex $\Delta^1$ is good, then every simplicial set $X$ is good. Writing $X$ as the filtered colimit its finite simplicial subsets (Remark 3.5.1.8), we can use (a) to reduce to the case where $X$ is finite. We now proceed by induction on the dimension of $X$. If $X = \emptyset$, then $c_X$ is an isomorphism (Example 4.5.5.4). Otherwise, the simplicial set $X$ has dimension $n \geq 0$. We now proceed by induction on the number of nondegenerate $n$-simplices of $X$. Using Proposition 1.1.3.13, we can choose a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \Delta^n \\
\downarrow & & \downarrow \\
X' & \rightarrow & X
\end{array}
$$

where $X' \subseteq X$ is a simplicial subset having one fewer nondegenerate $n$-simplex. It then follows from our inductive hypothesis that $\partial \Delta^n$ and $X'$ are good. By virtue of (b), it will suffice to show that $\Delta^n$ is good. This holds for $n = 1$ by assumption, and also for $n = 0$ because $\Delta^0$ is a retract of $\Delta^1$. We may therefore assume that $n \geq 2$, so that the horn inclusion $\Lambda^n_1 \hookrightarrow \Delta^n$ is inner anodyne. Our inductive hypothesis guarantees that $\Lambda^n_1$ is good, so that $\Delta^n$ is good by virtue of (c).

\[ \text{Lemma 4.5.5.10. The comparison map } c_{\Delta^1,\Delta^0} : \Delta^1 \circ \Delta^0 \rightarrow \Delta^1 \star \Delta^0 \text{ is a categorical equivalence.} \]

\[ \text{Proof. Unwinding the definitions, we can identify } \Delta^1 \circ \Delta^0 \text{ with the simplicial set } (\Delta^1 \times \Delta^1) \coprod_{\Delta^1 \times \{1\}} \Delta^0, \text{ which we represent informally by the diagram} \]

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet
\end{array} \]

Let $X$ denote the simplicial set $\Delta^2 \coprod_{\{1 \prec (1 \prec 2)\}} \Delta^0$ obtained from the standard 2-simplex by collapsing the final edge to a point. We then have an inclusion map $i : X \hookrightarrow \Delta^1 \circ \Delta^0$ (corresponding to the triangle in the upper right of the preceding diagram), which fits into a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
X & \rightarrow & \Delta^1 \\
\downarrow & & \downarrow \\
\Delta^1 \circ \Delta^0 & \rightarrow & \Delta^1 \star \Delta^0
\end{array}
$$
here \( u \) classifies to the “long edge” of the 2-simplex \( \Delta^1 \star \Delta^0 \simeq \Delta^2 \). Since the vertical maps are monomorphisms and \( r \) is a categorical equivalence (see Example 4.5.2.13), it follows that \( c_{\Delta^1, \Delta^0} \) is also a categorical equivalence (Remark 4.5.3.9).

**Proposition 4.5.5.11.** Let \( X \) be a simplicial set. Then the comparison map \( c_{X, \Delta^0} : X \diamond \Delta^0 \to X \star \Delta^0 = X^\triangledown \) is a categorical equivalence of simplicial sets.

**Proof.** Combine Lemmas 4.5.5.9 and 4.5.5.10.

**Corollary 4.5.5.12.** Let \( f : A \to B \) be a right anodyne morphism of simplicial sets. Then the induced map

\[
\theta : B \coprod_A (A \diamond \Delta^0) \to B \diamond \Delta^0
\]

is a categorical equivalence of simplicial sets.

**Proof.** Proposition 4.3.6.4 guarantees that the natural map \( B \coprod_A A^\circ \to B^\circ \) is inner anodyne, and therefore a categorical equivalence (Corollary 4.5.2.11). Using Proposition 4.5.3.7 we conclude that the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & A^\circ \\
\downarrow f & & \downarrow f^\circ \\
B & \longrightarrow & B^\circ
\end{array}
\]

is categorical pushout square. It then follows from Theorem 4.5.5.7 and Proposition 4.5.3.5 that the equivalent diagram

\[
\begin{array}{ccc}
A & \longrightarrow & A \diamond \Delta^0 \\
\downarrow f & & \downarrow f \circ \text{id}_{\Delta^0} \\
B & \longrightarrow & B \diamond \Delta^0
\end{array}
\]

is also categorical pushout square, so that \( \theta \) is a categorical equivalence by virtue of Proposition 4.5.3.7.

**Proof of Theorem 4.5.5.7.** Let \( X \) and \( Y \) be arbitrary simplicial sets; we wish to show that the collapse map \( c_{X, Y} : X \diamond Y \to X \star Y \) is a categorical equivalence. By virtue of Lemma 4.5.5.9 we may assume without loss of generality that \( X = \Delta^1 \). Note that the map the \( c_{X, Y} \)
fits into a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X \times Y & \longrightarrow & (X \circ \Delta^0) \times Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \circ Y \\
\end{array}
\]

\[
\begin{array}{ccc}
& & c_{X,\Delta^0 \times \text{id}_Y} \\
\downarrow & & \downarrow \\
& & X \star Y.
\end{array}
\]

Note that the morphism \(c_{X,\Delta^0 \times \text{id}_Y}\) is a categorical equivalence by virtue of Proposition 4.5.5.11 and Remark 4.5.2.7. Consequently, to show that \(c_{X,Y}\) is a categorical equivalence, it will suffice to show that the square on the right is a categorical pushout diagram (Proposition 4.5.3.6). Note that the square on the left is a pushout diagram in which the horizontal maps are monomorphisms, hence also a categorical pushout diagram (Proposition 4.5.3.7). We are therefore reduced to showing that the outer rectangle is a categorical pushout square (Proposition 4.5.3.4).

Specializing now to the case \(X = \Delta^1\), we wish to show that the lower square in the commutative diagram

\[
\begin{array}{ccc}
\{1\} \times Y & \longrightarrow & \{1\}^\circ \times Y \\
\downarrow & & \downarrow \\
\Delta^1 \times Y & \longrightarrow & (\Delta^1)^\circ \times Y \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & \Delta^1 \star Y
\end{array}
\]

is a categorical pushout diagram. We first claim that the upper square is a categorical pushout: by virtue of Proposition 4.5.3.7, this is equivalent to the assertion that the induced map

\[
\theta : (\Delta^1 \times Y) \coprod_{\{1\} \times Y} ((\{1\}^\circ \times Y) \rightarrow (\Delta^1)^\circ \times Y
\]

is a categorical equivalence. This follows from Remark 4.5.2.7 since \(\theta\) factors as a product of the identity map \(\text{id}_Y\) with the inner horn inclusion \(\Lambda^2_1 \hookrightarrow \Delta^2\). To complete the proof, it will suffice to show that the outer rectangle is a categorical pushout diagram. Using the criterion of Proposition 4.5.3.7, we are reduced to showing that the map

\[
\rho : \Delta^1 \coprod_{\{1\} \times Y} ((\{1\}^\circ \times Y) \rightarrow \Delta^1 \star Y
\]

is a categorical equivalence.
is a categorical equivalence. Unwinding the definitions, we can identify \( \rho \) with the composition
\[
\Delta^1 \coprod \{(1) \circ Y\} \xrightarrow{\rho'} \Delta^1 \coprod \{(1) \ast Y\} \xrightarrow{\rho''} \Delta^1 \ast Y.
\]
Here the map \( \rho' \) is a categorical equivalence by virtue of Proposition 4.5.5.11 (together with Remark 4.5.3.9), and the map \( \rho'' \) is inner anodyne by virtue of Proposition 4.3.6.4.

4.5.6 Lifting Property of Isofibrations

We now characterize isofibrations between \( \infty \)-categories by means of a lifting property.

**Proposition 4.5.6.1.** Let \( F : C \to D \) be a functor between \( \infty \)-categories. Then \( F \) is an isofibration if and only if it satisfies the following condition:

\((*)\) Let \( B \) be a simplicial set and let \( A \subseteq B \) be a simplicial subset for which the inclusion \( A \hookrightarrow B \) is a categorical equivalence. Then every lifting problem
\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow & & \downarrow F \\
B & \xrightarrow{g} & D
\end{array}
\]

admits a solution.

We begin by proving a weak form of Proposition 4.5.6.1.

**Lemma 4.5.6.2.** Let \( C \) be an \( \infty \)-category, let \( B \) be a simplicial set, and let \( A \subseteq B \) be a simplicial subset with the property that the inclusion \( A \hookrightarrow B \) is a categorical equivalence. Then every diagram \( f_0 : A \to C \) can be extended to a diagram \( f : B \to C \).

**Proof.** By virtue of Corollary 4.4.5.14 the restriction map \( \theta : \text{Fun}(B,C)^\simeq \to \text{Fun}(A,C)^\simeq \) is a Kan fibration. Since the inclusion \( A \hookrightarrow B \) is a categorical equivalence, the map \( \theta \) is a homotopy equivalence of Kan complexes (Proposition 4.5.2.8). Invoking Proposition 3.3.7.4 we conclude that \( \theta \) is a trivial Kan fibration. In particular, it is surjective on vertices.

**Lemma 4.5.6.3.** Let \( C \) be an \( \infty \)-category, let \( B \) be a simplicial set, let \( A \subseteq B \) be a simplicial subset, and suppose we are given a pair of diagrams \( f, g : B \to C \) together with a natural transformation \( u_0 : f|_A \to f'|_A \). If the inclusion \( A \hookrightarrow B \) is a categorical equivalence, then \( u_0 \) can be lifted to a natural transformation \( u : f \to f' \). Moreover, if \( u_0 \) is a natural isomorphism, then \( u \) is automatically a natural isomorphism.
Proof. The existence of the natural isomorphism $u$ follows by applying Lemma 4.5.6.2 to the inclusion of simplicial sets

$$(\Delta^1 \times A) \coprod_{(\partial \Delta^1 \times A)} (\partial \Delta^1 \times B) \hookrightarrow \Delta^1 \times B,$$

which is a categorical equivalence by virtue of Corollary 4.5.3.10. We will complete the proof by showing that if $u_0$ is a natural isomorphism, then $u$ is a natural isomorphism.

Let us identify $u$ with a morphism of simplicial sets $v : B \rightarrow \text{Fun}(\Delta^1, C)$, and let $\text{Isom}(C)$ denote the full subcategory of $\text{Fun}(\Delta^1, C)$ spanned by the isomorphisms in $C$. Since $u_0$ is a natural isomorphism, then restriction $v|_A$ factors through the full subcategory $\text{Isom}(C)$. Invoking Lemma 4.5.6.2, we conclude that $v|_A$ extends to a diagram $v' : B \rightarrow \text{Isom}(C)$. Since the inclusion $A \hookrightarrow B$ is a categorical equivalence, the equality $v|_A = v'|_A$ guarantees that $v$ and $v'$ are isomorphic as objects of the $\infty$-category $\text{Fun}(B, \text{Fun}(\Delta^1, C))$. Since the full subcategory $\text{Isom}(C) \subseteq \text{Fun}(\Delta^1, C)$ is replete (Example 4.4.1.13), we conclude that $v$ also factors through $\text{Isom}(C)$, so that $u$ is a natural isomorphism by virtue of Theorem 4.4.4.4.

Proof of Proposition 4.5.6.1. Let $F : C \rightarrow D$ be a functor of $\infty$-categories. Assume first that $F$ satisfies condition (*) of Proposition 4.5.6.1, we will prove that $F$ is an isofibration. For $0 < i < n$, the inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a categorical equivalence (Corollary 4.5.2.11), so condition (*) guarantees that $F$ is an inner fibration. Fix an object $C \in C$ and an isomorphism $u : D \rightarrow F(C)$ in the $\infty$-category $D$; we wish to show that $u$ can be lifted to an isomorphism $\overline{u} : \overline{D} \rightarrow C$ in the $\infty$-category $C$. By virtue of Corollary 4.4.3.9 we can assume that $u = G(v)$ for some functor $G : \mathcal{E} \rightarrow D$, where $\mathcal{E}$ is a contractible Kan complex and $v : X \rightarrow Y$ is a morphism in $\mathcal{E}$. Since the inclusion $\{Y\} \hookrightarrow \mathcal{E}$ is a categorical equivalence (Example 4.5.1.13), condition (*) guarantees the existence of a solution to the lifting problem

$$\begin{array}{ccc}
\{Y\} & \xrightarrow{Y \mapsto C} & C \\
\downarrow & & \downarrow F \\
\mathcal{E} & \xrightarrow{G} & D, \\
\end{array}$$

Then $\overline{u} = \overline{G}(v)$ is an isomorphism of $C$ having the desired property.

Now suppose that the functor $F : C \rightarrow D$ is an isofibration; we wish to show that condition (*) is satisfied. Let $B$ be a simplicial set and $A \subseteq B$ a simplicial subset for which
the inclusion $A \hookrightarrow B$ is a categorical equivalence. We wish to show that every lifting problem

![Diagram](image)

admits a solution. Invoking Lemma 4.5.6.2, we see that $f_0$ can be extended to a morphism of simplicial sets $f' : B \to \mathcal{C}$. Let $\mathcal{F}'$ denote the composition $B \xrightarrow{f} \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D}$, so that $\mathcal{F}'|_A = f'_|_A$. Invoking Lemma 4.5.6.3, we conclude that there exists an isomorphism $\pi : \mathcal{F} \to \mathcal{F}'$ in the diagram $\infty$-category $\text{Fun}(B, \mathcal{D})$ whose image in $\text{Fun}(A, \mathcal{D})$ is the identity transformation $\text{id}_{\mathcal{F}'|_A}$. Applying Proposition 4.4.5.6, we deduce that $\pi$ can be lifted to an isomorphism $u : f \to f'$ in the diagram $\infty$-category $\text{Fun}(B, \mathcal{C})$ whose image in $\text{Fun}(A, \mathcal{C})$ is the identity transformation $\text{id}_{f_0}$. The diagram $f : B \to \mathcal{C}$ then satisfies $f|_A = f_0$ and $F \circ f = \mathcal{F}'$, as desired. \hfill \square

### 4.5.7 Isofibrations of Simplicial Sets

We now use the characterization of Proposition 4.5.6.1 to generalize the notion of iso-fibration to arbitrary simplicial sets.

**Definition 4.5.7.1.** Let $q : X \to S$ be a morphism of simplicial sets. We will say that $q$ is an *iso-fibration* if it satisfies the following condition:

(∗) Let $B$ be a simplicial set and let $A \subseteq B$ be a simplicial subset for which the inclusion $A \hookrightarrow B$ is a categorical equivalence. Then every lifting problem

![Diagram](image)

admits a solution.

**Remark 4.5.7.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories. We have now given two *a priori* different definitions of an iso-fibration from $\mathcal{C}$ to $\mathcal{D}$:

- According to Definition 4.4.1.4, an iso-fibration $F : \mathcal{C} \to \mathcal{D}$ is an inner fibration with the property that every isomorphism $u : D \to F(C)$ in the $\infty$-category $\mathcal{D}$ can be lifted to an isomorphism $\pi : D \to C$ in the $\infty$-category $\mathcal{C}$. 

• According to Definition 4.5.7.1, an isofibration \( F : C \to D \) is a morphism of simplicial sets which has the right lifting property with respect to all monomorphisms \( A \hookrightarrow B \) which are categorical equivalences.

However, these definitions are equivalent: this is the content of Proposition 4.4.5.1.

**Remark 4.5.7.3.** Let \( q : X \to S \) be an isofibration of simplicial sets. Then \( q \) is an inner fibration: that is, it has the right lifting property with respect to every horn inclusion \( \Lambda^n_i \hookrightarrow \Delta^n \) for \( 0 < i < n \) (such inclusions are categorical equivalences, by virtue of Corollary 4.5.2.11). In particular, for each vertex \( s \in S \), the fiber \( X_s = \{s\} \times_S X \) is an \( \infty \)-category (Remark 4.1.1.6). Moreover, if \( S \) is an \( \infty \)-category, then \( X \) is also an \( \infty \)-category (Remark 4.1.1.9).

**Example 4.5.7.4.** Let \( q : X \to S \) be a Kan fibration of simplicial sets. Then \( q \) is an isofibration. To prove this, we note that if a monomorphism of simplicial sets \( i : A \hookrightarrow B \) is a categorical equivalence, then it is a weak homotopy equivalence (Remark 4.5.2.4) and therefore anodyne (Corollary 3.3.7.5), so that \( q \) has the right lifting property with respect to \( i \) (Remark 3.1.2.6).

**Remark 4.5.7.5.** Let \( q : X \to S \) be a morphism of simplicial sets. Then \( q \) is an isofibration if and only if the opposite morphism \( q^{\text{op}} : X^{\text{op}} \to S^{\text{op}} \) is an isofibration.

**Remark 4.5.7.6.** The collection of isofibrations is closed under retracts. That is, given a diagram of simplicial sets

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow q & & \downarrow q' \\
S & \longrightarrow & S'
\end{array}
\]

where both horizontal compositions are the identity, if \( q' \) is an isofibration, then so is \( q \).

**Remark 4.5.7.7.** The collection isofibrations is closed under pullback. That is, given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow q' & & \downarrow q \\
S' & \longrightarrow & S
\end{array}
\]

where \( q \) is an isofibration, the morphism \( q' \) is also an isofibration.
Warning 4.5.7.8. Suppose we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{q'} & & \downarrow{q} \\
S' & \xrightarrow{f} & S,
\end{array}
\]

where \(q\) is an isofibration. If \(f\) is an equivalence of \(\infty\)-categories, then \(f'\) is also an equivalence of \(\infty\)-categories (Corollary 4.5.4.3). Beware that if \(f\) is merely assumed to be a categorical equivalence of simplicial sets, then it is not necessarily true that \(f'\) is a categorical equivalence of simplicial sets.

Remark 4.5.7.9. Let \(p : X \rightarrow Y\) and \(q : Y \rightarrow Z\) be isofibrations of simplicial sets. Then the composite map \((q \circ p) : X \rightarrow Z\) is an isofibration of simplicial sets.

We have the following generalization of Proposition 4.4.5.1:

**Proposition 4.5.7.10.** Let \(q : X \rightarrow S\) be an isofibration of simplicial sets and let \(i : A \hookrightarrow B\) be a monomorphism of simplicial sets. Then the restriction map

\[q' : \text{Fun}(B, X) \rightarrow \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)\]

is also an isofibration of simplicial sets.

**Proof.** Let \(B'\) be a simplicial set and let \(A' \subseteq B'\) be a simplicial subset for which the inclusion \(A' \hookrightarrow B'\) is a categorical equivalence. We wish to show that every lifting problem

\[
\begin{array}{ccc}
A' & \xrightarrow{\text{Fun}(B, X)} & \text{Fun}(B, X) \\
\downarrow{q'} & & \downarrow{q} \\
B' & \xrightarrow{\text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)} & \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)
\end{array}
\]

admits a solution. Unwinding the definitions, we are reduced to the problem of solving an associated lifting problem

\[
\begin{array}{ccc}
(A \times B') \coprod_{(A \times A')} (B \times A') & \xrightarrow{\text{Fun}(B, X)} & X \\
\downarrow{q} & & \downarrow{q} \\
B \times B' & \xrightarrow{\text{Fun}(B, S)} & S.
\end{array}
\]

The left vertical map in this diagram is a categorical equivalence by virtue of Corollary 4.5.3.10, so the existence of the desired solution follows from our assumption that \(q\) is an isofibration. \(\square\)
Corollary 4.5.7.11. Let $q : X \to S$ be an isofibration of simplicial sets. For every simplicial set $B$, the induced map $\text{Fun}(B, X) \to \text{Fun}(B, S)$ is also an isofibration.

Proof. Apply Proposition 4.5.7.10 in the special case $A = \emptyset$.  

Proposition 4.5.7.12. Let $q : X \to S$ be an isofibration of simplicial sets and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. If $i$ is a categorical equivalence, then the restriction map

$q' : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)$

is a trivial Kan fibration.

Proof. Let $B'$ be a simplicial set and let $A' \subseteq B'$ be a simplicial subset. We wish to show that every lifting problem

```
\begin{array}{ccc}
A' & \xrightarrow{q'} & \text{Fun}(B, X) \\
\downarrow & & \downarrow q \\
B' & \xrightarrow{} & \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)
\end{array}
```

admits a solution. Unwinding the definitions, we are reduced to the problem of solving an associated lifting problem

```
\begin{array}{ccc}
(A \times B') \coprod_{(A \times A')(B \times A')}(B \times A') & \xrightarrow{} & X \\
\downarrow & & \downarrow q \\
B \times B' & \xrightarrow{} & S.
\end{array}
```

The left vertical map in this diagram is a categorical equivalence by virtue of Corollary 4.5.3.10, so the existence of the desired solution follows from our assumption that $q$ is an isofibration.

Corollary 4.5.7.13. Let $C$ be an $\infty$-category and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. If $i$ is a categorical equivalence, then the restriction functor $\text{Fun}(B, C) \to \text{Fun}(A, C)$ is a trivial Kan fibration of simplicial sets.

Proof. Apply Proposition 4.5.7.12 in the special case $S = \Delta^0$.  

Proposition 4.5.7.14. Let $q : X \to S$ be a morphism of simplicial sets. Then $q$ is a trivial Kan fibration if and only if it both an isofibration and a categorical equivalence.

Proof. If $q$ is a trivial Kan fibration, then it is an isofibration by virtue of Example 4.5.7.4 and a categorical equivalence by virtue of Proposition 4.5.2.9. Conversely, suppose that $q$ is both an isofibration and a categorical equivalence. Using Exercise 3.1.6.11 we can write $q$ as
a composition $X \xrightarrow{q'} Y \xrightarrow{q''} S$, where $q'$ is a monomorphism and $q''$ is a trivial Kan fibration. Then $q''$ is a categorical equivalence (Proposition 4.5.2.9), so that $q'$ is also a categorical equivalence (Remark 4.5.2.6). Invoking our assumption that $q$ is an isofibration, we conclude that the lifting problem

$$
\begin{array}{ccc}
X & \xrightarrow{\text{id}} & X \\
\downarrow^{q'} & & \downarrow^{q} \\
Y & \xrightarrow{q''} & S
\end{array}
$$

admits a solution. It follows that $q$ is a retract of the morphism $q''$, and is therefore also a trivial Kan fibration. \qed
Chapter 5

The ∞-Category of Spaces

5.1 Fibers of Left and Right Fibrations

Let \( q : X \to S \) be a left fibration of simplicial sets (Definition 4.2.1.1). For each vertex \( s \in S \), let \( X_s = \{s\} \times_S X \) denote the fiber of \( q \) over \( s \). Our assumption that \( q \) is a left fibration guarantees that each fiber \( X_s \) is a Kan complex (Corollary 4.4.2.2). Our goal in this section is to study the relationship between these Kan complexes as \( s \) varies. In §5.1.1, we associate to each edge \( e : s \to s' \) of \( S \) a morphism of Kan complexes \( e_! : X_s \to X_{s'} \), which is well-defined up to homotopy and which we refer to as the covariant transport morphism (Construction 5.1.1.5). By means of this construction, we can promote the construction \( s \mapsto X_s \) to a functor from the homotopy category \( hS \) to the homotopy category \( h\text{Kan} \) of Kan complexes, which we will refer to as the covariant transport functor (Construction 5.1.1.7). Dually, if \( q \) is a right fibration, we can associate to each edge \( e : s \to s' \) of \( S \) a morphism of Kan complexes \( e^* : X_{s'} \to X_{s} \), which we refer to as the contravariant transport morphism (Notation 5.1.1.11), which allows us to promote the construction \( s \mapsto X_s \) to a functor \( hS^{\text{op}} \to h\text{Kan} \) which we refer to as the contravariant transport functor (Construction 5.1.1.12).

Let \( q : X \to S \) be a left fibration between simplicial sets, and let \( e_! : X_s \to X_{s'} \) be the covariant transport morphism associated to an edge \( e : s \to s' \) of the simplicial set \( S \). If \( q \) is a Kan fibration, then the morphism \( e_! \) is a homotopy equivalence; more precisely, it is homotopy inverse to the contravariant transport morphism \( e^* : X_{s'} \to X_{s} \) (Proposition 5.1.2.1). In §5.1.2 we will establish a converse to this: if \( q : X \to S \) is a left fibration for which every covariant transport morphism \( e_! \) is a homotopy equivalence, then \( q \) is a Kan fibration (Theorem 5.1.2.2). Note that in some special cases, the latter hypothesis is automatically satisfied: for example, if \( S \) is a Kan complex (see Lemma 5.1.2.4) or if the fibers of \( q \) are contractible Kan complexes (see Lemma 5.1.2.7).

In §5.1.3 we study the relationship between different left fibrations over the same
simplicial set $S$. Suppose we are given a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & & S
\end{array}
$$

where $q_X$ and $q_Y$ are left fibrations. For each vertex $s \in S$, the morphism $f$ restricts to a map of Kan complexes $f_s : X_s \to Y_s$. Our main result asserts that, if each $f_s$ is a homotopy equivalence of Kan complexes, then $f$ is a homotopy equivalence relative to $S$ (Theorem 5.1.3.3): that is, there exists a morphism of simplicial sets $g : Y \to X$ compatible with the projection to $S$, such that $g \circ f$ and $f \circ g$ are homotopic to the identity morphisms $\text{id}_X$ and $\text{id}_Y$ via homotopies which are compatible with the projection to $S$ (see Definition 5.1.3.1). Beware that this implication need not hold if $q_X$ and $q_Y$ are not assumed to be left fibrations. To remedy the situation, it is convenient to introduce a weaker notion of relative homotopy equivalence which we refer to as covariant equivalence (Definition 5.1.4.1), which we introduce and study in §5.1.4.

**Remark 5.1.0.1.** Let $h\text{Kan}$ denote the homotopy category of Kan complexes (Construction 3.1.4.10). Then $h\text{Kan}$ can be identified with the homotopy category of the $\infty$-category $S$ of Construction 3.1.4.12. If $q : X \to S$ is a left fibration of simplicial sets, then Construction 5.1.1.7 determines a covariant transport functor

$$T : hS \to h\text{Kan}.$$ 

In §[?], we show that (up to equivalence) $T$ can be promoted to a map of simplicial sets $S \to S$. In other words, the formation of covariant transport morphisms $(e : s \to s') \mapsto (e! : X_s \to X_{s'})$ is compatible with composition not only up to homotopy, but up to coherent homotopy.

**5.1.1 Covariant Transport**

Let $q : X \to S$ be a left fibration of simplicial sets. Then, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a Kan complex (Corollary 4.4.2.2). Our goal in this section is to show that the construction $s \mapsto X_s$ determines a functor from the homotopy category $hS$ (Definition 1.2.5.1) to the homotopy category of Kan complexes $h\text{Kan}$ (Construction 5.1.1.7). For a homotopy-coherent refinement of this result, we refer the reader to §[?].

**Definition 5.1.1.1.** Let $q : X \to S$ be a left fibration of simplicial sets, let $e : s \to s'$ be an edge of the simplicial set $S$, and let $f : X_s \to X_{s'}$ be a morphism of simplicial sets. We will
say that \( f \) is given by covariant transport along \( e \) if there exists a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\Delta^1 \times X_s & \xrightarrow{h} & X \\
\downarrow & & \downarrow q \\
\Delta^1 & \xrightarrow{e} & S
\end{array}
\]

satisfying \( h|_{\{0\} \times X_s} = \text{id}_{X_s}, \ h|_{\{1\} \times X_s} = f \), and the left vertical map is given by projection onto the first factor. In this case, we also say that the morphism \( h \) witnesses \( f \) as given by covariant transport along \( e \).

**Example 5.1.1.2.** Let \( q : X \to S \) be a left fibration of simplicial sets and let \( s \in S \) be a vertex. Then a morphism of Kan complexes \( f : X_s \to X_s \) is given by covariant transport along the degenerate edge \( \text{id}_s : s \to s \) if and only if \( f \) is homotopic to the identity morphism \( \text{id}_{X_s} \).

**Remark 5.1.1.3.** Suppose we are given a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{q} & X \\
\downarrow \bar{\pi} & & \downarrow q \\
\bar{S} & \xrightarrow{\bar{q}} & S
\end{array}
\]

Let \( \bar{\pi} : \bar{s} \to \bar{s}' \) be an edge of the simplicial set \( \bar{S} \) having image \( e : s \to s' \) in the simplicial set \( S \). Then a morphism of Kan complexes \( f : X_s \to X_{s'} \) is given by covariant transport along \( \bar{\pi} \) if and only if, when viewed as a morphism from \( X_s \) to \( X_{s'} \), it is given by covariant transport along \( e \).

**Proposition 5.1.1.4.** Let \( q : X \to S \) be a left fibration of simplicial sets and let \( e : s \to s' \) be an edge of \( S \). Then:

1. There exists a morphism of Kan complexes \( f : X_s \to X_{s'} \) which is given by covariant transport along \( e \).
2. Let \( g : X_s \to X_{s'} \) be any morphism of Kan complexes. Then \( g \) is given by covariant transport along \( e \) if and only if it is homotopic to \( f \).

**Proof.** We first prove (1). Let \( s, s', \) and \( e \) denote the images of \( s, s', \) and \( e \) under the diagonal map \( S \to \text{Fun}(X_s, S) \). Let \( \bar{s} \) denote the vertex of the simplicial set \( \text{Fun}(X_s, X) \) corresponding to the inclusion map \( X_s \hookrightarrow X \). Applying Corollary 4.2.3.2, we see that postcomposition
with \( q \) induces a left fibration \( Q : \text{Fun}(X_s, X) \rightarrow \text{Fun}(X_s, S) \) satisfying \( Q(\overline{s}) = \underline{s} \). It follows
that \( \underline{e} \) can be lifted to an edge \( \overline{e} : \overline{s} \rightarrow \overline{s}' \) of the simplicial set \( \text{Fun}(X_s, X) \). Unwinding the definitions, we see that \( \overline{s}' \) can be identified with a morphism of Kan complexes \( f : X_s \rightarrow X_{s'} \) and that \( \overline{e} \) can be identified with a morphism \( h : \Delta^1 \times X_s \rightarrow X \) which witnesses \( f \) as given by covariant transport along the edge \( e \).

We now prove (2). Let \( g : X_s \rightarrow X_{s'} \) be a morphism of Kan complexes. Then \( g \) can be identified with a vertex \( \overline{s}'' \) of the simplicial set \( \text{Fun}(X_s, X) \) satisfying \( Q(\overline{s}'') = \underline{s}' \). Unwinding the definitions, we see that \( f \) is homotopic to \( g \) if and only if the following condition is satisfied:

\[
(\ast) \text{ There exists an edge } \overline{e}' : \overline{s}' \rightarrow \overline{s}'' \text{ of the simplicial set } \text{Fun}(X_s, X) \text{ satisfying } Q(\overline{e}') = \underline{s}''.
\]

Similarly, \( g \) is given by covariant transport along \( e \) if and only if the following alternative condition is satisfied:

\[
(\ast') \text{ There exists an edge } \overline{e}'' : \overline{s} \rightarrow \overline{s}'' \text{ of the simplicial set } \text{Fun}(X_s, X) \text{ satisfying } Q(\overline{e}'') = \underline{e}.
\]

To complete the proof, it suffices to show that conditions \((\ast)\) and \((\ast')\) are equivalent to one another. The implication \((\ast) \Rightarrow (\ast')\) follows from the observation that the left fibration \( Q : \text{Fun}(X_s, X) \rightarrow \text{Fun}(X_s, S) \) has the right lifting property with respect to the horn inclusion \( \Lambda^2_0 \hookrightarrow \Delta^2 \), and the implication \((\ast') \Rightarrow (\ast)\) follows from the observation that \( Q \) has the right lifting property with respect to the horn inclusion \( \Lambda^2_1 \hookrightarrow \Delta^2 \).

\[\square\]

**Notation 5.1.1.5.** Let \( q : X \rightarrow S \) be a left fibration of simplicial sets, and let \( e : s \rightarrow s' \) be an edge of the simplicial set \( S \). We will often use the symbol \( e_1 \) to denote a morphism of Kan complexes \( X_s \rightarrow X_{s'} \) which is given by covariant transport along \( e \). By virtue of Proposition 5.1.1.4, such a morphism exists and is uniquely determined up to homotopy.

**Proposition 5.1.1.6** (Transitivity). Let \( q : X \rightarrow S \) be a left fibration of simplicial sets and let \( \sigma \) be a 2-simplex of \( S \), whose faces we display as a diagram

\[
\begin{array}{ccc}
\text{s'} & \xrightarrow{e'} & \text{s''} \\
\downarrow{e} & & \downarrow{e''} \\
\text{s} & \xrightarrow{e''} & \text{s''}.
\end{array}
\]

Let \( f : X_s \rightarrow X_{s'} \) be a morphism of Kan complexes given by covariant transport along \( e \), and let \( f' : X_{s'} \rightarrow X_{s''} \) be a morphism of Kan complexes given by covariant transport along \( e' \). Then the composite map \( f' \circ f : X_s \rightarrow X_{s''} \) is given by covariant transport along \( e'' \).
5.1. FIBERS OF LEFT AND RIGHT FIBRATIONS

Proof. For every simplex \( \tau \) of the simplicial set \( S \), let \( \tau \) denote its image in the simplicial set \( \text{Fun}(X_s, S) \). Let \( q : \text{Fun}(X_s, X) \rightarrow \text{Fun}(X_s, S) \) be given by postcomposition with \( q \). Note that the inclusion map \( X_s \hookrightarrow X \), the morphism \( f : X_s \rightarrow X_{s'} \), and the composite map \( f' \circ f : X_s \rightarrow X_{s''} \) can be identified with vertices \( \pi, \pi', \) and \( \pi'' \) of the simplicial set \( \text{Fun}(X_s, X) \) satisfying \( Q(\pi) = s, Q(\pi') = s', \) and \( Q(\pi'') = s'' \). Let \( h : \Delta^1 \times X_s \rightarrow X \) witness \( f \) as given by covariant transport along \( e \), and let \( h' : \Delta^1 \times X_{s'} \rightarrow X \) witness \( f' \) as given by covariant transport along \( e' \). Then \( h \) can be identified with an edge \( e : s \rightarrow s' \) of the simplicial set \( \text{Fun}(X_s, X) \) satisfying \( Q(e) = e \). Similarly, the composite map

\[
\Delta^1 \times X_s \xrightarrow{id \times f} \Delta^1 \times X_{s'} \xrightarrow{h'} X
\]

can be identified with an edge \( e' \) of \( \text{Fun}(X_s, X) \) satisfying \( Q(e') = e' \). Since \( Q \) is a left fibration (Corollary 4.2.3.2), the lifting problem

\[
\begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{(\pi', \pi)} & \text{Fun}(X_s, X) \\
\downarrow & & \downarrow Q \\
\Delta^2 & \xrightarrow{\pi} & \text{Fun}(X_s, S)
\end{array}
\]

admits a solution. Unwinding the definitions, we see that \( d_1(\pi) \) can be identified with a map of simplicial sets \( \Delta^1 \times X_s \rightarrow X \) which witnesses \( f' \circ f \) as given by covariant transport along \( e'' \).

\( \Box \)

Construction 5.1.1.7 (The Covariant Transport Functor). Let \( q : X \rightarrow S \) be a left fibration of simplicial sets, and let \( h\text{Kan} \) denote the homotopy category of Kan complexes (Construction 3.1.4.10). Combining Proposition 5.1.1.4, Proposition 5.1.1.6, and Example 5.1.1.2, we deduce that there is a unique map of simplicial sets \( T : S \rightarrow N_\bullet(\text{hKan}) \) with the following properties:

- For each vertex \( s \in S \), we have \( T(s) = X_s \).
- For each edge \( e : s \rightarrow s' \) of the simplicial set \( S \), we have \( T(e) = [e] \). Here \( e : X_s \rightarrow X_{s'} \) denotes a morphism of Kan complexes given by covariant transport along \( e \), and \([e]\) denotes its homotopy class.

The morphism \( T \) determines a functor from the homotopy category \( hS \) (Notation 1.2.5.3) to the homotopy category \( h\text{Kan} \), which we will also denote by \( T : hS \rightarrow h\text{Kan} \). We will refer to \( T \) as the covariant transport functor.

We now consider a dual version of Definition 5.1.1.
Definition 5.1.1.8. Let \( q : X \to S \) be a right fibration of simplicial sets, let \( e : s \to s' \) be an edge of the simplicial set \( S \), and let \( g : X_{s'} \to X_s \) be a morphism of simplicial sets. We will say that \( g \) is given by contravariant transport along \( e \) if there exists a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
\Delta^1 \times X_{s'} & \xrightarrow{h} & X \\
\downarrow & & \downarrow q \\
\Delta^1 & \xrightarrow{e} & S
\end{array}
\]

satisfying \( h|_{\{1\} \times X_{s'}} = \text{id}_{X_{s'}} \), \( h|_{\{0\} \times X_s} = g \), and the left vertical map is given by projection onto the first factor.

Remark 5.1.1.9. In the situation of Definition 5.1.1.8, the morphism \( g \) is given by contravariant transport along \( e \) if and only if the opposite map \( g^\text{op} : X_{s'}^\text{op} \to X_s^\text{op} \) is given by covariant transport along \( e \) with respect to the left fibration \( q^\text{op} : X^\text{op} \to S^\text{op} \).

Proposition 5.1.1.10. Let \( q : X \to S \) be a left fibration of simplicial sets and let \( e : s \to s' \) be an edge of the simplicial set \( S \). Then:

1. There exists a morphism of Kan complexes \( g : X_{s'} \to X_s \) which is given by covariant transport along \( e \).
2. Let \( f : X_{s'} \to X_s \) be any morphism of Kan complexes. Then \( f \) is given by covariant transport along \( e \) if and only if it is homotopic to \( g \).

Proof. Apply Proposition 5.1.1.4 to the opposite map \( q^\text{op} : X^\text{op} \to S^\text{op} \).

Notation 5.1.1.11. Let \( q : X \to S \) be a right fibration of simplicial sets, and let \( e : s \to s' \) be an edge of the simplicial set \( S \). We will often use the symbol \( e^* \) to denote a morphism of Kan complexes \( X_{s'} \to X_s \) which is given by covariant transport along \( e \). By virtue of Proposition 5.1.1.10, such a morphism exists and is uniquely determined up to homotopy.

Construction 5.1.1.12 (The Contravariant Transport Functor). Let \( q : X \to S \) be a right fibration of simplicial sets, and let \( h\text{Kan} \) denote the homotopy category of Kan complexes (Construction 3.1.4.10). Dualizing Construction 5.1.1.7, we deduce that there is a unique map of simplicial sets \( T : S^\text{op} \to N_\bullet(h\text{Kan}) \) with the following properties:

- For each vertex \( s \in S \), we have \( T(s) = X_s \).
- For each edge \( e : s \to s' \) of the simplicial set \( S \), we have \( T(e) = [e^*] \). Here \( e^* : X_{s'} \to X_s \) denotes a morphism of Kan complexes given by contravariant transport along \( e \), and \([e^*]\) denotes its homotopy class.
The morphism $T$ determines a functor from the homotopy category $hS^{\text{op}}$ (Notation 1.2.5.3) to the homotopy category $h\text{Kan}$, which we will also denote by $T : hS^{\text{op}} \to h\text{Kan}$. We will refer to $T$ as the contravariant transport functor.

5.1.2 A Characterization of Kan Fibrations

Let $q : X \to S$ be a Kan fibration of simplicial sets. Then $q$ is both a left fibration and a right fibration (Example 4.2.1.5). Consequently, to each edge $e : s \to s'$ of the simplicial set $S$, we can associate both a covariant transport morphism $e_! : X_s \to X_{s'}$ (Notation 5.1.1.5) and a contravariant transport morphism $e^* : X_{s'} \to X_s$ (Notation 5.1.1.11).

Proposition 5.1.2.1. Let $q : X \to S$ be a Kan fibration of simplicial sets and let $e : s \to s'$ be an edge of $S$. Then the morphisms $e_! : X_s \to X_{s'}$ and $e^* : X_{s'} \to X_s$ are homotopy inverse to one another.

Proof. We will show that the composition $e^* \circ e_!$ is homotopic to the identity map $\text{id}_{X_s}$; a similar argument will show that $e_! \circ e^*$ is homotopic to the identity map $\text{id}_{X_{s'}}$. As in the proof of Proposition 5.1.1.4, we let $e$ denote the image of $e$ along the diagonal map $S \to \text{Fun}(X_s, S)$, and let $Q : \text{Fun}(X_s, X) \to \text{Fun}(X_s, S)$ be the morphism given by postcomposition with $q$. Let $h : \Delta^1 \times X_s \to X$ be a morphism of simplicial sets which exhibits $e_!$ as given by covariant transport along $e$, so that $h$ can be identified with an edge $\tau : \text{id}_{X_s} \to e_!$ of the simplicial set $\text{Fun}(X_s, X)$ satisfying $Q(\tau) = e$. Let $h' : \Delta^1 \times X_{s'} \to X$ be a morphism of simplicial sets which exhibits $e^*$ as given by contravariant transport along $e$. Then the composite map

$$\Delta^1 \times X_s \xrightarrow{\text{id} \times e_!} \Delta^1 \times X_{s'} \xrightarrow{h'} X$$

can be identified with an edge $\tau' : (e^* \circ e_!) \to e_!$ of $\text{Fun}(X_s, X)$ satisfying $Q(\tau') = e$. Since $Q$ is a Kan fibration (Corollary 3.1.3.2), the lifting problem

\[\Lambda^2_2 \xrightarrow{(\tau', \tau, \bullet)} \text{Fun}(X_s, X) \]

admits a solution. Here $\sigma$ is a 2-simplex of the simplicial set $\text{Fun}(X_s, X)$ which we can display as a diagram

\[
\begin{array}{ccc}
\Lambda^2_2 & \xrightarrow{(\tau', \tau, \bullet)} & \text{Fun}(X_s, X) \\
\sigma \downarrow & & \downarrow Q \\
\Delta^2 & \xrightarrow{s_0(e)} & \text{Fun}(X_s, S) \\
\end{array}
\]
By construction, the edge $h''$ can be regarded as a homotopy from $\text{id}_{X_s}$ to the composite map $e^* \circ e_!$. □

It follows from Proposition 5.1.2.1 that if $q : X \to S$ is a Kan fibration and $e : s \to s'$ is an edge of the simplicial set $S$, then the transport morphisms $e_! : X_s \to X_{s'}$ and $e^* : X_{s'} \to X_s$ are homotopy equivalences. Our goal in this section is to prove the converse:

**Theorem 5.1.2.2.** Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism $q$ is a Kan fibration.
2. The morphism $q$ is a left fibration and, for every edge $e : s \to s'$ of the simplicial set $S$, the covariant transport morphism $e_! : X_s \to X_{s'}$ is a homotopy equivalence.
3. The morphism $q$ is a right fibration and, for every edge $e : s \to s'$ of the simplicial set $S$, the contravariant transport morphism $e^* : X_{s'} \to X_s$ is a homotopy equivalence.

The proof of Theorem 5.1.2.2 will require some preliminaries.

**Lemma 5.1.2.3.** Let $q : X \to S$ be a left fibration of simplicial sets, where $S$ is a Kan complex. Suppose that $q$ induces a surjection $\pi_0(q) : \pi_0(X) \to \pi_0(S)$. Then $q$ is surjective on vertices.

**Proof.** Fix a vertex $s \in S$. Since $\pi_0(q)$ is surjective, there exists a vertex $x \in X$ for which $q(x)$ and $s$ belong to the same connected component of $\pi_0(S)$. Since $S$ is a Kan complex, we can choose an edge $e : q(x) \to s$ in the simplicial set $S$. Our assumption that $q$ is a left fibration guarantees that we can write $e = q(\overline{e})$ for some edge $\overline{e} : x \to \overline{s}$ of the simplicial set $X$. In particular, there exists a vertex $\overline{s} \in X$ satisfying $q(\overline{s}) = s$. □

**Lemma 5.1.2.4.** Let $q : X \to S$ be a morphism of simplicial sets, where $S$ is a Kan complex. Then the following conditions are equivalent:

1. The morphism $q$ is a Kan fibration.
2. The morphism $q$ is a left fibration.
3. The morphism $q$ is a right fibration.

**Proof.** We will show that (1) and (2) are equivalent (the equivalence of (1) and (3) follows from a similar argument). The implication (1) $\Rightarrow$ (2) is contained in Example 4.2.1.5 (and does not require the assumption that $S$ is a Kan complex). Conversely, suppose that $q$ is a left fibration. Then the projection map $X \to \Delta^0$ is a left fibration (since it can be written as a composition of $q$ with the Kan fibration $S \to \Delta^0$). Invoking Proposition 4.4.2.1, we
deduce that $X$ is a Kan complex. To prove that $q$ is a Kan fibration, we must show that for every anodyne morphism of simplicial sets $i : A \hookrightarrow B$, the induced map

$$\theta : \text{Fun}(B, X) \to \text{Fun}(A, X) \times_{\text{Fun}(A, S)} \text{Fun}(B, S)$$

is surjective on vertices (Remark 3.1.2.6). Note that we have a commutative diagram of simplicial sets

$$\begin{array}{ccc}
\text{Fun}(B, X) & \longrightarrow & \text{Fun}(B, S) \\
\downarrow & & \downarrow \\
\text{Fun}(A, X) & \longrightarrow & \text{Fun}(A, S),
\end{array} \tag{5.1}
$$

where the vertical maps are trivial Kan fibrations (since $i$ is anodyne and $X$ and $S$ are Kan complexes; see Corollary 3.1.3.6). Invoking Corollary 3.4.1.3, we see that the diagram (5.1) is homotopy Cartesian. It follows that $\theta$ is a weak homotopy equivalence of simplicial sets (Example 3.4.1.5), and is therefore bijective on connected components (Remark 3.1.5.4). Since $\theta$ is a left fibration of simplicial sets (Proposition 4.2.3.1), it is surjective on vertices by virtue of Lemma 5.1.2.3. \hfill \square

**Remark 5.1.2.5.** Assuming Theorem 5.1.2.2, one can give a more direct proof of Lemma 5.1.2.4. Let $q : X \to S$ be a left fibration of simplicial sets, where $S$ is a Kan complex. To show that $q$ is a Kan fibration, it will suffice (by virtue of Theorem 5.1.2.2) to show that the covariant transport functor $\text{h}S \to \text{hKan}$ carries every morphism in $\text{h}S$ to an invertible morphism in $\text{hKan}$. This is clear, since the homotopy category $\text{h}S$ is a groupoid (Proposition 1.3.6.11).

**Lemma 5.1.2.6.** Let $q : X \to S$ be a left fibration of simplicial sets, let $s \in S$ be a vertex having the property that the Kan complex $X_s = \{s\} \times_S X$ is contractible, and let $\sigma : \Delta^n \to S$ be an $n$-simplex of $S$ satisfying $\sigma(n) = s$. Then every lifting problem

$$\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{\sigma} & X \\
\downarrow & \nearrow & \downarrow q \\
\Delta^n & \xrightarrow{\sigma} & S
\end{array}$$

admits a solution.

**Proof.** When $n = 0$, the desired result follows from the fact that the fiber $X_s$ is nonempty. We may therefore assume without loss of generality that $n > 0$. Replacing $q$ by the projection
map $\Delta^n \times S X \to \Delta^n$, we may further reduce to the special case where $S = \Delta^n$ and $\sigma$ is the identity map. In this case, our assumption that $q$ is a left fibration guarantees that $X$ is an $\infty$-category (Remark 4.1.1.9).

Let $\eta : \Delta^1 \times \Delta^n \to \Delta^n$ be the morphism given on vertices by $h(i, j) = \begin{cases} j & \text{if } i = 0 \\ n & \text{if } i = 1. \end{cases}$ Since the inclusion $\{0\} \times \partial \Delta^n \to \Delta^1 \times \partial \Delta^n$ is left anodyne (Proposition 4.2.3.3), our assumption that $q$ is a left fibration guarantees the existence of a morphism $h' : \Delta^1 \times \partial \Delta^n \to X$ satisfying $\eta(h'|_{\{0\} \times \partial \Delta^n} = \sigma_0$ and $q \circ h' = \eta(h|_{\Delta^1 \times \partial \Delta^n})$. We will complete the proof by showing that $h'$ can be extended to a map $h : \Delta^1 \times \Delta^n \to X$ satisfying $q \circ h = \eta(h|_{\{0\} \times \Delta^n})$.

Let $Y(0) \subset Y(1) \subset Y(2) \subset \cdots \subset Y(n + 1) = \Delta^1 \times \Delta^n$ denote the filtration constructed in the proof of Lemma 3.1.2.9. Then $Y(0)$ can be described as the pushout

$$\left(\Delta^1 \times \partial \Delta^n\right) \coprod_{\{1\} \times \partial \Delta^n} \left(\{1\} \times \Delta^n\right).$$

Using our assumption that the fiber $X_s$ is a contractible Kan complex, we see that $h'$ can be extended to a morphism of simplicial sets $h_0 : Y(0) \to X$ satisfying $q \circ h_0 = \eta(h|_{Y(0)})$. We claim that $h_0$ can be extended to a compatible sequence of maps $h_i : Y(i) \to X$ satisfying $q \circ h_i = \eta(h|_{Y(i)})$. To prove this, we recall that each $Y(i + 1)$ can be realized as a pushout of the horn inclusion $\Lambda_{i+1}^n \to \Delta^{n+1}$, so that the construction of $h_{i+1}$ from $h_i$ can be rephrased as a lifting problem

$$\begin{array}{c}
\Lambda_{i+1}^n \searrow f_{i+1} \\
\downarrow \quad q \\
\Delta^{n+1} \to S.
\end{array}$$

For $0 \leq i < n$, this lifting problem is automatically solvable by virtue of our assumption that $q$ is a left fibration. In the case $i = n$, the edge

$$\Delta^1 \simeq N_{\bullet}(\{n, n + 1\}) \hookrightarrow \Lambda_{n+1}^{n+1} \xrightarrow{f_0} X$$

is an edge of the Kan complex $X_s$, and is therefore an isomorphism in the $\infty$-category $X$ (Proposition 1.3.6.11). In this case, the existence of the desired extension follows from Theorem 4.4.2.5. We complete the proof by taking $h = h_{n+1}$. \qed

**Lemma 5.1.2.7.** Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:

1. The morphism $q$ is a trivial Kan fibration.
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(2) The morphism \( q \) is a left fibration and, for every vertex \( s \in S \), the fiber \( X_s = \{s\} \times_S X \) is a contractible Kan complex.

(3) The morphism \( q \) is a right fibration and, for every vertex \( s \in S \), the fiber \( X_s = \{s\} \times_S X \) is a contractible Kan complex.

Proof. The implication (1) \( \Rightarrow \) (2) is immediate, and the converse follows from Lemma 5.1.2.6. The equivalence (1) \( \iff \) (3) follows by a similar argument.

Remark 5.1.2.8. Assuming Theorem 5.1.2.2, one can give a more direct proof of Lemma 5.1.2.7. Let \( q : X \to S \) be a left fibration of simplicial sets with the property that, for every vertex \( s \in S \), the fiber \( X_s \) is a contractible Kan complex. For each edge \( e : s \to s' \) of \( S \), the covariant transport \( e_1 : X_s \to X_{s'} \) is a morphism between contractible Kan complexes, and is therefore automatically a homotopy equivalence. Applying Theorem 5.1.2.2, we deduce that \( q \) is a Kan fibration. Since the fibers of \( q \) are contractible, Proposition 3.3.7.4 guarantees that \( q \) is a trivial Kan fibration.

Proof of Theorem 5.1.2.2. We will show that (1) \( \iff \) (2); the proof of the equivalence (1) \( \iff \) (3) is similar. The implication (1) \( \Rightarrow \) (2) is immediate from Proposition 5.1.2.1. For the converse, assume that \( q : X \to S \) is a left fibration of simplicial sets and that, for every edge \( e : s \to s' \) of \( S \), the covariant transport morphism \( e_1 : X_s \to X_{s'} \) is a homotopy equivalence. We wish to show that \( q \) is a Kan fibration. By virtue of Example 4.2.1.5, it will suffice to show that \( q \) is a right fibration. By virtue of Proposition 4.2.4.1, this is equivalent to the assertion that the induced map

\[
\theta : \text{Fun}(\Delta^1, X) \to \text{Fun}(\{1\}, X) \times_{\text{Fun}(\{1\}, S)} \text{Fun}(\Delta^1, S)
\]

is a trivial Kan fibration. Note that our assumption that \( q \) is a left fibration guarantees that \( \theta \) is also a left fibration (Proposition 4.2.3.1). It will therefore suffice to show that the fibers of \( \theta \) are contractible Kan complexes (Lemma 5.1.2.7).

Fix an edge \( e : s \to s' \) of the simplicial set \( S \) and set \( Y = \text{Fun}(\Delta^1, X) \times_{\text{Fun}(\Delta^1, S)} \{e\} \). Then evaluation at the vertex \( 1 \in \Delta^1 \) induces a morphism \( \theta_e : Y \to X_{s'} \). Note that \( \theta_e \) is a pullback of \( \theta \), and is therefore also a left fibration. Since \( X_{s'} \) is a Kan complex (Corollary 4.4.2.2), Lemma 5.1.2.4 guarantees that \( \theta_e \) is Kan fibration (so \( Y \) is also a Kan complex). Evaluation at the vertex \( 0 \in \Delta^1 \) induces another morphism of simplicial sets \( u : Y \to X_s \). Since \( q \) is a left fibration, the morphism \( u \) is a trivial Kan fibration. By construction, the covariant transport morphism \( e_1 : X_s \to X_{s'} \) can be realized as a composition

\[
X_s \xrightarrow{v} Y \xrightarrow{\theta_e} X_{s'},
\]

where \( v \) is a section of \( u \) (and therefore a homotopy equivalence). Consequently, our assumption that \( e_1 \) is a homotopy equivalence of Kan complexes guarantees that \( \theta_e \) is a
homotopy equivalence of Kan complexes (Remark 3.1.5.10). Applying Corollary 3.2.6.9, we deduce that the fibers of $\theta_e$ are contractible Kan complexes. Since every fiber of $\theta$ can also be viewed as a fiber of $\theta_e$ for some edge $e$ of the simplicial set $S$, it follows that the fibers of $\theta$ are also contractible Kan complexes, as desired.

5.1.3 Relative Homotopy Equivalences

Recall that a morphism of simplicial sets $f : X \to Y$ is said to be a homotopy equivalence if there exists a morphism $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity morphisms $\text{id}_Y$ and $\text{id}_X$, respectively (Definition 3.1.5.1). We now introduce a relative version of this condition.

**Definition 5.1.3.1.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & \xleftarrow{q_V} & 
\end{array}
$$

so that we can regard $f$ as a vertex of the simplicial set $\text{Fun}_/S(X,Y)$ of Construction 3.1.3.7. We say that $f$ is a homotopy equivalence relative to $S$ (Definition 3.1.3.1) if there exists a morphism $g \in \text{Fun}_/S(Y,X)$ such that $g \circ f$ and $\text{id}_X$ belong to the same connected component of $\text{Fun}_/S(X,X)$, and $f \circ g$ and $\text{id}_Y$ belong to the same connected component of $\text{Fun}_/S(Y,Y)$. In this case, we will refer to $g$ as a homotopy inverse of $f$ relative to $S$.

**Remark 5.1.3.2.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & \xleftarrow{q_V} & 
\end{array}
$$

The following conditions are equivalent:

- The morphism $f$ is a homotopy equivalence relative to $S$ (Definition 5.1.3.1).
- For every object $B$ of the slice category $(\text{Set}_\Delta)_/S$, postcomposition with $f$ induces a homotopy equivalence of simplicial sets $\text{Fun}_/S(B,X) \to \text{Fun}_/S(B,Y)$. 

• For every object $B$ of the slice category $(\text{Set}_\Delta)/S$, postcomposition with $f$ induces a bijection $\pi_0(\text{Fun}_/S(B,X)) \to \pi_0(\text{Fun}_/S(B,Y))$.

• For every object $Z$ of the slice category $(\text{Set}_\Delta)/S$, precomposition with $f$ induces a homotopy equivalence of simplicial sets $\text{Fun}_/S(Y,Z) \to \text{Fun}_/S(X,Z)$.

• For every object $Z$ of the slice category $(\text{Set}_\Delta)/S$, precomposition with $f$ induces a bijection $\pi_0(\text{Fun}_/S(Y,Z)) \to \pi_0(\text{Fun}_/S(X,Z))$.

In the special case where $q_X$ and $q_Y$ are both left fibrations (or both right fibrations), then we can improve upon Remark 5.1.3.2.

**Theorem 5.1.3.3.** Suppose we are given a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & \xleftarrow{q_s} & 
\end{array}
$$

Assume either that $q_X$ and $q_Y$ are both left fibrations, or that they are both right fibrations. Then the following conditions are equivalent:

1. The morphism $f$ is a homotopy equivalence relative to $S$ (Definition 5.1.3.1).

2. For every object $B$ of the slice category $(\text{Set}_\Delta)/S$, the induced map $f_B : \text{Fun}_/S(B,X) \to \text{Fun}_/S(B,Y)$ is a homotopy equivalence.

3. For each vertex $s \in S$, the induced map $f_s : X_s \to Y_s$ is a homotopy equivalence of Kan complexes.

The proof of Theorem 5.1.3.3 will require some preliminaries.

**Proposition 5.1.3.4.** Let $q_X : X \to S$ and $q_Y : Y \to S$ be morphisms of simplicial sets. If $q_Y$ is either a left fibration or a right fibration, then the simplicial set $\text{Fun}_/S(X,Y)$ is a Kan complex.

**Proof.** Without loss of generality, we may assume that $q_Y$ is a left fibration. Then postcomposition with $q_Y$ induces a left fibration $Q_Y : \text{Fun}(X,Y) \to \text{Fun}(X,S)$ (Corollary 4.2.3.2). By construction, $\text{Fun}_/S(X,Y)$ is a fiber of the morphism $Q_Y$, and is therefore a Kan complex by virtue of Corollary 4.4.2.2.

**Proposition 5.1.3.5.** Let $q_X : X \to S$ and $q_Y : Y \to S$ be morphisms of simplicial sets, and let $X' \subseteq X$ be a simplicial subset. If $q_Y$ is either a left fibration or a right fibration, then the restriction map $\theta : \text{Fun}_/S(X,Y) \to \text{Fun}_/S(X',Y)$ is a Kan fibration.
Proof. Without loss of generality, we may assume that \( q_Y \) is a left fibration. By construction, we have a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Fun}_S(X, Y) & \longrightarrow & \text{Fun}(X, Y) \\
\downarrow \theta & & \downarrow \theta' \\
\text{Fun}_S(X', Y) & \longrightarrow & \text{Fun}(X', Y) \times_{\text{Fun}(X', S)} \text{Fun}(X, S).
\end{array}
\]

Proposition 4.2.3.1 guarantees \( \theta' \) is a left fibration, so that \( \theta \) is also a left fibration (Remark 4.2.1.8). Since \( \text{Fun}_S(X', Y) \) is a Kan complex (Proposition 5.1.3.4), it follows that \( \theta \) is also a Kan fibration (Lemma 5.1.2.4).

Remark 5.1.3.6. In the situation of Proposition 5.1.3.5, suppose that \( q_Y \) is a left fibration and the inclusion \( X' \hookrightarrow X \) is left anodyne. Then the restriction map \( \theta : \text{Fun}_S(X, Y) \rightarrow \text{Fun}_S(X', Y) \) is a trivial Kan fibration (this follows immediately from Proposition 4.2.3.4). Similarly, if \( q_Y \) is a right fibration and the inclusion \( X' \hookrightarrow X \) is right anodyne, then \( \theta \) is a trivial Kan fibration.

Remark 5.1.3.7. Let \( q : Y \rightarrow S \) be a morphism of simplicial sets. Then the construction \( X \mapsto \text{Fun}_S(X, Y) \) carries colimits in the slice category \( (\text{Set}_\Delta)_S \) to limits in the category of simplicial sets. In particular, \( \text{Fun}_S(\emptyset, Y) \) can be identified with the 0-simplex \( \Delta^0 \).

Corollary 5.1.3.8. Let \( q : Y \rightarrow S \) be a morphism of simplicial sets, and suppose we are given a pushout diagram

\[
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow i & & \downarrow i' \\
B & \longrightarrow & B'
\end{array}
\]

in the slice category \( (\text{Set}_\Delta)_S \). If \( i \) is a monomorphism and \( q \) is either a left or right fibration, then the induced diagram

\[
\begin{array}{ccc}
\text{Fun}_S(A', Y) & \longleftarrow & \text{Fun}_S(A, Y) \\
\downarrow & & \downarrow \\
\text{Fun}_S(B', Y) & \longleftarrow & \text{Fun}_S(B, Y)
\end{array}
\]

is a homotopy pullback square of Kan complexes.
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Proof. Diagram (5.2) is automatically a pullback square (Remark 5.1.3.7). If \( q \) is a right fibration and \( i \) is a monomorphism, then the horizontal maps in (5.2) are Kan fibrations (Proposition 5.1.3.5). Invoking Example 3.4.1.5, we deduce that (5.2) is also a homotopy pullback square.

Proof of Theorem 5.1.3.3. The equivalence (1) \( \iff \) (2) follows from Remark 5.1.3.2 and the implication (2) \( \implies \) (3) is immediate (Example 3.1.3.12). We will complete the proof by showing that (3) \( \implies \) (2). Assume that we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & \xleftarrow{f_s} & S'
\end{array}
\]

where \( q_X \) and \( q_Y \) are left fibrations and, for every vertex \( s \in S \), the induced map of fibers \( f : X_s \to Y_s \) is a homotopy equivalence of Kan complexes. We wish to show that, for each object \( B \in (\text{Set}_{\Delta})/S \), the induced map \( f_B : \text{Fun}_/S(B, X) \to \text{Fun}_/S(B, Y) \) is a homotopy equivalence. For each \( n \geq 0 \), let \( \text{sk}_n(B) \) denote the \( n \)-skeleton of \( B \) (Construction 1.1.3.5). Then \( f_A \) can be realized as the inverse limit of a tower of morphisms \( \{f_{\text{sk}_n(B)} : \text{Fun}_/S(\text{sk}_n(B), X) \to \text{Fun}_/S(\text{sk}_n(B), Y)\}_n \geq 0 \). Using Proposition 5.1.3.5, we see that the transition morphisms \( \text{Fun}_/S(\text{sk}_{n+1}(B), X) \to \text{Fun}_/S(\text{sk}_n(B), X) \) and \( \text{Fun}_/S(\text{sk}_{n+1}(B), Y) \to \text{Fun}_/S(\text{sk}_n(B), Y) \) are Kan fibrations. Consequently, to show that \( f_B \) is a homotopy equivalence, it will suffice to show that each of the morphisms \( f_{\text{sk}_n(B)} \) is a homotopy equivalence (Proposition 3.3.9.6). We may therefore replace \( B \) by \( \text{sk}_n(B) \) and thereby reduce to the case where the simplicial set \( B \) has dimension \( \leq n \), for some \( n \geq -1 \).

The proof now proceeds by induction on \( n \). In the case \( n = -1 \), the simplicial set \( B \) is empty, and the morphism \( f_B \) is an isomorphism (see Remark 5.1.3.7). Assume that \( n \geq 0 \), let \( B' \) denote the \((n-1)\)-skeleton of \( B \), and let \( T \) denote the collection of all nondegenerate \( n \)-simplices of \( B \). Then Proposition 1.1.3.13 supplies a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
A' & \xrightarrow{} & A \\
\downarrow & & \downarrow \\
B' & \xrightarrow{} & B
\end{array}
\]

where \( A = \coprod_{\tau \in T} \Delta^n \) and \( A' = \coprod_{\tau \in T} \partial \Delta^n \). This pushout square determines a commutative
diagram of Kan complexes

\[
\begin{array}{cccc}
\text{Fun}_/S(B, X) & \longrightarrow & \text{Fun}_/S(B', X) \\
\downarrow f_B & & \downarrow f_{B'} \\
\text{Fun}_/S(A, X) & \longrightarrow & \text{Fun}_/S(A', X) \\
\downarrow f_A & & \downarrow f_{A'} \\
\text{Fun}_/S(B, Y) & \longrightarrow & \text{Fun}_/S(B', Y) \\
\downarrow f_B & & \downarrow f_{B'} \\
\text{Fun}_/S(A, Y) & \longrightarrow & \text{Fun}_/S(A', Y) \\
\downarrow f_A & & \downarrow f_{A'} \\
\end{array}
\]

where the upper and lower squares are homotopy Cartesian by virtue of Corollary 5.1.3.8. Consequently, to show that $f_B$ is a homotopy equivalence of Kan complexes, it will suffice to show that $f_{B'}$, $f_{A'}$, and $f_A$ are homotopy equivalences (Corollary 3.4.1.10). In the first two cases, this follows from our inductive hypothesis. We may therefore replace $B$ by $A$, and thereby reduce to the case where is a coproduct of simplices. Using Remarks 5.1.3.7 and 3.1.5.8 we can further reduce to the case where $B = \Delta^n$ is a simplex of dimension $n$. In this case, the inclusion of the initial vertex $0 \in \Delta^n$ is left anodyne (Example 4.3.7.10). Let $s \in S$ denote the image of the vertex $0 \in \Delta^n$, so that we have a commutative diagram of Kan complexes

\[
\begin{array}{cccc}
\text{Fun}_/S(B, X) & \longrightarrow & \text{Fun}_/S(B, Y) \\
\downarrow f_B & & \downarrow f_s \\
\text{Fun}_/S(\{s\}, X) & \longrightarrow & \text{Fun}_/S(\{s\}, Y). \\
\end{array}
\]

Invoking Remark 5.1.3.6 we conclude that the vertical maps in this diagram are trivial Kan fibrations. Since $f_s$ is a homotopy equivalence by assumption, it follows that $f_B$ is also a homotopy equivalence.

5.1.4 Covariant and Contravariant Equivalences
5.1. FIBERS OF LEFT AND RIGHT FIBRATIONS

Fix a simplicial set $S$. In §5.1.3 we showed that the notion of homotopy equivalence relative to $S$ (Definition 5.1.3.1) is well-behaved when restricted to simplicial sets $X$ which are equipped with a left (or right) fibration $q : X \to S$. When working with more general objects of $(\text{Set}_\Delta)_{/S}$, it is convenient to consider a more liberal notion of equivalence.

**Definition 5.1.4.1.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xleftarrow{} & S
\end{array}
\]

We say that $f$ is a **covariant equivalence relative to $S$** if, for every left fibration $q : Z \to S$, precomposition with $f$ induces a bijection $\pi_0(\text{Fun}_{/S}(Y, Z)) \to \pi_0(\text{Fun}_{/S}(X, Z))$. We say that $f$ is a **contravariant equivalence relative to $S$** if, for every right fibration $q : Z \to S$, precomposition with $f$ induces a bijection $\pi_0(\text{Fun}_{/S}(Y, Z)) \to \pi_0(\text{Fun}_{/S}(X, Z))$.

**Remark 5.1.4.2.** In the situation of Definition 5.1.4.1, the morphism $f : X \to Y$ is a covariant equivalence relative to $S$ if and only if the opposite morphism $f^{\text{op}} : X^{\text{op}} \to Y^{\text{op}}$ is a contravariant equivalence relative to $S^{\text{op}}$.

**Example 5.1.4.3.** Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xleftarrow{} & S
\end{array}
\]

If $f$ is a homotopy equivalence relative to $S$ (Definition 5.1.3.1), then it is both a covariant and contravariant equivalence relative to $S$ (see Remark 5.1.3.2).

**Example 5.1.4.4.** Let $f : X \to Y$ be a morphism of simplicial sets. It follows from Proposition 4.4.2.1 that the following conditions are equivalent:

- The morphism $f$ is a covariant equivalence relative to $\Delta^0$.
- The morphism $f$ is a contravariant equivalence relative to $\Delta^0$.
- The morphism $f$ is a weak homotopy equivalence.

See Proposition 5.1.4.16 for a more general statement.
CHAPTER 5. THE ∞-CATEGORY OF SPACES

**Warning 5.1.4.5.** The notions of covariant and contravariant equivalence are generally distinct from one another. For example, the inclusion map $i_1 : \{1\} \hookrightarrow \Delta^1$ is a contravariant equivalence with respect to $\Delta^1$ (see Proposition 5.1.4.13 below). However, $i_1$ is not a covariant equivalence with respect to $\Delta^1$: note that $i_1$ is a left fibration, but the restriction map

$$\pi_0(\text{Fun}_{/\Delta^1}(\Delta^1, \{1\})) \rightarrow \pi_0(\text{Fun}_{/\Delta^1}(\{1\}, \{1\}))$$

is not bijective (the left hand side is empty and the right hand side is not).

**Warning 5.1.4.6.** In the situation of Definition 5.1.4.1, we will often abuse terminology by referring to the morphism $f : X \rightarrow Y$ in the slice category $(\text{Set}_\Delta)/S$ as a covariant equivalence (contravariant equivalence) if it is a covariant equivalence relative to $S$. Beware that there is some danger of confusion, because the notion of covariant equivalence depends on $S$. For example, the inclusion map $\{1\} \hookrightarrow \Delta^1$ is not a covariant equivalence relative to $\Delta^1$ (Warning 5.1.4.5), but is a covariant equivalence relative to the final object $\Delta^0 \in \text{Set}_\Delta$ (Example 5.1.4.4).

**Remark 5.1.4.7.** Suppose we given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & S
\end{array}
\]

and a morphism $g : S \rightarrow T$. If $f$ is a covariant equivalence relative to $S$, then it is also a covariant equivalence relative to $T$. This follows from the observation that for every left fibration $q : Z \rightarrow T$, we have canonical isomorphisms

$$\text{Fun}_T(Y, Z) \simeq \text{Fun}_{/S}(Y, S \times_T Z) \quad \text{Fun}_T(X, Z) \simeq \text{Fun}_{/S}(X, S \times_T Z).$$

Similarly, if $f$ is a contravariant equivalence relative to $S$, then it is also a contravariant equivalence relative to $T$.

**Remark 5.1.4.8.** Suppose we given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & S
\end{array}
\]

If $f$ is either a covariant or contravariant equivalence relative to $S$, then it is a weak homotopy equivalence. This follows by combining Remark 5.1.4.7 with Example 5.1.4.4.
Remark 5.1.4.9 (Two-out-of-Six). Let $S$ be a simplicial set, and let $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ be morphisms in the slice category $(\text{Set}_\Delta)_S$. If $g \circ f$ and $h \circ g$ are covariant equivalences relative to $S$, then $f$, $g$, and $h$ are all covariant equivalences relative to $S$. Similarly, if $g \circ f$ and $h \circ g$ are contravariant equivalences relative to $S$, then $f$, $g$, and $h$ are contravariant equivalences relative to $S$.

Remark 5.1.4.10 (Two-out-of-Three). Let $S$ be a simplicial set, and let $f : X \to Y$ and $g : Y \to Z$ be morphisms in the slice category $(\text{Set}_\Delta)_S$. If any two of the morphisms $f$, $g$, and $g \circ f$ is a covariant equivalence relative to $S$, then so is the third. Similarly, if any two of the morphisms $f$, $g$, and $g \circ f$ is a contravariant equivalence relative to $S$, then so is the third.

Proposition 5.1.4.11. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{\theta} & \text{Fun}(K,S)
\end{array}
\]

Let $q : Z \to S$ be a morphism of simplicial sets and let $\theta : \text{Fun}_S(Y,Z) \to \text{Fun}_S(X,Z)$ be the morphism given by precomposition with $f$. If $f$ is a covariant equivalence relative to $S$ and $q$ is a left fibration, then $\theta$ is homotopy equivalence. Similarly, if $f$ is a contravariant equivalence relative to $S$ and $q$ is a right fibration, then $\theta$ is a homotopy equivalence.

Proof. We will prove the first assertion; the second follows from the same argument. We will prove that $\theta$ is a homotopy equivalence by showing that, for every simplicial set $K$, the induced map $\theta_K : \pi_0(\text{Fun}(K,\text{Fun}_S(Y,Z))) \to \pi_0(\text{Fun}(K,\text{Fun}_S(X,Z)))$ is a bijection. Form a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
Z' & \xrightarrow{q'} & \text{Fun}(K,Z) \\
\downarrow & & \downarrow \circ q \\
S & \xrightarrow{\theta_K} & \text{Fun}(K,S).
\end{array}
\]

The assumption that $q$ is a left fibration guarantees that the right vertical map is also a left fibration (Corollary 4.2.3.2), so that $q' : Z' \to S$ is also a left fibration. Unwinding the definitions, we see that $\theta_K$ can be identified with the map $\pi_0(\text{Fun}_S(Y,Z')) \xrightarrow{\circ f} \pi_0(\text{Fun}_S(X,Z'))$, and is therefore bijective by virtue of our assumption that $f$ is a covariant equivalence relative to $S$. □
Corollary 5.1.4.12. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S & & S
\end{array}
\]

where \( f \) is a monomorphism. If \( q : Z \rightarrow S \) is a left fibration of simplicial sets and \( f \) is a covariant equivalence relative to \( S \), then the induced map \( \theta : \text{Fun}_{/S}(Y, Z) \rightarrow \text{Fun}_{/S}(X, Z) \) is a trivial Kan fibration. Similarly, if \( q : Z \rightarrow S \) is a right fibration and \( f \) is a contravariant equivalence relative to \( S \), then \( \theta \) is a trivial Kan fibration.

Proof. We will prove the first assertion; the proof of the second is similar. If \( q \) is a left fibration, then the morphism \( \theta : \text{Fun}_{/S}(Y, Z) \rightarrow \text{Fun}_{/S}(X, Z) \) is a Kan fibration (Proposition 5.1.3.5). If \( f \) is a covariant equivalence relative to \( S \), then \( \theta \) is also a homotopy equivalence (Proposition 5.1.4.11), and therefore a trivial Kan fibration (Proposition 3.3.7.4).

Proposition 5.1.4.13. Let \( f : X \rightarrow Y \) be a monomorphism of simplicial sets. The following conditions are equivalent:

1. The morphism \( f \) is left anodyne.
2. The morphism \( f \) is a covariant equivalence relative to \( Y \).
3. For every morphism of simplicial sets \( Y \rightarrow S \), the morphism \( f \) is a covariant equivalence relative to \( S \).

Proof. The implication (1) \( \Rightarrow \) (2) follows from Remark 5.1.3.6 and the implication (2) \( \Rightarrow \) (3) from Remark 5.1.4.7. We will show that (3) implies (1). For this, we must show that every lifting problem

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow & & \downarrow q \\
Y & \xleftarrow{q} & S \\
\end{array}
\]

admits a solution, provided that \( q \) is a left fibration of simplicial sets (Corollary 4.2.2.9). To prove this, it will suffice to show that the natural map \( \theta : \text{Fun}_{/S}(Y, Z) \rightarrow \text{Fun}_{/S}(X, Z) \) is surjective on vertices. In fact, our assumption that \( f \) is a monomorphism and a covariant equivalence relative to \( S \) guarantees that \( \theta \) is a trivial Kan fibration of simplicial sets (Corollary 5.1.4.12).
Proposition 5.1.4.14. Suppose we are given a commutative diagram of simplicial sets

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
S & \xleftarrow{\id} & S
\end{array}
\]

where \(q_X\) and \(q_Y\) are left fibrations. Then the following conditions are equivalent:

1. The morphism \(f\) is a homotopy equivalence relative to \(S\).
2. The morphism \(f\) is a covariant equivalence relative to \(S\).
3. For each vertex \(s \in S\), the induced map \(f_s : X_s \to Y_s\) is a homotopy equivalence of Kan complexes.

Proof. The implications (1) \(\Rightarrow\) (2) and (1) \(\Rightarrow\) (3) follow from Example 5.1.4.3 and Remark 5.1.3.2 (and do not require the assumption that \(q_X\) and \(q_Y\) are left fibrations). The implication (3) \(\Rightarrow\) (1) follows from Theorem 5.1.3.3. We will complete the proof by showing that (2) \(\Rightarrow\) (1). Assume that \(f\) is a covariant equivalence relative to \(S\). Then precomposition with \(f\) induces a bijection \(\pi_0(\text{Fun}_{/S}(Y,X)) \to \pi_0(\text{Fun}_{/S}(X,X))\). We can therefore choose a morphism \(g : Y \to X\) in the slice category \((\text{Set}_{\Delta})_{/S}\) such that \(g \circ f\) and \(\id_X\) belong to the same connected component of \(\text{Fun}_{/S}(X,X)\). Then \(f \circ g \circ f\) and \(f\) belong to the same connected component of the simplicial set \(\text{Fun}_{/S}(X,Y)\). Since the map \(\pi_0(\text{Fun}_{/S}(Y,Y)) \to \pi_0(\text{Fun}_{/S}(X,Y))\) is injective, it follows that \(f \circ g\) and \(\id_Y\) belong to the same connected component of \(\text{Fun}_{/S}(Y,Y)\). It follows that \(f\) is a homotopy equivalence relative to \(S\) (with homotopy inverse \(g\)). \(\square\)

Remark 5.1.4.15 (Axioms for Covariant Equivalence). Fix a simplicial set \(S\). Combining Remark 5.1.4.10, Proposition 5.1.4.13, and Proposition 5.1.4.14, we see that the notion of covariant equivalence (relative to \(S\)) satisfies the following properties:

(a) If \(f : X \to Y\) and \(g : Y \to Z\) are morphisms in \((\text{Set}_{\Delta})_{/S}\) and two of the morphisms \(f, g,\) and \(g \circ f\) are covariant equivalences, then so is the third.

(b) Every left anodyne morphism of \((\text{Set}_{\Delta})_{/S}\) is a covariant equivalence.

(c) If \(q_X : X \to S\) and \(q_Y : Y \to S\) are left fibrations, then a morphism \(f : X \to Y\) in \((\text{Set}_{\Delta})_{/S}\) is a covariant equivalence if and only if the induced map \(f_s : X_s \to Y_s\) is a homotopy equivalence of Kan complexes, for each \(s \in S\).
In fact, the notion of covariant equivalence is completely characterized by \((a), (b), (c)\). To prove this, suppose we are given an arbitrary commutative diagram of simplicial sets

![Diagram](image)

Using Proposition 4.2.2.7, we can factor \(q_Y\) as a composition \(Y \xrightarrow{q_Y'} Y' \xrightarrow{q_Y''} S\) where \(q_Y'\) is left anodyne and \(q_Y''\) is a left fibration. Similarly, we can factor the composite map \(q_Y \circ f\) as a composition \(X \xrightarrow{q_X} X' \xrightarrow{f'} Y'\), where \(q_X\) is left anodyne and \(f'\) is a left fibration. We then have a commutative diagram

![Diagram](image)

where the vertical maps are left anodyne (and therefore covariant equivalences, by virtue of \((b)\)). Using \((a)\), we conclude that \(f\) is a covariant equivalence if and only if \(f'\) is a covariant equivalence relative to \(S\). By virtue of \((c)\), this is equivalent to the requirement that for each vertex \(s \in S\), the induced map \(f'_s : X'_s \to Y'_s\) is a homotopy equivalence.

We conclude by establishing a partial converse to Remark 5.1.4.8:

**Proposition 5.1.4.16.** Suppose we given a commutative diagram of simplicial sets

![Diagram](image)

where \(S\) is a Kan complex. The following conditions are equivalent:

1. The morphism \(f\) is a covariant equivalence relative to \(S\).
2. The morphism \(f\) is a contravariant equivalence relative to \(S\).
3. The morphism \(f\) is a weak homotopy equivalence.
5.2. MORPHISM SPACES

Proof. The implications \((1) \Rightarrow (3)\) and \((2) \Rightarrow (3)\) follow from Remark \(5.1.4.8\) (and do not require the assumption that \(S\) is a Kan complex). We will complete the proof by establishing the implication \((3) \Rightarrow (1)\) (the proof of the implication \((3) \Rightarrow (2)\) is similar). Assume that \(f\) is a weak homotopy equivalence and let \(Z \rightarrow S\) be a left fibration; we wish to show that the induced map \(\theta : \text{Fun}_S(Y, Z) \rightarrow \text{Fun}_S(X, Z)\) is bijective on connected components. Our assumption that \(S\) is a Kan complex guarantees that \(q\) is a Kan fibration (Lemma \(5.1.2.4\)), so that \(Z\) is also a Kan complex. Consider the diagram of simplicial sets

\[
\begin{array}{ccc}
\text{Fun}(Y, Z) & \xrightarrow{\circ f} & \text{Fun}(X, Z) \\
\downarrow{q_0} & & \downarrow{q_0} \\
\text{Fun}(Y, S) & \xrightarrow{\circ f} & \text{Fun}(X, S).
\end{array}
\]

(5.3)

Our assumption that \(f\) is a weak homotopy equivalence guarantees that the horizontal maps in this diagram are homotopy equivalences (Corollary \(3.1.6.5\)), so that diagram (5.3) is a homotopy Cartesian (Corollary \(3.4.1.3\)). Corollary \(3.1.3.2\) guarantees that the vertical maps in the diagram (5.3) are Kan fibrations. Invoking Example \(3.4.1.4\) we conclude that the induced map

\[
\theta : \text{Fun}_S(Y, Z) = \{q_Y\} \times_{\text{Fun}(Y,S)} \text{Fun}(Y, Z) \rightarrow \{q_X\} \times_{\text{Fun}(X,S)} \text{Fun}(X, Z) = \text{Fun}_S(X, Z)
\]

is also a homotopy equivalence, and is therefore bijective on connected components. \(\square\)

5.2 Morphism Spaces

Let \(\mathcal{C}\) be an \(\infty\)-category containing a pair of objects \(X\) and \(Y\). Recall that a morphism from \(X\) to \(Y\) is an edge \(f\) of \(\mathcal{C}\) satisfying \(d_1(f) = X\) and \(d_0(f) = Y\) (Definition \(1.3.1.1\)). Morphisms from \(X\) to \(Y\) can be identified with vertices of a simplicial set \(\text{Hom}_\mathcal{C}(X, Y)\), given by the iterated fiber product

\[
\{X\} \times_{\text{Fun}(\{0\}, \mathcal{C})} \text{Fun}(\Delta^1, \mathcal{C}) \times_{\text{Fun}(\{1\}, \mathcal{C})} \{Y\}.
\]

In \(5.2.1\), we show that the simplicial set \(\text{Hom}_\mathcal{C}(X, Y)\) is a Kan complex (Proposition \(5.2.1.8\)), which we refer to as the space of morphisms from \(X\) to \(Y\) (Construction \(5.2.1.1\)).

Let \(F : \mathcal{C} \rightarrow \mathcal{D}\) be a functor of \(\infty\)-categories. We say that \(F\) is fully faithful if, for every pair of objects \(X, Y \in \mathcal{C}\), the induced map \(\text{Hom}_\mathcal{C}(X, Y) \rightarrow \text{Hom}_\mathcal{D}(F(X), F(Y))\) is a homotopy equivalence of Kan complexes (Definition \(5.2.2.1\)). We say that \(F\) is essentially surjective if it induces a surjection \(\pi_0(\mathcal{C}^\infty) \rightarrow \pi_0(\mathcal{D}^\infty)\) on isomorphism classes of objects. In \(5.2.2\) we show that \(F\) is an equivalence of \(\infty\)-categories if and only if it is both fully...
faithful and essentially surjective (Theorem 5.2.15). This is essentially a reformulation of
the criterion of Theorem 4.5.4.1. Nevertheless, it can be quite useful: the mapping spaces
\( \text{Hom}_C(X, Y) \) are often more amenable to calculation than the Kan complex \( \text{Fun}(\Delta^1, C) \). For reasons of
convenience, it is often useful to work with a variant of Construction 5.2.1.1. Let \( C \) be an
\( \infty \)-category containing a pair of objects \( X \) and \( Y \). We define simplicial sets \( \text{Hom}^L_C(X, Y) \) and \( \text{Hom}^R_C(X, Y) \) by the formulae
\[
\text{Hom}^L_C(X, Y) = \mathcal{C}/X \times_C \{Y\} \quad \text{Hom}^R_C(X, Y) = \{X\} \times_C \mathcal{C}/Y .
\]
We will refer to \( \text{Hom}^L_C(X, Y) \) as the left-pinched space of morphisms from \( X \) to \( Y \), and to \( \text{Hom}^R_C(X, Y) \) as the right-pinched space of morphisms from \( X \) to \( Y \). These simplicial sets
are also Kan complexes, which can often be described very explicitly:

- Let \( C \) be a \((2,1)\)-category containing objects \( X \) and \( Y \), and let \( \mathcal{N}^D(C) \) denote the
Duskin nerve of \( C \) (Construction 2.3.1.1). Then there are canonical isomorphisms of
simplicial sets
\[
\text{Hom}^L_{\mathcal{N}^D(C)}(X, Y) \simeq \mathcal{N}_\bullet(\text{Hom}_C(X, Y)) \simeq \text{Hom}^R_{\mathcal{N}^D(C)}(X, Y)^{\text{op}};
\]
see Example 5.2.5.11.

- Let \( C \) be a differential graded category containing objects \( X \) and \( Y \), and let \( \mathcal{N}^{dg}(C) \) denote the
differential graded nerve of \( C \) (Definition 2.5.3.7). Then there is a canonical
isomorphism of simplicial sets
\[
\text{Hom}^R_{\mathcal{N}^{dg}(C)}(X, Y) \simeq K(\text{Hom}_C(X, Y)_*),
\]
where \( K(\text{Hom}_C(X, Y)_*) \) denotes the Eilenberg-MacLane space associated to the chain
complex \( \text{Hom}_C(X, Y)_* \) (Example 5.2.5.13).

- Let \( C \) be a locally Kan simplicial category containing a pair of objects \( X \) and \( Y \), and let \( \mathcal{N}^{hc}(C) \) denote the homotopy coherent nerve of \( C \) (Definition 2.4.3.5). Then there are canonical
homotopy equivalences
\[
\text{Hom}^L_{\mathcal{N}^{hc}(C)}(X, Y)^{\text{op}} \leftrightarrow \text{Hom}_C(X, Y)_* \rightarrow \text{Hom}^R_{\mathcal{N}^{hc}(C)}(X, Y);
\]
see Theorem 5.2.6.4. This is a special case of a more general result (where the simplicial
set \( \text{Hom}_C(X, Y)_* \) is assumed to be an \( \infty \)-category rather than a Kan complex), which
we prove in 5.2.6.

In 5.2.5 we construct comparison maps
\[
\iota^L_{X,Y} : \text{Hom}^L_C(X, Y) \hookrightarrow \text{Hom}_C(X, Y) \quad \iota^R_{X,Y} : \text{Hom}^R_C(X, Y) \hookrightarrow \text{Hom}_C(X, Y),
\]
which we refer to as the *pinch inclusion maps*, and show that they are homotopy equivalences of Kan complexes (Proposition 5.2.5.9). This follows from a more general statement about the relationship between (co)slice ∞-categories and comma objects (Theorem 5.2.4.14), which we formulate and prove in §5.2.4. Our proof will make use of a general detection principle for natural isomorphisms of diagrams (Theorem 5.2.3.8), which we explain in §5.2.3.

### 5.2.1 Morphism Spaces

Let \( C \) be a category. To every pair of objects \( X, Y \in \text{Ob}(C) \), one can associate a set \( \text{Hom}_C(X,Y) \) of *morphisms from \( X \) to \( Y \).* Our goal in this section is to explain a counterpart of this construction in the setting of ∞-categories.

**Construction 5.2.1.1.** Let \( C \) be a simplicial set containing a pair of vertices \( X \) and \( Y \). We let \( \text{Hom}_C(X,Y) \) denote the simplicial set given by the fiber product

\[
\{X\} \times_{\text{Fun}(\{0\},C)} \text{Fun}(\Delta^1,C) \times_{\text{Fun}(\{1\},C)} \{Y\}.
\]

We will typically be interested in this construction only in the case where \( C \) is an ∞-category; if this condition is satisfied, we will refer to \( \text{Hom}_C(X,Y) \) as the *space of morphisms from \( X \) to \( Y \).*

**Remark 5.2.1.2.** Let \( C \) be an ∞-category containing a pair of objects \( X \) and \( Y \). Recall that a *morphism* from \( X \) to \( Y \) is an edge \( e : \Delta^1 \to C \) satisfying \( e(0) = X \) and \( e(1) = Y \) (Definition 1.3.1.1). It follows that morphisms from \( X \) to \( Y \) can be identified with vertices of the morphism space \( \text{Hom}_C(X,Y) \) of Construction 5.2.1.1.

**Example 5.2.1.3.** Let \( C \) be an ordinary category containing objects \( X \) and \( Y \), which we will identify with objects of the ∞-category \( \text{N}^\bullet(C) \). Then the morphism space \( \text{Hom}_{\text{N}^\bullet(C)}(X,Y) \) of Construction 5.2.1.1 can be identified with the constant simplicial set having the value \( \text{Hom}_C(X,Y) \) (see Example 5.2.4.4).

**Example 5.2.1.4.** Let \( X \) be a topological space containing a pair of points \( x \) and \( y \), which we regard as objects of the ∞-category \( \text{Sing}^\bullet(X) \). Then we have a canonical isomorphism of Kan complexes

\[
\text{Hom}_{\text{Sing}^\bullet(X)}(x,y) \cong \text{Sing}^\bullet(P_{x,y}),
\]

where \( P_{x,y} \) denotes the topological space of continuous paths \( p : [0,1] \to X \) satisfying \( p(0) = x \) and \( p(1) = y \) (equipped with the compact-open topology). See Example 5.2.4.5.
Example 5.2.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories, so that the join $\mathcal{C} \star \mathcal{D}$ is also an $\infty$-category (Proposition 4.3.3.21). Then the morphism spaces in $\mathcal{C} \star \mathcal{D}$ are described by the formula

$$
\text{Hom}_{\mathcal{C} \star \mathcal{D}}(X,Y) \simeq \begin{cases} 
\text{Hom}_{\mathcal{C}}(X,Y) & \text{if } X,Y \in \mathcal{C} \\
\text{Hom}_{\mathcal{D}}(X,Y) & \text{if } X,Y \in \mathcal{D} \\
\Delta^0 & \text{if } X \in \mathcal{C}, Y \in \mathcal{D} \\
\emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}.
\end{cases}
$$

Example 5.2.1.6. Let $\mathcal{C}$ be a simplicial set containing vertices $X$ and $Y$. Let $K$ be a simplicial set, and let $X, Y : K \to \mathcal{C}$ be the constant maps taking the values $X$ and $Y$, respectively. Then there is a canonical isomorphism of simplicial sets

$$
\text{Hom}_{\text{Fun}(K,\mathcal{C})}(X,Y) \simeq \text{Fun}(K,\text{Hom}_{\mathcal{C}}(X,Y)).
$$

Remark 5.2.1.7. Let $\mathcal{C}$ be a simplicial set containing vertices $X$ and $Y$, which we also regard as vertices of the opposite simplicial set $\mathcal{C}^{\text{op}}$. Then there is a canonical isomorphism of simplicial sets $\text{Hom}_{\mathcal{C}^{\text{op}}}(X,Y) \simeq \text{Hom}_{\mathcal{C}}(Y,X)^{\text{op}}$.

Proposition 5.2.1.8. Let $\mathcal{C}$ be an $\infty$-category. For every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\text{Hom}_{\mathcal{C}}(X,Y)$ is a Kan complex.

Proof. By definition, the morphism space $\text{Hom}_{\mathcal{C}}(X,Y)$ can be identified with the fiber of the restriction map

$$
\theta : \text{Fun}(\Delta^1,\mathcal{C}) \to \text{Fun}(\partial \Delta^1,\mathcal{C})
$$

over the vertex $(X,Y) \in \text{Fun}(\partial \Delta^1,\mathcal{C})$. Corollary 4.4.5.3 guarantees that $\theta$ is an isofibration, so that $\text{Hom}_{\mathcal{C}}(X,Y)$ is an $\infty$-category. To show that it is a Kan complex, it will suffice to show that every morphism $u$ in $\text{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism (Proposition 4.4.2.1). By virtue of Corollary 4.4.3.15, it will suffice to show that the image of $u$ in the $\infty$-category $\text{Fun}(\Delta^1,\mathcal{C})$ is an isomorphism. This follows from Theorem 4.4.4.4 since evaluation functors carry $u$ to the identity morphisms $\text{id}_X$ and $\text{id}_Y$, respectively.

Remark 5.2.1.9. Let $\mathcal{C}$ be an $\infty$-category containing a pair of morphisms $f, g : X \to Y$ having the same source and target. Then $f$ and $g$ are homotopic (Definition 1.3.3.1) if and only if they belong to the same connected component of the Kan complex $\text{Hom}_{\mathcal{C}}(X,Y)$: this follows from the characterization of Corollary 1.3.3.7. Consequently, we obtain a bijection $\text{Hom}_{h\mathcal{C}}(X,Y) \simeq \pi_0(\text{Hom}_{\mathcal{C}}(X,Y))$. 


Example 5.2.1.10 (Loop Spaces). Let \((X, x)\) be a pointed Kan complex. The Kan complex \(\text{Hom}_X(x, x)\) is often denoted by \(\Omega(X)\) and referred to as the based loop space of \(X\). Note that it can be identified with the fiber over \(x\) of the evaluation map 

\[
q : \{x\} \times_{\text{Fun}(\{0\}, X)} \text{Fun}(\Delta^1, X) \to \text{Fun}(\{1\}, X) = X.
\]

By virtue of Example 3.1.6.10, this map is a Kan fibration whose domain is a contractible Kan complex. It follows that the long exact sequence of Theorem 3.2.5.1 yields isomorphisms 

\[
\pi_n(\text{Hom}_X(x, x), \text{id}_x) \cong \pi_{n+1}(X, x) \text{ for } n \geq 0.
\]

Example 5.2.1.11. Let \(\mathcal{C}\) be a 2-category, let \(N^\bullet_{\mathcal{D}}(\mathcal{C})\) denote its Duskin nerve (Construction 2.3.1.1), and let \(X\) and \(Y\) be objects of \(\mathcal{C}\) (which we identify with vertices of the simplicial set \(N^\bullet_{\mathcal{D}}(\mathcal{C})\)). For any category \(\mathcal{E}\), Corollary 2.3.5.14 supplies a bijection

\[
\{\text{Functors } U : \text{Tw}(\mathcal{E})^{\text{op}} \to \text{Hom}_{\mathcal{C}}(X, Y)\} 
\cong \{\text{Morphisms of simplicial sets } N^\bullet_{\mathcal{E}}(\mathcal{E}) \to \text{Hom}_{N^\bullet_{\mathcal{D}}(\mathcal{C})}(X, Y)\}
\]

where \(\text{Tw}(\mathcal{E})\) denotes the twisted arrow category of Construction 2.3.5.1. Specializing to categories of the form \([n] = \{0 < 1 < \cdots < n\}\), we see that the simplicial set \(\text{Hom}_{N^\bullet_{\mathcal{D}}(\mathcal{C})}(X, Y)\) is given by the construction

\[
([n] \in \Delta) \mapsto \{\text{Functors from } \text{Tw}([n])^{\text{op}} \text{ to } \text{Hom}_{\mathcal{C}}(X, Y)\}.
\]

In other words, we can identify \(n\)-simplices of the simplicial set \(\text{Hom}_{N^\bullet_{\mathcal{D}}(\mathcal{C})}(X, Y)\) with commutative diagrams

\[
\begin{array}{cccccccccc}
& & f_{0,n} & & & & & & & \\
& f_{0,n-1} & & & f_1,n & & & & & \\
& f_{0,n-2} & & f_{1,n-1} & & f_{2,n} & & & & \\
& & & \ddots & \ddots & \ddots & \ddots & & & \\
& f_0,0 & & f_1,1 & & \cdots & & f_{n-1,n-1} & & f_{n,n}
\end{array}
\]

in the category \(\text{Hom}_{\mathcal{C}}(X, Y)\).
5.2.2 Fully Faithful and Essentially Surjective Functors

Let \( C \) and \( D \) be categories. Recall that a functor \( F : C \to D \) is an equivalence of categories if and only if it satisfies the following pair of conditions:

1. The functor \( F \) is fully faithful: that is, for every pair of objects \( X, Y \in C \), the induced map \( \text{Hom}_C(X,Y) \to \text{Hom}_D(F(X), F(Y)) \) is bijective.

2. The functor \( F \) is essentially surjective: that is, for every object \( X \in D \), there exists an object \( Y \in C \) and an isomorphism \( X \simeq F(Y) \) in the category \( D \).

Our goal in this section is to give an analogous characterization of equivalences in the setting of \( \infty \)-categories (Theorem 5.2.2.15). We begin by formulating \( \infty \)-categorical analogues of conditions (1) and (2).

**Definition 5.2.2.1.** Let \( F : C \to D \) be a functor of \( \infty \)-categories. We say that \( F \) is fully faithful if, for every pair of objects \( X, Y \in C \), the induced map of morphism spaces \( \text{Hom}_C(X,Y) \to \text{Hom}_D(F(X), F(Y)) \) is a homotopy equivalence of Kan complexes.

**Example 5.2.2.2.** Let \( C \) be an \( \infty \)-category and let \( C' \subseteq C \) be a full subcategory (Definition 4.1.2.15). Then the inclusion map \( \iota : C' \to C \) is fully faithful. In fact, for every pair of objects \( X, Y \in C' \), the inclusion \( \iota \) induces an isomorphism of simplicial sets \( \text{Hom}_{C'}(X,Y) \simeq \text{Hom}_C(X,Y) \).

**Example 5.2.2.3.** Let \( F : C \to D \) be a functor between ordinary categories. Then \( F \) is fully faithful if and only if the induced map \( N_\bullet(F) : N_\bullet(C) \to N_\bullet(D) \) is fully faithful (in the sense of Definition 5.2.2.1). Consequently, we can regard Definition 5.2.2.1 as a generalization of the classical notion of fully faithful functor.

**Remark 5.2.2.4.** Let \( F : C \to D \) be a functor between \( \infty \)-categories, so that \( F \) induces a functor of homotopy categories \( f : hC \to hD \). If \( F \) is fully faithful, then \( f \) is also fully faithful (see Remark 5.2.1.9). Beware that the converse is generally false.

**Remark 5.2.2.5 (Transitivity).** Let \( F : C \to D \) and \( G : D \to E \) be functors of \( \infty \)-categories, where \( G \) is fully faithful. Then \( F \) is fully faithful if and only if \( G \circ F \) is fully faithful. In particular, the collection of fully faithful functors is closed under composition.

**Proposition 5.2.2.6.** Let \( F : C \to D \) be a fully faithful functor of \( \infty \)-categories. Then \( F \) is conservative (Definition 4.4.2.6). That is, if \( u : X \to Y \) is a morphism in \( C \) for which \( F(u) \) is an isomorphism in the \( \infty \)-category \( D \), then \( u \) is an isomorphism in the \( \infty \)-category \( C \).

**Proof.** Let \( \sigma : F(Y) \to F(X) \) be a homotopy inverse to \( F(u) \). Since \( F \) is fully faithful, the natural map \( \text{Hom}_C(Y,X) \to \text{Hom}_D(F(Y), F(X)) \) is a homotopy equivalence. We may
therefore assume without loss of generality that $v = F(u)$, for some morphism $v : Y \to X$ in the $\infty$-category $C$. Let $v \circ u$ be a composition of $u$ and $v$ in the $\infty$-category $C$. Since $F(u)$ is homotopy inverse to $F(v)$, the morphism $F(v \circ u)$ is homotopic to $\text{id}_{F(C)} = F(\text{id}_C)$. Since the map $\text{Hom}_C(X, X) \to \text{Hom}_D(F(X), F(X))$ is a homotopy equivalence, it follows that $v \circ u$ is homotopic to $\text{id}_C$: that is, $v$ is a left homotopy inverse to $u$. A similar argument (with the roles of $u$ and $v$ reversed) shows that $v$ is also a right homotopy inverse to $u$. It follows that $u$ is an isomorphism.

\begin{proof} \end{proof}

**Corollary 5.2.2.7.** Let $F : C \to D$ be a fully faithful functor of $\infty$-categories. Then the induced map of cores $C^\simeq \to D^\simeq$ is also fully faithful.

**Proof.** Fix objects $X, Y \in C^\simeq$. Our assumption that $F$ is fully faithful guarantees that the induced map $\theta : \text{Hom}_C(X, Y) \to \text{Hom}_D(F(X), F(Y))$ is a homotopy equivalence of Kan complexes. By virtue of Proposition 5.2.2.6, $\theta$ restricts to a homotopy equivalence from the summand of $\text{Hom}_C(X, Y)$ spanned by the isomorphisms from $X$ to $Y$ to the summand of $\text{Hom}_D(F(X), F(Y))$ spanned by the isomorphisms from $F(X)$ to $F(Y)$. Unwinding the definitions, we conclude that $F^\simeq$ induces a homotopy equivalence $\text{Hom}_{C^\simeq}(X, Y) \to \text{Hom}_{D^\simeq}(F(X), F(Y))$. \end{proof}

**Definition 5.2.2.8.** Let $F : C \to D$ be a functor of $\infty$-categories. The essential image of $F$ is the full subcategory of $D$ spanned by those objects $D \in D$ for which there exists an object $C \in C$ and an isomorphism $F(C) \simeq D$. We say that $F$ is essentially surjective if its essential image is the entire $\infty$-category $D$: that is, if the map of sets $\pi_0(C^\simeq) \to \pi_0(D^\simeq)$ is surjective.

**Remark 5.2.2.9.** Let $F : C \to D$ be a functor of $\infty$-categories, and let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of $F$. Then $\mathcal{D}'$ is a replete full subcategory of $\mathcal{D}$, and $F$ can be regarded as an essentially surjective functor from $C$ to $\mathcal{D}'$. Moreover, the essential image $\mathcal{D}'$ is uniquely determined by these properties.

**Remark 5.2.2.10.** Let $F : C \to D$ be a functor between $\infty$-categories. Then $F$ is essentially surjective if and only if the induced functor of homotopy categories $f : hC \to hD$ is essentially surjective (in the sense of classical category theory).

**Remark 5.2.2.11.** Let $F : C \to D$ be a functor between $\infty$-categories. Then $F$ is essentially surjective if and only if the induced map of Kan complexes $F^\simeq : C^\simeq \to D^\simeq$ is essentially surjective.

**Example 5.2.2.12.** Let $F : C \to D$ be a functor between ordinary categories. Then $F$ is essentially surjective if and only if the induced map $N_\bullet(F) : N_\bullet(C) \to N_\bullet(D)$ is an essentially surjective functor of $\infty$-categories (in the sense of Definition 5.2.2.8).
Example 5.2.2.13. Let \( f : X \to Y \) be a morphism of Kan complexes. Then \( f \) is essentially surjective (in the sense of Definition 5.2.2.8) if and only if the induced map \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \) is a surjection.

Remark 5.2.2.14 (Transitivity). Let \( F : C \to D \) and \( G : D \to E \) be functors of \( \infty \)-categories, where \( F \) is essentially surjective. Then \( G \) is essentially surjective if and only if \( G \circ F \) is essentially surjective. In particular, the collection of essentially surjective functors is closed under composition.

Theorem 5.2.2.15. Let \( F : C \to D \) be functor of \( \infty \)-categories. Then \( F \) is an equivalence of \( \infty \)-categories if and only if it is fully faithful and essentially surjective.

We begin by considering the special case of Theorem 5.2.2.15 where \( C \) and \( D \) are Kan complexes.

Lemma 5.2.2.16. Let \( f : X \to Y \) be a morphism of Kan complexes which is fully faithful and essentially surjective. Then \( f \) is a homotopy equivalence.

Proof. Since \( f \) is essentially surjective, the underlying map of connected components \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \) is surjective. We claim that it is also injective. To prove this, suppose that \( x \) and \( x' \) are vertices of \( X \) such that \( f(x) \) and \( f(x') \) belong to the same connected component of \( Y \). Then the morphism space \( \text{Hom}_Y(f(x), f(x')) \) is nonempty. Since \( f \) is fully faithful, it induces a homotopy equivalence \( \text{Hom}_X(x, x') \to \text{Hom}_Y(f(x), f(x')) \). It follows that \( \text{Hom}_X(x, x') \) is nonempty, so that \( x \) and \( x' \) belong to the same connected component of \( X \). This completes the proof that \( \pi_0(f) \) is a bijection.

By virtue of Whitehead’s theorem (Theorem 3.2.6.1), it will suffice to show that for every vertex \( x \in X \) having image \( y = f(x) \in Y \) and every integer \( n \geq 0 \), the induced map \( \theta : \pi_{n+1}(X, x) \to \pi_{n+1}(Y, y) \) is an isomorphism. Using Example 5.2.1.10 we can identify \( \theta \) with the natural map \( \pi_n(\text{Hom}_X(x, x), \text{id}_x) \to \pi_n(\text{Hom}_Y(y, y), \text{id}_y) \), which is bijective by virtue of our assumption that \( f \) induces a homotopy equivalence \( \text{Hom}_X(x, x) \to \text{Hom}_Y(y, y) \).

Proof of Theorem 5.2.2.15. Assume first that \( F : C \to D \) is an equivalence of \( \infty \)-categories. Then \( F \) induces a homotopy equivalence of Kan complexes \( F^\simeq : C^\simeq \to D^\simeq \) (Remark 4.5.1.18). Passing to connected components, we conclude that the induced map \( \pi_0(C^\simeq) \to \pi_0(D^\simeq) \) is bijective. In particular, \( F \) is essentially surjective. We have a commutative diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Fun}(\Delta^1, C)^\simeq & \xrightarrow{\theta} & \text{Fun}(\Delta^1, D)^\simeq \\
\downarrow & & \downarrow \\
\text{Fun}(& \partial \Delta^1, C)^\simeq & \xrightarrow{\theta_{\partial_1}} & \text{Fun}(& \partial \Delta^1, D)^\simeq,
\end{array}
\]
where the horizontal maps are homotopy equivalences (Theorem 4.5.4.1) and the vertical maps are Kan fibrations (Corollary 4.4.5.4). Applying Proposition 3.2.7.1, we conclude that for every vertex \((X,Y) \in \text{Fun}(\partial \Delta^1, \mathcal{C})\), the induced map of fibers
\[
\text{Hom}_\mathcal{C}(X,Y) = \{(X,Y)\} \times_{\text{Fun}(\partial \Delta^1, \mathcal{C})} \text{Fun}(\Delta^1, \mathcal{C}) \cong \text{Fun}(\partial \Delta^1, \mathcal{D}) \cong \text{Fun}(\Delta^1, \mathcal{D}) \cong \text{Hom}_\mathcal{D}(F(X), F(Y))
\]
is a homotopy equivalence. It follows that \(F\) is fully faithful.

Now suppose that \(F : \mathcal{C} \to \mathcal{D}\) is a functor of \(\infty\)-categories which is fully faithful and essentially surjective. Using Corollary 5.2.2.7 and Remark 5.2.2.11, we see that the induced map \(F^\sim : \mathcal{C}^\sim \to \mathcal{D}^\sim\) is also fully faithful and essentially surjective, and is therefore a homotopy equivalence of Kan complexes (Lemma 5.2.2.16). It follows that the morphism \(\theta_0\) in (5.4) is a homotopy equivalence of Kan complexes. Combining our assumption that \(F\) is fully faithful with Proposition 3.2.7.1, we conclude that \(\theta\) is also a homotopy equivalence. Applying Theorem 4.5.4.1, we conclude that \(F\) is an equivalence of \(\infty\)-categories.

**Corollary 5.2.2.17.** Let \(F : \mathcal{C} \to \mathcal{D}\) be a functor of \(\infty\)-categories, and let \(\mathcal{D}' \subseteq \mathcal{D}\) be the essential image of \(F\). Then \(F\) is fully faithful if and only if it induces an equivalence of \(\infty\)-categories \(\mathcal{C} \to \mathcal{D}'\).

**Proof.** Combine Corollary 5.2.2.17 with Exercise 4.4.1.12.

**Corollary 5.2.2.18.** Let \(f : X \to Y\) be a morphism of Kan complexes. Then \(f\) is fully faithful (when regarded as a functor of \(\infty\)-categories) if and only if it induces a homotopy equivalence from \(X\) to a summand of \(Y\).

**Proof.** Combine Corollary 5.2.2.17 with Exercise 4.4.1.12.

### 5.2.3 Digression: Categorical Mapping Cylinders

**Proposition 5.2.3.1.** Let \(\mathcal{C}\) be an \(\infty\)-category and let \(f_0, f_1 : B \to \mathcal{C}\) be diagrams in \(\mathcal{C}\) indexed by a simplicial set \(B\). Recall that \(f_0\) and \(f_1\) are naturally isomorphic if they are isomorphic as objects of the diagram \(\infty\)-category \(\text{Fun}(B, \mathcal{C})\) (Definition 4.4.4.1). Our goal in this section is to establish a detection criterion for natural isomorphisms.

**Proposition 5.2.3.1.** Let \(\mathcal{C}\) be an \(\infty\)-category and let \(f_0, f_1 : B \to \mathcal{C}\) be a pair of diagrams. The following conditions are equivalent:

1. The diagrams \(f_0\) and \(f_1\) are isomorphic when regarded as objects of the \(\infty\)-category \(\text{Fun}(B, \mathcal{C})\).

2. There exists a factorization of the fold map \((\text{id}_B, \text{id}_B) : B \coprod B \to B\) as a composition

\[
B \coprod B \xrightarrow{(\iota_0, \iota_1)} B \xrightarrow{\pi} B,
\]
where \( \pi \) is a categorical equivalence, and a diagram \( \overline{f} : \overline{B} \to C \) satisfying \( f_0 = \overline{f} \circ s_0 \) and \( f_1 = \overline{f} \circ s_1 \).

(3) For every factorization of the fold map \((\text{id}_B, \text{id}_B) : B \coprod B \to B\) as a composition

\[
B \coprod B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B,
\]

where \( s_0 \) and \( s_1 \) have disjoint images, there exists a diagram \( \overline{f} : \overline{B} \to C \) satisfying \( f_0 = \overline{f} \circ s_0 \) and \( f_1 = \overline{f} \circ s_1 \).

We will deduce Proposition \ref{prop:mappingcylinder} from a more general statement (Theorem \ref{thm:moregeneral}), which we prove at the end of this section.

**Remark 5.2.3.2.** Proposition \ref{prop:mappingcylinder} has an interpretation in the language of model categories. Let us regard the category \( \text{Set}_\Delta \) of simplicial sets as equipped with the Joyal model structure of Remark \[?\]. Conditions (2) and (3) of Proposition \ref{prop:mappingcylinder} are equivalent to the requirement that the morphisms \( f_0, f_1 : B \to C \) are homotopic with respect to the Joyal model structure (in the sense of Definition \[?\]). Proposition \ref{prop:mappingcylinder} asserts that this is equivalent to the requirement that \( f_0 \) and \( f_1 \) are naturally isomorphic (in the sense of Definition \ref{def:natiso}).

Let us introduce a bit of terminology which is useful for exploiting Proposition \ref{prop:mappingcylinder}.

**Definition 5.2.3.3.** Let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets. A *categorical mapping cylinder for \( B \) relative to \( A \)* is a simplicial set \( \overline{B} \) equipped with a morphism \( \pi : \overline{B} \to B \) together with a pair of sections \( s_0, s_1 : B \to \overline{B} \) having the following properties:

1. The morphism \( \pi : \overline{B} \to B \) is a categorical equivalence of simplicial sets.
2. The morphisms \( s_0, s_1 : B \to \overline{B} \) satisfy \( s_0 \circ i = s_1 \circ i \), and the induced map \((s_0, s_1) : (B \coprod_A B) \to \overline{B}\) is a monomorphism.

If these conditions are satisfied in the special case \( A = \emptyset \), we will simply refer to \( \overline{B} \) (together with the morphisms \( \pi, s_0, \) and \( s_1 \)) as a *categorical mapping cylinder for \( B \).

**Remark 5.2.3.4.** In the situation of Definition \ref{def:mappingcylinder}, condition (2) is equivalent to the requirement that the diagram of simplicial sets

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{i} & & \downarrow{s_0} \\
B & \xrightarrow{s_1} & \overline{B}
\end{array}
\]

commutes and is a pullback square (note that the morphisms \( s_0 \) and \( s_1 \) are automatically monomorphisms, since they are left inverse to the map \( \pi : \overline{B} \to B \)).
Remark 5.2.3.5. Let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets, and let \( (\text{id}_B, \text{id}_B) : (B \coprod_A B) \to B \) be the fold map. Unwinding the definitions, we see that a categorical mapping cylinder for \( B \) relative to \( A \) can be identified with a factorization of \( (\text{id}_B, \text{id}_B) \) as a composition

\[
B \coprod_A B \xrightarrow{\iota} \overline{B} \xrightarrow{\pi} B,
\]

where \( \iota \) is a monomorphism of simplicial sets and \( \pi \) is a categorical equivalence. Such factorizations always exist: by virtue of Exercise 3.1.6.11, we can even arrange that \( \pi \) is a trivial Kan fibration of simplicial sets (hence a categorical equivalence by virtue of Proposition 4.5.2.9).

Example 5.2.3.6. Let \( i : A \hookrightarrow B \) be a monomorphism of simplicial sets, and let \( Q \) be a Kan complex containing vertices \( x_0, x_1 \in Q \) with \( x_0 \neq x_1 \). Set \( \overline{B} = A \coprod_{(Q \times A)} (Q \times B) \). The commutative diagram

\[
\begin{array}{ccc}
Q \times A & \xrightarrow{i} & Q \times B \\
\downarrow & & \downarrow \\
A & \xrightarrow{i} & B
\end{array}
\]

is a categorical pushout square (since the vertical maps are categorical equivalences), and therefore induces a categorical equivalence \( \pi : \overline{B} \to B \) (Proposition 4.5.3.7). Let \( s_0 : B \to \overline{B} \) be the section of \( \pi \) given by the composition

\[
B \simeq \{x_0\} \times B \hookrightarrow Q \times B \to \overline{B},
\]

and define \( s_1 : B \to \overline{B} \) similarly. Then the quadruple \( (\overline{B}, \pi, s_0, s_1) \) is a categorical mapping cylinder of \( B \) relative to \( A \).

Corollary 5.2.3.7. Let \( \mathcal{C} \) be an \( \infty \)-category, and let \( f_0, f_1 : B \to \mathcal{C} \) be a pair of diagrams indexed by a simplicial set \( B \). Let

\[
(B \coprod_B) \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B
\]

be a categorical mapping cylinder for \( B \) (Definition 5.2.3.3). The following conditions are equivalent:

(a) The diagrams \( f_0 \) and \( f_1 \) are isomorphic when regarded as objects of the \( \infty \)-category \( \text{Fun}(B, \mathcal{C}) \).

(b) There exists a diagram \( \overline{f} : \overline{B} \to \mathcal{C} \) satisfying \( f_0 = \overline{f} \circ s_0 \) and \( f_1 = \overline{f} \circ s_1 \).

In particular, condition (b) does not depend on the choice of categorical mapping cylinder.
Proof. The implication \((a) \Rightarrow (b)\) follows from the implication \((1) \Rightarrow (3)\) of Proposition 5.2.3.1, and the implication \((b) \Rightarrow (a)\) from the implication \((2) \Rightarrow (1)\) of Proposition 5.2.3.1.

We will deduce Proposition 5.2.3.1 from a more general relative statement.

**Theorem 5.2.3.8.** Let \(q : X \to S\) be an isofibration of simplicial sets, let \(g : B \to S\) be a morphism of simplicial sets, and let \(f_0, f_1 : B \to X\) be morphisms satisfying \(q \circ f_0 = g = q \circ f_1\). Let \(A \subseteq B\) be a simplicial subset satisfying \(f_0|_A = f_1|_A\). The following conditions are equivalent:

1. The diagrams \(f_0\) and \(f_1\) are isomorphic when regarded as objects of the \(\infty\)-category \(\text{Fun}_{A/S}(B, X)\) (see Proposition 4.1.4.6).

2. There exists a factorization of the fold map \((\text{id}_B, \text{id}_B) : B \amalg A \to B\) as a composition

\[
B \amalg_A B \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B,
\]

where \(\pi\) is a categorical equivalence and the lifting problem

\[
\begin{array}{ccc}
B \amalg_A B & \xrightarrow{(f_0, f_1)} & X \\
\downarrow^{(s_0, s_1)} & & \downarrow^{q} \\
B & \xrightarrow{g \circ \pi} & S
\end{array}
\]

admits a solution.

3. For every factorization of the fold map \((\text{id}_B, \text{id}_B) : B \amalg_A B \to B\) as a composition

\[
B \amalg_A B \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B,
\]

where the map \((s_0, s_1) : B \amalg_A B \to B\) is a monomorphism, the lifting problem

\[
\begin{array}{ccc}
B \amalg_A B & \xrightarrow{(f_0, f_1)} & X \\
\downarrow^{(s_0, s_1)} & & \downarrow^{q} \\
B & \xrightarrow{g \circ \pi} & S
\end{array}
\]

admits a solution.
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Proof. By virtue of Corollary 4.4.3.10, condition (1) is satisfied if and only if there exists a morphism of simplicial sets $u : Q \to \text{Fun}_{A/\mathcal{S}}(B, X)$, where $Q$ is a contractible Kan complex, and a pair of vertices $x_0, x_1 \in Q$ satisfying $u(x_0) = f_0$ and $u(x_1) = f_1$. Moreover, we may assume (modifying $Q$ if necessary) that the vertices $x_0$ and $x_1$ are distinct. In this case, let

$$\mathcal{B} = A \coprod_{(Q \times A)} (Q \times B)$$

and let

$$B \coprod_A B \xrightarrow{(s_0, s_1)} \mathcal{B} \xrightarrow{\pi} B$$

be the categorical mapping cylinder described in Example 5.2.3.6. Unwinding the definitions, we see that morphisms $u : Q \to \text{Fun}_{A/\mathcal{S}}(B, X)$ satisfying $u(x_0) = f_0$ and $u(x_1) = f_1$ can be identified with solutions to the lifting problem

$$B \coprod_A B \xrightarrow{(f_0, f_1)} X \xrightarrow{q} \mathcal{B} \xrightarrow{g \circ \pi} S.$$ 

This proves that $(3) \Rightarrow (1) \Rightarrow (2)$.

We will complete the proof by showing that $(2) \Rightarrow (3)$. Assume that $(2)$ is satisfied, so that the fold map $(\text{id}_B, \text{id}_B) : B \coprod_A B \to B$ factors as a composition

$$B \coprod_A B \xrightarrow{(s_0, s_1)} \mathcal{B} \xrightarrow{\pi} B$$

where $\pi$ is a categorical equivalence and there exists a morphism $\overline{f} : \mathcal{B} \to X$ for which the diagram

$$B \coprod_A B \xrightarrow{(f_0, f_1)} X \xrightarrow{q} \mathcal{B} \xrightarrow{g \circ \pi} S$$

commutes. Using Exercise 3.1.6.11, we can factor $\pi$ as a composition

$$\mathcal{B} \xrightarrow{j} \mathcal{B}' \xrightarrow{\pi'} B,$$

where $j$ is a monomorphism and $\pi'$ is a trivial Kan fibration. Then $\pi'$ is also a categorical equivalence (Proposition 4.5.2.9), so the morphism $j$ is a categorical equivalence (Remark
Our assumption that \( q \) is an isofibration guarantees that the lifting problem

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{f} & X \\
\downarrow j & & \downarrow q \\
\mathcal{B} & \xrightarrow{g \circ \pi'} & S
\end{array}
\]

admits a solution \( f' : \mathcal{B} \to X \).

We now show that condition (3) is satisfied. Suppose that we are given another factorization of the fold map \((\text{id}_B, \text{id}_B) : (\text{id}_B, \text{id}_B) : B \coprod_A B \to B\) as a composition

\[
B \coprod_A B \xrightarrow{(f_0, f_1)} \mathcal{B}'' \xrightarrow{\pi''} B,
\]

where \( \iota \) is a monomorphism. We wish to show that the lifting problem

\[
\begin{array}{ccc}
B \coprod_A B & \xrightarrow{(f_0, f_1)} & X \\
\downarrow \iota & & \downarrow q \\
\mathcal{B}'' & \xrightarrow{g \circ \pi''} & S
\end{array}
\]

admits a solution \( f'' : \mathcal{B}'' \to X \). We first observe that the lifting problem

\[
\begin{array}{ccc}
B \coprod_A B & \xrightarrow{j \circ (s_0, s_1)} & \mathcal{B}' \\
\downarrow \iota & & \downarrow \pi' \\
\mathcal{B}'' & \xrightarrow{\pi''} & B
\end{array}
\]

admits a solution \( v : \mathcal{B}'' \to \mathcal{B}' \), since \( \iota \) is a monomorphism and \( \pi' \) is a trivial Kan fibration. We now conclude the proof by setting \( f'' = f' \circ v \).

**Corollary 5.2.3.9.** Let \( \mathcal{C} \) be an \( \infty \)-category, let \( f_0, f_1 : B \to \mathcal{C} \) be a pair of diagrams indexed by a simplicial set \( B \), and let \( A \subseteq B \) be a simplicial subset satisfying \( f_0|_A = f_1|_A \). The following conditions are equivalent:

1. The diagrams \( f_0 \) and \( f_1 \) are isomorphic when regarded as objects of the \( \infty \)-category \( \operatorname{Fun}_{\mathcal{A}/}(B, \mathcal{C}) \).

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(2) There exists a factorization of the fold map \((\text{id}_B, \text{id}_B) : B \coprod_A B \to B\) as a composition

\[ B \coprod_A B \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B, \]

where \(\pi\) is a categorical equivalence, and a morphism \(\overline{f} : B \to C\) satisfying \(f_0 = \overline{f} \circ s_0\) and \(f_1 = \overline{f} \circ s_1\).

(3) For every factorization of the fold map \((\text{id}_B, \text{id}_B) : B \coprod_A B \to B\) as a composition

\[ B \coprod_A B \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B, \]

where the map \((s_0, s_1) : B \coprod_A B \to B\) is a monomorphism, there exists a morphism \(\overline{f} : B \to C\) satisfying \(f_0 = \overline{f} \circ s_0\) and \(f_1 = \overline{f} \circ s_1\).

Proof. Apply Theorem 5.2.3.8 in the special case where \(S = \Delta^0\).

Proof of Proposition 5.2.3.9. Apply Corollary 5.2.3.9 in the special case \(A = \emptyset\).

For later use, we record a relative version of Corollary 5.2.3.7.

**Corollary 5.2.3.10.** Let \(q : X \to S\) be an isofibration of simplicial sets, let \(g : B \to S\) be a morphism of simplicial sets, and let \(f_0, f_1 : B \to X\) be morphisms satisfying \(q \circ f_0 = g = q \circ f_1\). Let \(A\) be a simplicial subset of \(B\) satisfying \(f_0|_A = f_1|_A\), and let

\[ (B \coprod_A B) \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B \]

be a categorical mapping cylinder of \(B\) relative to \(A\). The following conditions are equivalent:

(a) The diagrams \(f_0\) and \(f_1\) are isomorphic when regarded as objects of the \(\infty\)-category \(\text{Fun}_{A//S}(B, X)\).

(b) The lifting problem

\[ \begin{array}{ccc}
B \coprod_A B & \xrightarrow{(f_0, f_1)} & X \\
\downarrow (s_0, s_1) \quad \overline{f} \quad q \\
B & \xrightarrow{g \circ \pi} & S
\end{array} \]

admits a solution.

In particular, condition (b) does not depend on the choice of categorical mapping cylinder.
Proof. The implication \((a) \Rightarrow (b)\) follows from the implication \((1) \Rightarrow (3)\) of Theorem 5.2.3.8 and the implication \((b) \Rightarrow (a)\) from the implication \((2) \Rightarrow (1)\) of Theorem 5.2.3.8.

Corollary 5.2.3.11. Let \(C\) be an \(\infty\)-category and let \(f_0, f_1 : B \to C\) be a pair of diagrams indexed by a simplicial set \(B\). Let \(A\) be a simplicial subset of \(B\) satisfying \(f_0|_A = f_1|_A\), and let
\[
\left( B \coprod_A B \right)^{(s_0, s_1)} \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B
\]
be a categorical mapping cylinder of \(B\) relative to \(A\). The following conditions are equivalent:

(a) The diagrams \(f_0\) and \(f_1\) are isomorphic when regarded as objects of the \(\infty\)-category \(\text{Fun}_{A/}(B, C)\).

(b) There exists a diagram \(\bar{f} : B \to C\) satisfying \(f_0 = \bar{f} \circ s_0\) and \(f_1 = \bar{f} \circ s_1\).

In particular, condition (b) does not depend on the choice of categorical mapping cylinder.

Proof. Apply Corollary 5.2.3.10 in the special case \(S = \Delta^0\).

5.2.4 The Comma Construction

Let \(C\), \(D\), and \(E\) be categories. To every pair of functors \(F : C \to E\) and \(G : D \to E\), one can associate the \textit{comma category} \((F \downarrow G)\), whose objects are triples \((C, D, \eta)\) where \(C\) is an object of \(C\), \(D\) is an object of \(D\), and \(\eta : F(C) \to G(D)\) is a morphism in the category \(C\) (Notation 2.1.4.19). This construction has an obvious analogue in the setting of \(\infty\)-categories. We now generalize this construction to the setting of \(\infty\)-categories.

Construction 5.2.4.1. Let \(F : C \to \mathcal{E}\) and \(G : D \to \mathcal{E}\) be morphisms of simplicial sets. We let \((F \downarrow G)\) denote the simplicial set given by the iterated fiber product
\[
C \times_{\text{Fun}(\{0\}, \mathcal{E})} \text{Fun}(\Delta^1, \mathcal{E}) \times_{\text{Fun}(\{1\}, \mathcal{E})} D.
\]
We will refer to \((F \downarrow G)\) as the \textit{comma construction} on \(F\) and \(G\).

In the special case where \(C = \mathcal{E}\) and \(F = \text{id}_C\) is the identity map, we will denote the comma construction \((F \downarrow G)\) by \((C \downarrow G)\); it can be described more simply as the fiber product \(\text{Fun}(\Delta^1, C) \times_{\text{Fun}(\{1\}, C)} D\). Similarly, in the special case where \(D = \mathcal{E}\) and \(G = \text{id}_D\) is the identity map, we denote the comma construction \((F \downarrow G)\) by \((F \downarrow D)\); it can be described more simply as the fiber product \(C \times_{\text{Fun}(\{0\}, D)} \text{Fun}(\Delta^1, D)\).

As our notation suggests, we will be primarily interested in the special case of Construction 5.2.4.1 where the simplicial sets \(C\), \(D\), and \(E\) are \(\infty\)-categories.
Proposition 5.2.4.2. Let \( \mathcal{E} \) be an \( \infty \)-category, and suppose we are given morphisms of simplicial sets \( F : \mathcal{C} \rightarrow \mathcal{E} \) and \( G : \mathcal{D} \rightarrow \mathcal{E} \). Then the projection map \( \theta : (F \downarrow G) \rightarrow \mathcal{C} \times \mathcal{D} \) is an isofibration of simplicial sets.

Proof. By construction, we have a pullback diagram of simplicial sets

\[
\begin{array}{ccc}
(F \downarrow G) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{E}) \\
\downarrow \theta & & \downarrow \theta_0 \\
\mathcal{C} \times \mathcal{D} & \longrightarrow & \text{Fun}(\partial \Delta^1, \mathcal{E}).
\end{array}
\]

Since \( \mathcal{E} \) is an \( \infty \)-category, the restriction map \( \theta_0 \) is an isofibration of \( \infty \)-categories (Corollary 4.4.5.3). Invoking Remark 4.5.7.7, we conclude that \( \theta \) is an isofibration of simplicial sets.

Corollary 5.2.4.3. Let \( F : \mathcal{C} \rightarrow \mathcal{E} \) and \( G : \mathcal{D} \rightarrow \mathcal{E} \) be functors of \( \infty \)-categories. Then the comma construction \((F \downarrow G)\) is also an \( \infty \)-category.

Proof. By virtue of Proposition 5.2.4.2, the projection map \((F \downarrow G) \rightarrow \mathcal{C} \times \mathcal{D}\) is an isofibration. Since \( \mathcal{C} \times \mathcal{D} \) is an \( \infty \)-category, it follows that \((F \downarrow G)\) is also an \( \infty \)-category (Remark 4.5.7.3).

Example 5.2.4.4. Let \( F : \mathcal{C} \rightarrow \mathcal{E} \) and \( G : \mathcal{D} \rightarrow \mathcal{E} \) be functors between ordinary categories, and let \((F \downarrow G)\) denote the comma category of Notation 2.1.4.19. Since the nerve construction is compatible with the formation of inverse limits and functor categories, we have a canonical isomorphism of simplicial sets

\[ \text{Sing}_\bullet (F \downarrow G) \simeq (\text{Sing}_\bullet (F) \downarrow \text{Sing}_\bullet (G)). \]

Consequently, Construction 5.2.4.1 can be viewed as a generalization of the classical comma construction in category theory.

Example 5.2.4.5. Let \( X, Y, \) and \( Z \) be topological spaces, and suppose we are given a pair of continuous functions \( f : X \rightarrow Z \) and \( g : Y \rightarrow Z \). We let \((f \downarrow g)\) denote the set of all triples \((x, y, \eta)\) where \( x \) is a point of \( X \), \( y \) is a point of \( Y \), and \( \eta : [0, 1] \rightarrow Z \) is a continuous function satisfying \( \eta(0) = f(x) \) and \( \eta(1) = g(y) \). The set \((f \downarrow g)\) carries a natural topology, which is defined by viewing it as a subspace of the product \( X \times Y \times \text{Hom}_{\text{Top}}([0, 1], Z) \) (where we endow the path space \( \text{Hom}_{\text{Top}}([0, 1], Z) \) with the compact-open topology). We then have a canonical isomorphism of simplicial sets

\[ \text{Sing}_\bullet (f \downarrow g) \simeq \text{Sing}_\bullet (f) \downarrow \text{Sing}_\bullet (g), \]

where the right hand side is given by Construction 5.2.4.1.
Remark 5.2.4.6. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets, and let $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{E}^{\text{op}}$ and $G^{\text{op}} : \mathcal{D}^{\text{op}} \to \mathcal{E}^{\text{op}}$ be the opposite morphisms. Then we have a canonical isomorphism of simplicial sets

$$(F \downarrow G)^{\text{op}} \simeq (G^{\text{op}} \downarrow F^{\text{op}}).$$

We will be particularly interested in the special case of Construction 5.2.4.1.

Notation 5.2.4.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let $X$ be a vertex of $\mathcal{D}$, classified by a morphism of simplicial sets $G : \Delta^0 \to \mathcal{D}$. In this case, we denote the comma constructions $(F \downarrow G)$ and $(G \downarrow F)$ by $(F \downarrow X)$ and $(X \downarrow F)$, respectively. In the special case where $\mathcal{C} = \mathcal{D}$ and $F = \text{id}_\mathcal{C}$ is the identity morphism, we will also denote $(F \downarrow G)$ and $(G \downarrow F)$ by $(\mathcal{C} \downarrow X)$ and $(X \downarrow \mathcal{C})$, respectively.

Remark 5.2.4.8. Let $\mathcal{C}$ be a simplicial set and let $X$ be a vertex of $\mathcal{C}$. For any simplicial set $K$, we have canonical isomorphisms of simplicial sets

$$\text{Fun}(K, (\mathcal{C} \downarrow X)) \simeq \text{Fun}_{\{x\}}/(K \circ \{x\}, \mathcal{C})$$

$$\text{Fun}(K, (X \downarrow \mathcal{C})) \simeq \text{Fun}_{\{x\}}/\{x\} \circ K, \mathcal{C},$$

where $K \circ \{x\}$ and $\{x\} \circ K$ denote the simplicial sets defined in Notation 4.5.5.3. Restricting to vertices, we obtain bijections

$$\{\text{Morphisms } K \to (\mathcal{C} \downarrow X)\} \simeq \{\text{Morphisms } f : (K \circ \{x\}) \to \mathcal{C} \text{ with } f(x) = X\}$$

$$\{\text{Morphisms } K \to (X \downarrow \mathcal{C})\} \simeq \{\text{Morphisms } f : (\{x\} \circ K) \to \mathcal{C} \text{ with } f(x) = X\}.$$

Example 5.2.4.9. Let $\mathcal{C}$ be a simplicial set containing vertices $X$ and $Y$, which we identify with morphisms of simplicial sets $X, Y : \Delta^0 \to \mathcal{C}$. Then the simplicial set $\text{Hom}_\mathcal{C}(X, Y)$ of Construction 5.2.1.1 can be identified with the comma construction

$$(X \downarrow \mathcal{C}) \times_{\mathcal{C}} \{Y\} = (X \downarrow Y) = \{X\} \times_{\mathcal{C}} (\mathcal{C} \downarrow Y).$$

The following result is a relative version of Proposition 5.2.1.8.

Proposition 5.2.4.10. Let $\mathcal{C}$ be an $\infty$-category containing an object $X$. Then the projection map $(X \downarrow \mathcal{C}) \to \mathcal{C}$ is a left fibration and the projection map $(\mathcal{C} \downarrow X) \to \mathcal{C}$ is a right fibration.

Proof. We will prove the second assertion; the second follows by a similar argument. Let $A \hookrightarrow B$ be a right anodyne morphism of simplicial sets; we wish to show that every lifting problem

\[ \begin{array}{ccc}
A & \to & (\mathcal{C} \downarrow X) \\
\downarrow & & \downarrow \\
B & \to & \mathcal{C}
\end{array} \]
admits a solution. Unwinding the definitions, we are reduced to showing that a map of simplicial sets
\[ \sigma_0 : B \coprod_A (A \diamond \{X\}) \to C \]
can be extended to a map \( \sigma : B \diamond \{X\} \to C \) (see Notation 4.5.5.3). By virtue of Lemma 4.5.6.2, it will suffice to show that the inclusion map
\[ \iota : B \coprod_A (A \diamond \{X\}) \hookrightarrow B \diamond \{X\} \]
is a categorical equivalence of simplicial sets, which follows from Corollary 4.5.5.12.

**Corollary 5.2.4.11.** Let \( \mathcal{D} \) be an \( \infty \)-category containing an object \( X \), and let \( F : \mathcal{C} \to \mathcal{D} \) be a morphism of simplicial sets. Then the projection map \( (X \downarrow F) \to \mathcal{C} \) is a left fibration, and the projection map \( (F \downarrow X) \to \mathcal{C} \) is a right fibration.

**Proof.** Unwinding the definition, we have pullback diagrams
\[
\begin{array}{ccc}
(X \downarrow F) & \rightarrow & (X \downarrow \mathcal{D}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\quad
\begin{array}{ccc}
(F \downarrow X) & \rightarrow & (\mathcal{D} \downarrow X) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D},
\end{array}
\]
The desired result now follows by combining Proposition 5.2.4.10 with Remark 4.2.1.8.

Recall that, if \( \mathcal{C} \) is an ordinary category containing an object \( X \), then the slice and coslice categories \( \mathcal{C}/X \) and \( \mathcal{C}\downarrow X \) can be identified with the comma categories \( \mathcal{C} \downarrow X \) and \( (X \downarrow \mathcal{C}) \), respectively (Remark 4.3.1.6). Our next goal is to establish a similar result in the setting of \( \infty \)-categories. Here the situation is a bit more subtle: if \( X \) is an object of an \( \infty \)-category \( \mathcal{C} \), then the simplicial sets \( \mathcal{C}/X \) and \( \mathcal{C} \downarrow X \) are generally not isomorphic. However, we will show that they are nevertheless equivalent as \( \infty \)-categories.

**Construction 5.2.4.12.** For each \( n \geq 0 \), let
\[
c_n^L : \Delta^1 \times \Delta^n \simeq \Delta^0 \times \Delta^1 \times \Delta^n \rightarrow \Delta^0 \star \Delta^n \simeq \Delta^{n+1}
\]
\[
c_n^R : \Delta^n \times \Delta^1 \simeq \Delta^n \times \Delta^1 \times \Delta^0 \rightarrow \Delta^n \star \Delta^0 \simeq \Delta^{n+1}
\]
denote the collapse morphisms given by Construction 4.5.5.1, given concretely on vertices by the formulae
\[
c_n^L(i, j) = \begin{cases} 
0 & \text{if } i = 0 \\
1 & \text{if } i = 1 
\end{cases}
\]
\[
c_n^R(j, i) = \begin{cases} 
0 & \text{if } i = 0 \\
1 & \text{if } i = 1 
\end{cases}
\]
Let $C$ be a simplicial set containing a vertex $X$, and let $\sigma$ be an $n$-simplex of the coslice simplicial set $C_{X/}$, which we identify with a morphism of simplicial sets $\sigma : \Delta^{n+1} \to C$ satisfying $\sigma(0) = X$. The composite map $\Delta^1 \times \Delta^n \xrightarrow{c^L_n} \Delta^{n+1} \xrightarrow{\sigma} C$ can be identified with an $n$-simplex of the coslice construction $(X \downarrow C)$ which we will denote by $i^L_X(\sigma)$. The construction $\sigma \mapsto i^L_X(\sigma)$ determines a morphism of simplicial sets $i^L_X : C_{X/} \to (X \downarrow C)$, which we will refer to as the left-pinch inclusion map.

Similarly, let $\tau$ be an $n$-simplex of the slice simplicial set $C_{/X}$, which we identify with a morphism of simplicial sets $\tau : \Delta^{n+1} \to C$ satisfying $\tau(n+1) = X$. The composite map $\Delta^n \times \Delta^1 \xrightarrow{c^R_n} \Delta^{n+1} \xrightarrow{\tau} C$ can be identified with an $n$-simplex of the comma construction $(C \downarrow X)$ which we will denote by $i^R_X(\tau)$. The construction $\sigma \mapsto i^R_X(\tau)$ determines a morphism of simplicial sets $i^R_X : C_{/X} \to (C \downarrow X)$, which we will refer to as the right-pinch inclusion map.

**Remark 5.2.4.13.** Let $C$ be a simplicial set containing a vertex $X$. Then the pinch inclusion maps

$$i^L_X : C_{X/} \to (X \downarrow C) \quad i^R_X : C_{/X} \to (C \downarrow X)$$

are monomorphisms of simplicial sets: this follows immediately from surjectivity of the collapse morphisms

$$c^L_n : \Delta^1 \times \Delta^n \to \Delta^{n+1} \quad c^R_n : \Delta^n \times \Delta^1 \to \Delta^{n+1}.$$

**Theorem 5.2.4.14.** Let $C$ be an $\infty$-category and let $X$ be an object of $C$. Then the pinch inclusion maps

$$i^L_X : C_{X/} \hookrightarrow (X \downarrow C) \quad i^R_X : C_{/X} \hookrightarrow (C \downarrow X)$$

are equivalences of $\infty$-categories.

The proof of Theorem 5.2.4.14 will make use of the following:

**Lemma 5.2.4.15.** Let $C$ be an $\infty$-category, let $g : A \to C$ be a diagram indexed by a simplicial set $A$. Suppose we are given a pair of diagrams $f_0, f_1 : B \to C_{g/}$ indexed by a simplicial set $B$, which we identify with diagrams $f'_0, f'_1 : (A \star B) \to C$ satisfying $f'_0|_A = g = f'_1|_A$. The following conditions are equivalent:

1. The diagrams $f_0$ and $f_1$ are isomorphic when regarded as objects of the diagram $\infty$-category $\text{Fun}(B, C_{g/})$.

2. The diagrams $f'_0$ and $f'_1$ are isomorphic when regarded as objects of the $\infty$-category $\text{Fun}_A(A \star B, C)$. 

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Proof. Choose a categorical mapping cylinder

$$B \coprod B \xrightarrow{(s_0, s_1)} B \xrightarrow{\pi} B$$

for the simplicial set $B$ (Definition 5.2.3.3). Using Corollary 4.5.5.8, we deduce that the resulting diagram

$$(A \star B) \coprod (A \star B) \xrightarrow{(s_0', s_1')} A \star B \xrightarrow{\pi'} A \star B$$

is a categorical mapping cylinder for the join $A \star B$ relative to $A$. Using the criterion of Corollary 5.2.3.11, we see that (1) and (2) can be reformulated as follows:

(1′) There exists a diagram $\mathcal{F} : \mathcal{B} \to C$ satisfying $f_0 = \mathcal{F} \circ s_0$ and $f_1 = \mathcal{F} \circ s_1$.

(2′) There exists a diagram $\mathcal{F}' : A \star B \to C$ satisfying $f'_0 = \mathcal{F}' \circ s'_0$ and $f'_1 = \mathcal{F}' \circ s'_1$.

The equivalence of (1′) and (2′) now follows immediately from the universal property of the coslice $\infty$-category $C_{g/}$.

Proof of Theorem 5.2.4.14. Let $\mathcal{C}$ be an $\infty$-category containing an object $X$. We will show that the left-pinching inclusion morphism $\iota_X^L : C_{X/} \hookrightarrow (X \downarrow \mathcal{C})$ is an equivalence of $\infty$-categories; the analogous statement for $\iota_X^R$ follows by a similar argument. Let $K$ be a simplicial set and $\rho : \pi_0(\text{Fun}(K, C_{X/})^\simeq) \to \pi_0(\text{Fun}(K, X \downarrow \mathcal{C})^\simeq)$ be the map induced by composition with $\iota_X^L$; we wish to show that $\rho$ is a bijection. Using Lemma 5.2.4.15 and Remark 5.2.4.8, we can identify $\rho$ with the map

$$\pi_0(\text{Fun}_{\{x\}}(\{x\} \star K, C)^\simeq) \to \pi_0(\text{Fun}_{\{x\}}(\{x\} \diamond K, C)^\simeq)$$

given by composition with the comparison map $c : \{x\} \diamond K \to \{x\} \star K$. We will complete the proof by showing that composition with $c$ induces an equivalence of $\infty$-categories $\text{Fun}_{\{x\}}(\{x\} \star K, C) \to \text{Fun}_{\{x\}}(\{x\} \diamond K, C)$. This follows by applying Corollary 4.5.4.4 to the commutative diagram

$$\begin{align*}
\text{Fun}(\{x\} \star K, C) &\xrightarrow{\circ c} \text{Fun}(\{x\} \diamond K, C) \\
\text{Fun}(\{x\}, C) &\xrightarrow{} \text{Fun}(\{x\}, C);
\end{align*}$$

here the vertical maps are isofibrations (Corollary 4.4.5.3) and the upper horizontal map is an equivalence of $\infty$-categories by Proposition 4.5.2.8 (and the fact that $c$ is a categorical equivalence, by Theorem 4.5.5.7).
5.2.5 Pinched Morphism Spaces

Let \( C \) be an \( \infty \)-category. In §5.2.1, we associated to every pair of objects \( X, Y \in C \) a Kan complex \( \text{Hom}_C(X, Y) \), which we refer to as the space of morphisms from \( X \) to \( Y \) (Construction 5.2.1.1). In this section, we discuss a variant of this construction which is often more technically convenient to work with.

Construction 5.2.5.1. Let \( C \) be a simplicial set containing vertices \( X \) and \( Y \). We let \( \text{Hom}^L_C(X, Y) \) denote the fiber product \( C/X \times_C \{ Y \} \), and we let \( \text{tHom}^R_C(X, Y) \) denote the fiber product \( \{ X \} \times_C C/Y \). We will be primarily interested in these constructions in the situation where \( C \) is an \( \infty \)-category. In this case, we refer to \( \text{Hom}^L_C(X, Y) \) as the left-pinched space of morphisms from \( X \) to \( Y \) and to \( \text{Hom}^R_C(X, Y) \) as the right-pinched space of morphisms from \( X \) to \( Y \).

Remark 5.2.5.2. Let \( C \) be a simplicial set containing vertices \( X \) and \( Y \). For every integer \( n \geq 0 \), one can identify \( n \)-simplices of the left-pinched morphism space \( \text{Hom}^L_C(X, Y) \) with \((n + 1)\)-simplices \( \sigma : \Delta^{n+1} \to C \) for which \( \sigma(0) = X \) and the face \( d_0(\sigma) \) is the constant map \( \Delta^n \to \{ Y \} \leftarrow C \). Similarly, one can identify \( n \)-simplices of the right-pinched morphism space \( \text{Hom}^R_C(X, Y) \) with \((n + 1)\)-simplices \( \sigma' : \Delta^{n+1} \to C \) for which \( \sigma(n + 1) = Y \) and the face \( d_{n+1}(\sigma) \) is the constant map \( \Delta^n \to \{ X \} \leftarrow C \). In particular, we have canonical bijections

\[
\{ \text{Vertices of } \text{Hom}^L_C(X, Y) \} \simeq \{ \text{Edges } f : X \to Y \text{ in } C \} \simeq \{ \text{Vertices of } \text{Hom}^R_C(X, Y) \}.
\]

Remark 5.2.5.3. Let \( C \) be a simplicial set containing vertices \( X \) and \( Y \), which we also regard as vertices of the opposite simplicial set \( C^{op} \). Then we have canonical isomorphisms of simplicial sets

\[
\text{Hom}^L_{C^{op}}(X, Y) \simeq \text{Hom}^R_C(Y, X)^{op} \quad \text{and} \quad \text{Hom}^R_{C^{op}}(X, Y) \simeq \text{Hom}^L_C(Y, X)^{op}.
\]

Proposition 5.2.5.4. Let \( C \) be an \( \infty \)-category. For every pair of objects \( X, Y \in C \), the pinched morphism spaces \( \text{Hom}^L_C(X, Y) \) and \( \text{Hom}^R_C(X, Y) \) are Kan complexes.

Proof. By virtue of Proposition 4.3.6.1, the projection map \( C/X \to C \) is a left fibration. Applying Corollary 4.4.2.2 we deduce that the fiber \( \text{Hom}^L_C(X, Y) = C/X \times_C \{ Y \} \) is a Kan complex. A similar argument shows that \( \text{Hom}^R_C(X, Y) \) is a Kan complex.

Remark 5.2.5.5. Let \( C \) be an \( \infty \)-category containing a pair of morphisms \( f, g : X \to Y \) having the same source and target. Then the datum of an edge \( e : f \to g \) in the left-pinched morphism space \( \text{Hom}^L_C(X, Y) \) is equivalent to the datum of a homotopy from \( f \) to \( g \), in the sense of Definition 1.3.3.1. In particular, \( f \) and \( g \) are homotopic if and only if they belong to the same connected component of \( \text{Hom}^L_C(X, Y) \). We therefore have a canonical bijection \( \text{Hom}_{hc}(X, Y) \simeq \pi_0(\text{Hom}^L_C(X, Y)) \).
We now compare the pinched morphism spaces of Construction 5.2.5.1 with the morphism spaces of Construction 5.2.1.1.

**Construction 5.2.5.6.** Let $C$ be a simplicial set containing vertices $X$ and $Y$, and let

$$
i^L_X : C_{X/} \hookrightarrow (X \downarrow C) \quad \ni^R_Y : C_{/Y} \hookrightarrow (C \downarrow Y)$$

be the pinch inclusion morphisms of Construction 5.2.4.12. Restricting to the fibers over the objects $Y, X \in C$, we obtain morphisms of Kan complexes

$$\hom_L^C(X,Y) = C_{X/} \times_C \{Y\} \to (X \downarrow C) \times_C Y = \hom_C(X,Y)$$

$$\hom_R^C(X,Y) = \{X\} \times_C C_{/Y} \to \{X\} \times_C (C \downarrow Y) = \hom_C(X,Y),$$

which we will denote by $\ni^L_{X,Y}$ and $\ni^R_{X,Y}$, respectively. We will refer to $\ni^L_{X,Y}$ as the \textit{left-pinch inclusion map} and to $\ni^R_{X,Y}$ as the \textit{right-pinch inclusion map}.

**Remark 5.2.5.7.** Let $C$ be a simplicial set containing vertices $X$ and $Y$. Then the pinch inclusion maps

$$\hom_L^C(X,Y) \xrightarrow{\ni^L_{X,Y}} \hom_C(X,Y) \xleftarrow{\ni^R_{X,Y}} \hom_R^C(X,Y)$$

are monomorphisms (see Remark 5.2.4.13).

**Remark 5.2.5.8.** Let $C$ be an $\infty$-category containing objects $X$ and $Y$. Then the pinch inclusion maps

$$\hom_L^C(X,Y) \xrightarrow{\ni^L_{X,Y}} \hom_C(X,Y) \xleftarrow{\ni^R_{X,Y}} \hom_R^C(X,Y)$$

are bijective on vertices: vertices of each simplicial set can be identified with morphisms from $X$ to $Y$ in the $\infty$-category $C$ (Remarks 5.2.1.2 and 5.2.5.2). However, they are generally not bijective on edges. Note that edges of the simplicial set $\hom_C(X,Y)$ can be identified with diagrams

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{\sigma} & \searrow^{\tau} & \downarrow^{\id_Y} \\
X & \xrightarrow{f'} & Y
\end{array}$$
in the ∞-category \( C \). Such a diagram belongs to the image of the left-pinching inclusion map \( \iota^L_{X,Y} \) if and only \( \tau = s_0(g) \) (so that the simplex \( \tau \) is degenerate, \( f' = g \), and the entire diagram is determined by \( \sigma \)). Similarly, the diagram belongs to the image of the right-pinching inclusion map \( \iota^R_{X,Y} \) if and only if \( \sigma = s_1(g) \) (so that the simplex \( \sigma \) is degenerate, \( f = g \), and the entire diagram is determined by \( \tau \)).

**Proposition 5.2.5.9.** Let \( C \) be an ∞-category. For every pair of objects \( X, Y \in C \), the pinch inclusion morphisms

\[
\begin{align*}
\text{Hom}^L_C(X,Y) & \xrightarrow{\iota^L_{X,Y}} \text{Hom}_C(X,Y) \xleftarrow{\iota^R_{X,Y}} \text{Hom}^R_C(X,Y)
\end{align*}
\]

are homotopy equivalences of Kan complexes.

**Proof.** We will prove that the left-pinching inclusion morphism \( \iota^L_{X,Y} \) is a homotopy equivalence; the proof for the right-pinching inclusion morphism \( \iota^R_{X,Y} \) is similar. Note that we have a commutative diagram of ∞-categories

\[
\begin{array}{ccc}
\mathcal{C}_{X/} & \xrightarrow{\iota^L_X} & (X \downarrow \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\text{id}} & \mathcal{C},
\end{array}
\]

where the horizontal maps are equivalences of ∞-categories (Theorem 5.2.4.14) and the vertical maps are left fibrations (Propositions 4.3.6.1 and 5.2.4.10), hence isofibrations (Example 4.4.1.10). Applying Corollary 4.5.4.4, we deduce that the induced map of fibers

\[
\iota^L_{X,Y} : \text{Hom}^L_C(X,Y) = (\mathcal{C}_{X/}) \times_{\mathcal{C}} \{Y\} \to (X \downarrow \mathcal{C}) \times_{\mathcal{C}} \{Y\} = \text{Hom}^L_C(X,Y)
\]

is an equivalence of ∞-categories, hence a homotopy equivalence of Kan complexes (Remark 4.5.1.3).

**Example 5.2.5.10.** Let \( C \) be an ordinary category containing objects \( X \) and \( Y \). Then the pinch inclusion morphisms

\[
\begin{align*}
\iota^L_X : N_\bullet(C)_{X/} & \to (X \downarrow N_\bullet(C)) \\
\iota^R_Y : N_\bullet(C)/Y & \to (N_\bullet(C) \downarrow Y)
\end{align*}
\]

are isomorphisms (see Remark 4.3.1.6). In particular, we can identify the pinched morphism spaces \( \text{Hom}^L_{N_\bullet(C)}(X,Y) \) and \( \text{Hom}^R_{N_\bullet(C)}(X,Y) \) with the constant simplicial set \( \text{Hom}_{N_\bullet(C)}(X,Y) \) associated to the usual morphism set \( \text{Hom}_C(X,Y) \).
5.2. MORPHISM SPACES

Let $C$ be an $\infty$-category containing a pair of objects $X$ and $Y$. By virtue of Proposition 5.2.5.9, the pinched morphism spaces $\text{Hom}_C^L(X,Y)$ and $\text{Hom}_C^R(X,Y)$ of Construction 5.2.5.1 contain the same homotopy-theoretic information as the morphism space $\text{Hom}_C(X,Y)$ of Construction 5.2.1.1. However, they package this information in a more efficient way: an $n$-simplex of the Kan complex $\text{Hom}_C^L(X,Y)$ can be identified with a single $(n+1)$-simplex of the $\infty$-category $C$ (see Remark 5.2.5.2), but to specify an $n$-simplex of $\text{Hom}_C(X,Y)$ one must supply $n+1$ different $(n+1)$-simplices of $C$ (see Remark 5.2.5.8 for the case $n=1$).

Example 5.2.5.11 (Pinched Morphism Spaces in the Duskin Nerve). Let $C$ be a 2-category (Definition 2.2.1.1) and let $N^\bullet_D(C)$ denote its Duskin nerve (Construction 2.3.1.1). Let $X$ and $Y$ be objects of $C$, which we identify with vertices of the simplicial set $N^\bullet_D(C)$. Using Example 5.2.1.11, we can identify $n$-simplices $\sigma$ of the simplicial set $\text{Hom}_{N^\bullet_D(C)}(X,Y)$ with commutative diagrams

\[ \begin{array}{c}
  f_{0,0} \quad f_{1,1} \quad f_{n,n} \\
  \vdots \quad \vdots \quad \vdots \\
  f_{0,n-1} \quad f_{1,n} \\
  \vdots \quad \vdots \quad \vdots \\
  f_{0,n} \\
  \end{array} \]

in the category of 1-morphisms $\text{Hom}_C(X,Y)$. The image of the left-pinch inclusion morphism

$\iota^L_{X,Y} : \text{Hom}_{N^\bullet_D(C)}^L(X,Y) \hookrightarrow \text{Hom}_{N^\bullet_D(C)}(X,Y)$

consists of those simplices which correspond (under this identification) to commutative diagrams in which each of the leftward pointing 2-morphisms $f_{i,j} \Rightarrow f_{i-1,j}$ is an identity map. In this case, the entire diagram is determined by the sequence of composable morphisms $f_{0,0} \Rightarrow f_{0,1} \Rightarrow f_{0,2} \Rightarrow \cdots \Rightarrow f_{0,n}$ in the category $\text{Hom}_C(X,Y)$. Similarly, the image of the right-pinch inclusion morphism

$\iota^R_{X,Y} : \text{Hom}_{N^\bullet_D(C)}^R(X,Y) \hookrightarrow \text{Hom}_{N^\bullet_D(C)}(X,Y)$

consists of those simplices which correspond to commutative diagrams in which the rightward pointing 2-morphisms $f_{i,j} \Rightarrow f_{i,j+1}$ is an identity map, which ensures that the entire diagram is determined by the sequence of composable morphisms $f_{n,n} \Rightarrow f_{n-1,n} \Rightarrow f_{n-2,n} \Rightarrow \cdots \Rightarrow$.
$f_{0,n}$ in $\text{Hom}_C(X,Y)$. Allowing $[n] \in \Delta$ to vary, we obtain canonical isomorphisms of simplicial sets

$$\text{Hom}_{N\Delta(C)}^L(X,Y) \simeq N_\bullet(\text{Hom}_C(X,Y)) \quad \text{Hom}_{N\Delta(C)}^R(X,Y) \simeq N_\bullet(\text{Hom}_C(X,Y))^{op}.$$ 

**Example 5.2.5.12.** Let $X$ be a topological space containing a pair of points $x$ and $y$, which we regard as objects of the $\infty$-category $\text{Sing}_\bullet(X)$. Using Example 4.3.5.9, we obtain canonical isomorphisms of Kan complexes

$$\text{Hom}_{\text{Sing}_\bullet(X)}^L(x,y) \simeq \text{Sing}_\bullet(P_{x,y}) \simeq \text{Hom}_{\text{Sing}_\bullet(X)}^R(x,y),$$

where $P_{x,y}$ denotes the topological space of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$ (equipped with the compact-open topology). Combining this observation with Example 5.2.1.4, we can identify the pinch inclusion maps $\iota_{x,y}^L$ and $\iota_{x,y}^R$ with monomorphisms from the simplicial set $\text{Sing}_\bullet(P_{x,y})$ to itself. Beware that these maps are not the identity (though one can show that they are homotopic to the identity).

**Example 5.2.5.13 (Pinched Morphism Spaces in the Differential Graded Nerve).** Let $C$ be a differential graded category (Definition 2.5.2.1), let $N_{d^g}(C)$ denote the differential graded nerve of $C$ (Definition 2.5.3.7), and let $X$ and $Y$ be objects of $C$ (which we also view as objects of the $\infty$-category $N_{d^g}(C)$), and let $\text{Hom}_C(X,Y)_*$ denote the chain complex of morphisms from $X$ to $Y$. For $n \geq 0$, we can identify $n$-simplices of the right-pinched morphism space $\text{Hom}_{N_{d^g}(C)}^R(X,Y)$ with $(n+1)$-simplices $\sigma : \Delta^{n+1} \to N_{d^g}(C)$ for which the face $d_{n+1}(\sigma)$ is the constant $n$-simplex with value $X$ and $\sigma(n+1)$ is the vertex $Y$ (Remark 5.2.5.2). Unwinding the definitions, we see that such an $n$-simplex can be identified with a collection of elements $g_J \in \text{Hom}(X,Y)_k$ for each nonempty subset $J = \{j_k > j_{k-1} > \cdots > j_0\} \subseteq [n]$ which satisfy the equation

$$\partial g_J = \sum_{0 \leq a \leq k} (-1)^a g_J \setminus \{j_a\};$$

here $g_J$ corresponds to the chain $f_{J \cup \{n+1\}}$ appearing in Construction 2.5.3.1. The construction $J \mapsto g_J$ can then be identified with a morphism from the normalized chain complex $N_\bullet(\Delta^n)$ of Construction 2.5.5.9 to the chain complex $\text{Hom}_C(X,Y)_*$. This identification depends functorially on $[n] \in \Delta$, and therefore determines an isomorphism of simplicial sets

$$\text{Hom}_{N_{d^g}(C)}^R(X,Y) \simeq K(\text{Hom}_C(X,Y)_*),$$

where $K(\text{Hom}_C(X,Y)_*)$ denotes the Eilenberg-MacLane space associated to the chain complex $\text{Hom}_C(X,Y)_*$ (Construction 2.5.6.3). In particular, the right-pinched morphism space $\text{Hom}_{N_{d^g}(C)}^R(X,Y)$ has the structure of a simplicial abelian group.
5.2.6 Morphism Spaces in the Homotopy Coherent Nerve

Let \( C \) be a simplicial category and let \( \operatorname{N}^{\text{hc}}(C) \) denote the homotopy coherent nerve of \( C \) (Definition 2.4.3.5). Suppose that \( C \) is locally Kan, so that the simplicial set \( \operatorname{N}^{\text{hc}}(C) \) is an \( \infty \)-category (Theorem 2.4.5.1). Our goal in this section is to describe the morphism spaces in the \( \infty \)-category \( \operatorname{N}^{\text{hc}}(C) \). Our main result (Theorem 5.2.6.4) implies that, for every pair of objects \( X, Y \in C \), there is a canonical homotopy equivalence

\[
\operatorname{Hom}_C(X,Y) \to \operatorname{Hom}_{\operatorname{N}^{\text{hc}}(C)}(X,Y),
\]

where \( \operatorname{Hom}_C(X,Y) \) denotes the Kan complex of morphisms from \( X \) to \( Y \) in \( C \), and \( \operatorname{Hom}_{\operatorname{N}^{\text{hc}}(C)}(X,Y) \) is given by Construction 5.2.1.1.

**Notation 5.2.6.1.** Let \( K \) be a simplicial set. We define a simplicial category \( E[K] \) as follows:

- The category \( E[K] \) has exactly two objects, which we will denote by \( x \) and \( y \).
- The morphism spaces in \( E[K] \) are given by the formulae
  \[
  \operatorname{Hom}_{E[K]}(x,x) = \{ \operatorname{id}_x \} \\
  \operatorname{Hom}_{E[K]}(y,y) = \{ \operatorname{id}_y \} \\
  \operatorname{Hom}_{E[K]}(x,y) = K \\
  \operatorname{Hom}_{E[K]}(y,x) = \emptyset.
  \]

**Remark 5.2.6.2.** The simplicial category \( E[K] \) is characterized by the following universal property: if \( C \) is any simplicial category containing a pair of objects \( X \) and \( Y \), then the natural map

\[
\{ \text{Simplicial Functors } F : E[K] \to C \text{ with } F(x) = X \text{ and } F(y) = Y \} \to \operatorname{Hom}_{\Delta}(K, \operatorname{Hom}_C(X,Y))
\]

is a bijection (see Proposition 2.4.5.9).

**Construction 5.2.6.3.** Fix an integer \( n \geq 0 \), and let \( \operatorname{Path}[n+1] \) denote the simplicial path category of Notation 2.4.3.1. We define a simplicial functor \( \pi : \operatorname{Path}[n+1] \to E[\Delta^n] \) as follows:

- On objects, the functor \( \pi \) is given by the formula
  \[
  \pi(i) = \begin{cases} 
  x & \text{if } 0 \leq i \leq n \\
  y & \text{if } i = n + 1.
  \end{cases}
  \]
• For $0 \leq i \leq n$, the morphism of simplicial sets

$$\text{Hom}_{\text{Path}[n+1]}(i, n + 1) \to \text{Hom}_{\mathcal{C}}[\Delta^n](x, y) = \Delta^n$$

is given by the map of partially ordered sets

$$\{ \text{Subsets } S = \{ i = i_0 < i_1 < \cdots < i_k < n + 1 \} \subseteq [n+1] \} \to [n] \quad S \mapsto i_k.$$

Let $\mathcal{C}$ be a simplicial category containing a pair of objects $X$ and $Y$. Then every $n$-simplex $\sigma \in \text{Hom}_{\mathcal{C}}(X, Y)_n$ determines a simplicial functor $F_{\sigma} : \mathcal{E}[\Delta^n] \to \mathcal{C}$, given on objects by $F_{\sigma}(x) = X$ and $F_{\sigma}(y) = Y$. The composition $F_{\sigma} \circ \pi$ is a simplicial functor from $\text{Path}[n+1] \to \mathcal{C}$, which (by Proposition 2.4.4.15) we can view as a map of simplicial sets $f_{\sigma} : \Delta^{n+1} \to \mathcal{N}_{\mathcal{C}}^h$. By construction, $f_{\sigma}$ carries the last vertex of $\Delta^{n+1}$ to $Y$, and the restriction $f_{\sigma}|_{\mathcal{N}_{\mathcal{C}}^h(\{0<1<\cdots<n\})}$ is the constant map taking the value $X$. We can therefore identify $f_{\sigma}$ with an $n$-simplex $\theta(\sigma)$ of the right-pinched morphism space $\text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y)$ introduced in Construction 5.2.5.1 (see Remark 5.2.5.2). The construction $\sigma \mapsto \theta(\sigma)$ depends functorially on the object $[n] \in \Delta$, and therefore determines a map of simplicial sets

$$\theta : \text{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y),$$

which we will refer to as the comparison map.

Our comparison result can now be formulated as follows:

**Theorem 5.2.6.4.** Let $\mathcal{C}$ be a locally Kan simplicial category containing a pair of objects $X, Y \in \mathcal{C}$. Then the comparison map

$$\theta : \text{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y)$$

of Construction 5.2.6.3 is a homotopy equivalence of Kan complexes.

**Remark 5.2.6.5.** Let $\mathcal{C}$ be a locally Kan simplicial category containing a pair of objects $X, Y \in \mathcal{C}$. Combining Theorem 5.2.6.4 with Proposition 5.2.5.9, we obtain a homotopy equivalence of Kan complexes $\text{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y)$, given by composing the comparison map $\theta$ of Construction 5.2.6.3 with the right-pinch inclusion map of Construction 5.2.5.6.

Theorem 5.2.6.4 is an immediate consequence of the following more general result:

**Theorem 5.2.6.6.** Let $\mathcal{C}$ be a simplicial category containing a pair of objects $X$ and $Y$, and suppose that the simplicial set $\text{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is an $\infty$-category. Then the right-pinched morphism space $\text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y)$ is also an $\infty$-category, and the comparison map

$$\theta : \text{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \text{Hom}_{\mathcal{N}_{\mathcal{C}}^h}(X, Y)$$

of Construction 5.2.6.3 is an equivalence of $\infty$-categories.
Remark 5.2.6.7. Let $C$ be a simplicial category containing a pair of objects $X$ and $Y$, and suppose that the simplicial set $\text{Hom}_C(X,Y)_\bullet$. Applying Theorem 5.2.6.6 to the opposite simplicial category $C^{\text{op}}$ (and using Remark 5.2.5.3), we obtain an equivalence of $\infty$-categories

$$\theta': \text{Hom}_C(X,Y)^{\text{op}} \to \text{Hom}_{\text{N}^L_{\bullet}(\mathcal{C})}(X,Y),$$

which can be described explicitly using a variant of Construction 5.2.6.3.

The remainder of this section is devoted to the proof of Theorem 5.2.6.6.

Construction 5.2.6.8. Let $K$ be a simplicial set. We let $\Sigma(K)$ denote the pushout $\{x\} \amalg_K (K \star \{y\})$ (this is a model for the unreduced suspension of $K$). Let $\text{Path}[\Sigma(K)]_\bullet$ denote the simplicial path category of $\Sigma(K)$ (Notation 2.4.4.2). Then $\text{Path}[\Sigma(K)]_\bullet$ has exactly two objects, which we denote by $x$ and $y$. We let $\Phi(K)$ denote the simplicial set $\text{Hom}_{\text{Path}[\Sigma(K)]}(x,y)_\bullet$.

Example 5.2.6.9. If $K = \Delta^0$, then the suspension $\Sigma(K)$ can be identified with $\Delta^1$. In this case, the simplicial path category $\text{Path}[\Sigma(K)]_\bullet$ can be identified with the ordinary category $[1]$, and the simplicial set $\Phi(K)$ is isomorphic to $\Delta^0$.

Remark 5.2.6.10. Let $K$ be a simplicial set, and let $D$ be another simplicial set containing vertices $X$ and $Y$. Unwinding the definitions, we have a canonical bijection

$$\{\text{Morphisms } K \to \text{Hom}_D^R(X,Y)\} \sim \{\text{Morphisms } F : \Sigma(K) \to D \text{ with } F(x) = X \text{ and } F(y) = Y\}.$$

Remark 5.2.6.11. Let $K$ be a simplicial set. Note that, for $n > 0$, every nondegenerate simplex $\sigma : \Delta^n \to \Sigma(K)$ satisfies $\sigma(0) = x$ and $\sigma(n) = y$. Using Theorem 2.4.4.10, we see that for each $m \geq 0$, $\text{Path}[\Sigma(K)]_m$ can be identified with the path category of a directed graph $G_m$ with vertex set $\text{Vert}(G_m) = \{x,y\}$, where each edge of $G_m$ has source $x$ and target $y$. These path categories are easy to describe: they satisfy

$$\text{Hom}_{\text{Path}[G_m]}(x,x) = \{\text{id}_x\} \quad \text{Hom}_{\text{Path}[G_m]}(y,y) = \{\text{id}_y\}$$

$$\text{Hom}_{\text{Path}[G_m]}(x,y) = \text{Edge}(G_m) \quad \text{Hom}_{\text{Path}[G_m]}(y,x) = \emptyset.$$

Allowing $m$ to vary, we conclude that the simplicial category $\text{Path}[\Sigma(K)]_\bullet$ satisfies

$$\text{Hom}_{\text{Path}[\Sigma(K)]}(x,x)_\bullet = \{\text{id}_x\} \quad \text{Hom}_{\text{Path}[\Sigma(K)]}(y,x)_\bullet = \emptyset \quad \text{Hom}_{\text{Path}[\Sigma(K)]}(y,y)_\bullet = \{\text{id}_y\}.$$ 

That is, $\text{Path}[\Sigma(K)]_\bullet$ can be identified with the simplicial category $\mathcal{E}[\Phi(X)]$ of Notation 5.2.6.1. Moreover, we can identify $m$-simplices $\bullet$ of $\Phi(X)$ with elements of the set $E(\Sigma(K),m)$ defined in Notation 2.4.4.9.
**Remark 5.2.6.12.** Let \( u : K \to K' \) be a monomorphism of simplicial sets. Then the induced map \( \Phi(u) : \Phi(K) \to \Phi(K') \) is also a monomorphism (this follows immediately from the description given in Remark 5.2.6.11).

**Lemma 5.2.6.13.** Let \( K \) be a simplicial set and let \( \mathcal{C} \) be a simplicial category containing objects \( X \) and \( Y \). Then the natural map

\[
\{ \text{Functors } F : \text{Path}[\Sigma(K)] \to X \text{ and } F(y) = Y \} \to \{ \text{Morphisms } \Phi(K) \to \text{Hom}_\mathcal{C}(X,Y) \}
\]

is a bijection.

**Proof.** Combine Remarks 5.2.6.2 and 5.2.6.11. □

Combining Remark 5.2.6.10, Lemma 5.2.6.13, and the universal property of simplicial path categories, we obtain the following:

**Corollary 5.2.6.14.** Let \( K \) be a simplicial set and let \( \mathcal{C} \) be a simplicial category containing objects \( X \) and \( Y \). Then we have a canonical bijection

\[
\{ \text{Morphisms } K \to \text{Hom}_{R^\mathbb{N}_e(\mathcal{C})}(X,Y) \} \simeq \{ \text{Morphisms } \Phi(K) \to \text{Hom}_\mathcal{C}(X,Y) \}.
\]

**Corollary 5.2.6.15.** Let \( K \) and \( L \) be simplicial sets, and let \( \mathcal{E}[L] \) be the simplicial category of Notation 5.2.6.1. Then we have a canonical bijection

\[
\{ \text{Morphisms } K \to \text{Hom}_{R^\mathbb{N}_e(\mathcal{E}[L])}(x,y) \} \simeq \{ \text{Morphisms } \Phi(K) \to L \}.
\]

**Proof.** Apply Corollary 5.2.6.14 in the special case \( \mathcal{C} = \mathcal{E}[L] \). □

**Corollary 5.2.6.16.** The functor

\[
\text{Set}_\Delta \to \text{Set}_\Delta, \quad L \mapsto \text{Hom}_{R^\mathbb{N}_e(\mathcal{E}[L])}(x,y)
\]

has a left adjoint, given by the functor \( K \mapsto \Phi(K) \) of Construction 5.2.6.8.

**Remark 5.2.6.17.** The adjunction of Corollary 5.2.6.16 has an interpretation in the framework of Proposition 1.1.8.22. Let \( Q^\bullet \) denote the cosimplicial object of \( \text{Set}_\Delta \) given by the construction \([n] \mapsto \Phi(\Delta^n)\). For every simplicial set \( L \), Corollary 5.2.6.15 supplies a canonical isomorphism of simplicial sets

\[
\text{Hom}_{R^\mathbb{N}_e(\mathcal{E}[L])}(x,y) \simeq \text{Sing}_{Q^\bullet}(L),
\]
where $\text{Sing}^Q(L)$ is the simplicial set defined in Variant 1.1.7.6. It follows that $\Phi$ can be identified with the generalized geometric realization functor $K \mapsto |K|^Q$ of Proposition 1.1.8.22.

**Corollary 5.2.6.18.** The functor $\Phi : \text{Set}_\Delta \to \text{Set}_\Delta$ of Construction 5.2.6.8 preserves colimits.

**Construction 5.2.6.19.** Let $K$ be a simplicial set, let $\mathcal{E}[K]$ be the simplicial category of Notation 5.2.6.1, and let

$$\theta : K = \text{Hom}_{\mathcal{E}[K]}(x, y)_\bullet \to \text{Hom}^R_{\text{N}^\bullet_{\text{hc}}(\mathcal{E}[K])}(x, y)$$

be the comparison map of Construction 5.2.6.3. We let $\rho_K : \Phi(K) \to K$ denote the image of $\theta$ under the bijection of Corollary 5.2.6.15.

We will deduce Theorem 5.2.6.6 from the following result, which we prove at the end of this section:

**Proposition 5.2.6.20.** Let $K$ be a simplicial set. Then the morphism $\rho_K : \Phi(K) \to K$ of Construction 5.2.6.19 is a categorical equivalence of simplicial sets.

**Corollary 5.2.6.21.** Let $u : K \to K'$ be a categorical equivalence of simplicial sets. Then the induced map $\Phi(u) : \Phi(K) \to \Phi(K')$ is also a categorical equivalence of simplicial sets.

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
\Phi(K) & \xrightarrow{\Phi(u)} & \Phi(K') \\
\downarrow{\rho_K} & & \downarrow{\rho_{K'}} \\
K & \xrightarrow{u} & K'
\end{array}
$$

where $u$ is a categorical equivalence by hypothesis and the vertical maps are categorical equivalences by Proposition 5.2.6.20. Using Remark 4.5.2.6, we conclude that $\Phi(u)$ is a categorical equivalence as well. □

**Corollary 5.2.6.22.** Let $\mathcal{C}$ be a simplicial categories containing a pair of objects $X$ and $Y$, and assume that the simplicial set $\text{Hom}_\mathcal{C}(X,Y)_\bullet$ is an $\infty$-category. Then the simplicial set $\text{Hom}^R_{\text{N}^\bullet_{\text{hc}}(\mathcal{C})}(X,Y)$ is also an $\infty$-category.
Proof. Let \( i : A \to B \) be an inner anodyne morphism of simplicial sets. We wish to show that every lifting problem

\[
\begin{array}{ccc}
A & \to & \text{Hom}_{\mathcal{N}(C)}(X, Y) \\
\downarrow i & \downarrow & \downarrow \\
B & \to & \Delta^0
\end{array}
\]

admits a solution. By virtue of Corollary 5.2.6.14, we can rephrase this as a lifting problem

\[
\begin{array}{ccc}
\Phi(A) & \to & \text{Hom}_C(X, Y)_\bullet \\
\downarrow \Phi(i) & & \downarrow \\
\Phi(B) & \to & \Delta^0.
\end{array}
\]

Note that \( \Phi(i) \) is a monomorphism (Remark 5.2.6.12) and a categorical equivalence (Corollary 5.2.6.21), so the desired result follows from Lemma 4.5.6.2.

**Corollary 5.2.6.23.** Let \( C \) be a simplicial categories containing a pair of objects \( X \) and \( Y \), and assume that the simplicial set \( \text{Hom}_C(X, Y)_\bullet \) is an \( \infty \)-category. Let \( K \) be another simplicial set, and suppose we are given a pair of morphisms \( f_0, f_1 : K \to \text{Hom}_{\mathcal{N}(C)}(X, Y) \), which correspond (under the bijection of Corollary 5.2.6.14) to diagrams \( f'_0, f'_1 : \Phi(K) \to \text{Hom}_C(X, Y)_\bullet \). The following conditions are equivalent:

1. The diagrams \( f_0 \) and \( f_1 \) are isomorphic when regarded as objects of the \( \infty \)-category \( \text{Fun}(K, \text{Hom}_{\mathcal{N}(C)}(X, Y)) \).
2. The diagrams \( f'_0 \) and \( f'_1 \) are isomorphic when regarded as objects of the \( \infty \)-category \( \text{Fun}(\Phi(K), \text{Hom}_C(X, Y)_\bullet) \).

Proof. Choose a categorical mapping cylinder

\[
K \coprod_{\Phi(K)} \Delta^0 \xrightarrow{\pi} K
\]

for the simplicial set \( K \) (Definition 5.2.3.3). Using Remark 5.2.6.12 Corollary 5.2.6.18 and Corollary 5.2.6.21 we conclude that the induced diagram

\[
\Phi(K) \coprod_{\Phi(K)} \Delta^0 \xrightarrow{\Phi(\pi)} \Phi(K)
\]

exhibits \( \Phi(K) \) as a categorical mapping cylinder of \( K \). Using Corollary 5.2.3.7 we see that (1) and (2) are equivalent to the following:
There exists a diagram \( \mathcal{F} : K \to \text{Hom}_{\mathcal{C}}^{R}(X, Y) \) satisfying \( f_0 = \mathcal{F} \circ s_0 \) and \( f_1 = \mathcal{F} \circ s_1 \).

There exists a diagram \( \mathcal{F}' : \Phi(K) \to \text{Hom}_{\mathcal{C}}(X, Y) \) satisfying \( f'_0 = \mathcal{F}' \circ \Phi(s_0) \) and \( f'_1 = \mathcal{F}' \circ \Phi(s_1) \).

The equivalence of (1') and (2') follows from Corollary 5.2.6.14.

**Proof of Theorem 5.2.6.6.** Let \( C \) be a simplicial category containing a pair of objects \( X, Y \in \mathcal{C} \) for which the simplicial set \( \text{Hom}_{\mathcal{C}}(X, Y) \) is an \( \infty \)-category. Applying Corollary 5.2.6.22, we deduce that the right-pinched morphism space \( \text{Hom}_{\mathcal{C}}^{R}(X, Y) \) is also an \( \infty \)-category.

We wish to show that the comparison map \( \theta : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}^{R}(X, Y) \) of Construction 5.2.6.3 is an equivalence of \( \infty \)-categories. To prove this, it will suffice to show that for every simplicial set \( K \), postcomposition with \( \theta \) induces a bijection \( \theta_K : \pi_0(\text{Fun}(K, \text{Hom}_{\mathcal{C}}(X, Y))) \to \pi_0(\text{Fun}(K, \text{Hom}_{\mathcal{C}}^{R}(X, Y))) \).

By virtue of Corollary 5.2.6.23, we can identify \( \pi_0(\text{Fun}(K, \text{Hom}_{\mathcal{C}}^{R}(X, Y))) \) with the set \( \pi_0(\text{Fun}(\Phi(K), \text{Hom}_{\mathcal{C}}(X, Y))) \). Under this identification, \( \theta_K \) corresponds to the map

\[ \pi_0(\text{Fun}(K, \text{Hom}_{\mathcal{C}}(X, Y))) \to \pi_0(\text{Fun}(\Phi(K), \text{Hom}_{\mathcal{C}}(X, Y))) \]

given by precomposition with the map \( \rho_K : \Phi(K) \to K \) of Construction 5.2.6.19, which is bijective by virtue of the fact that \( \rho_K \) is a categorical equivalence (Proposition 5.2.6.20).

We now turn to the proof of Proposition 5.2.6.20. Our strategy is to use formal arguments to reduce to the case where the simplicial set \( K \) is a standard simplex, which can be analyzed explicitly.

**Example 5.2.6.24.** Let \( m \) and \( n \) be nonnegative integers. By virtue of Remark 5.2.6.11, we can identify \( m \)-simplices of the simplicial set \( \Phi(\Delta^n) \) with the set \( E(\Sigma(\Delta^n), m) \) defined in Notation 2.4.4.9. By definition, the elements of \( E(\Sigma(\Delta^n), m) \) are given by pairs \((\sigma, \overrightarrow{t})\), where \( \sigma : \Delta^k \to \Sigma(\Delta^n) \) is a nondegenerate simplex of dimension \( k > 0 \) and \( \overrightarrow{t} = (I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m) \) is a chain of subsets of \([k]\) satisfying \( I_0 = \{0, k\} \) and \( I_m = [k] \).

For each \( k > 0 \), there is a canonical bijection

\[ \{\text{Subsets } S \subseteq [n] \text{ of cardinality } k\} \to \{\text{Nondegenerate }k\text{-Simplices of }\Sigma(\Delta^n)\}, \]

which carries a subset \( S \) to the \( k \)-simplex \( \sigma_S \) given by the composite map

\[ \Delta^k \simeq \mathcal{N}_\bullet(S) \star \{y\} \xrightarrow{i_S \star \text{id}} \Delta^n \star \{y\} \to \Sigma(\Delta^n). \]
For every such subset $S$, let $\iota_S : [k - 1] \to S$ be the unique order-preserving bijection. Then construction

$$(\sigma_S, \overrightarrow{T}) \mapsto (\iota_S(I_0 \setminus \{k\}) \subseteq \iota_S(I_1 \setminus \{k\}) \subseteq \cdots \subseteq \iota_S(I_m \setminus \{k\}))$$

induces a bijection from $E(\Sigma(\Delta^n), m)$ to the collection of chains $\overrightarrow{S} = (S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m)$ of subsets of $[n]$ which satisfy the following pair of conditions:

(a) The set $S_0$ contains exactly one element.

(b) For $0 \leq i \leq m$, the unique element of $S_0$ is a least element of $S_i$.

Let us henceforth use this bijection to identify $m$-simplices of $\Phi(\Delta^n)$ with chains $\overrightarrow{S}$ satisfying (a) and (b). In these terms, the face and degeneracy operators for the simplicial set $\Phi(\Delta^n) = \Phi(\Delta^n)_\bullet$ can be described explicitly as follows:

- For $0 \leq i \leq m$, the degeneracy operator $s_i : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m+1}$ is given by
  $$s_i(S_0 \subseteq \cdots \subseteq S_m) = (S_0 \subseteq \cdots \subseteq S_{i-1} \subseteq S_i \subseteq S_{i+1} \subseteq \cdots \subseteq S_m).$$

- For $0 < i \leq m$, the face operator $d_i : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m-1}$ is given by the construction
  $$d_i(S_0 \subseteq \cdots \subseteq S_m) = (S_0 \subseteq \cdots \subseteq S_{i-1} \subseteq S_i \subseteq S_{i+1} \subseteq \cdots \subseteq S_m).$$

- For $m > 0$, the face operator $d_0 : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m-1}$ is given by
  $$d_0(S_0 \subseteq \cdots \subseteq S_m) = (S_1' \subseteq S_2' \subseteq \cdots \subseteq S_m'),$$

  where $S_i' = \{j \in S_i : j \geq \text{max}(S_1)\}$.

See Remark 2.4.4.17.

**Construction 5.2.6.25.** Let $m$ and $n$ be nonnegative integers. Suppose we are given an $m$-simplex of $\Phi(\Delta^n)$, which we identify with a chain of subsets $\overrightarrow{S} = (S_0 \subseteq \cdots \subseteq S_m)$ satisfying conditions (a) and (b) of Example 5.2.6.24. Let $\tau : [m] \to [1]$ be a nondecreasing function. Let $\overrightarrow{S'} = (S_0' \subseteq \cdots \subseteq S'_m)$ be the chain of subsets of $[n + 1]$ given by the formula

$$S_i' = \begin{cases} S_i & \text{if } \tau(i) = 0 \\ \{n + 1\} & \text{if } \tau(0) = 1 \\ S_i \cup \{n + 1\} & \text{otherwise.} \end{cases}$$

The construction $(\overrightarrow{S}, \tau) \mapsto \overrightarrow{S'}$ is compatible with the formation of face and degeneracy operators, and therefore determines a morphism of simplicial sets $\pi : \Phi(\Delta^n) \times \Delta^1 \to \Phi(\Delta^{n+1})$. 
Lemma 5.2.6.26. Let \( n \geq 0 \) be an integer, and let \( \pi : \Phi(\Delta^n) \times \Delta^1 \to \Phi(\Delta^{n+1}) \) be the morphism of simplicial sets defined in Construction 5.2.6.25. Then \( \pi \) fits into a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\Phi(\Delta^n) \times \{1\} & \longrightarrow & \Phi(\Delta^n) \times \Delta^1 \\
\downarrow & & \downarrow \pi \\
\Delta^0 & \longrightarrow & \Phi(\Delta^{n+1}). \\
\end{array}
\]

Remark 5.2.6.27. It follows from Lemma 5.2.6.26 that the morphism \( \pi \) of Construction 5.2.6.25 induces an isomorphism of simplicial sets \( \Phi(\Delta^n) \circ \Delta^0 \to \Phi(\Delta^{n+1}) \), where the operation \( \circ \) is defined in Notation 4.5.5.3.

Proof of Lemma 5.2.6.26. Fix an integer \( m \geq 0 \). By construction, the restriction \( \pi|_{\Phi(\Delta^n) \times \{1\}} \) is the constant map which carries each \( m \)-simplex of \( \Phi(\Delta^n) \) to the element of \( \Phi(\Delta^{n+1}) \) given by the constant chain \( \overrightarrow{S_0} = \{n+1\} \subseteq \{n+1\} \subseteq \cdots \subseteq \{n+1\} \). To complete the proof, we must show that \( \pi \) induces a bijection. To complete the proof, we must show that for each \( m \geq 0 \), the map \( \pi \) induces a bijection

\[
\Phi(\Delta^n)_m \times \{\text{Functions } \tau : [n] \to [1] \text{ with } \tau(0) = 0\}.
\]

The inverse bijection can be described explicitly as follows: it carries an \( n \)-simplex \( (S_0 \subseteq \cdots \subseteq S_m) \neq \overrightarrow{S_0} \) of \( \Phi(\Delta^{n+1}) \) to the pair \( (\overrightarrow{S'}, \tau) \), where \( \overrightarrow{S'} \) is the \( m \)-simplex of \( \Phi(\Delta^n) \) given by the sequence \( (S_0 \cap [n] \subseteq S_1 \cap [n] \subseteq \cdots \subseteq S_m \cap [n]) \) and \( \tau : [n] \to [1] \) is given by the formula

\[
\tau(i) = \begin{cases} 
0 & \text{if } n+1 \notin S_i \\
1 & \text{if } n+1 \in S_i.
\end{cases}
\]

Proof of Proposition 5.2.6.20. Let \( K \) be a simplicial set. We wish to show that the map \( \rho_K : \Phi(K) \to K \) of Construction 5.2.6.19 is a categorical equivalence of simplicial sets. Using Corollary 5.2.6.18 we can write \( \rho_K \) as a filtered colimit of morphisms \( \rho_{K_\alpha} : \Phi(K_\alpha) \to K_\alpha \), where \( K_\alpha \) ranges over the collection of all finite simplicial subsets of \( K \) (Remark 3.5.1.8). Since the collection of categorical equivalences is closed under the formation of filtered colimits (Corollary 4.5.4.2), it will suffice to show that each \( \rho_{K_\alpha} \) is a categorical equivalence. We may therefore replace \( K \) by \( K_\alpha \) and thereby reduce to the case where the simplicial set \( K \) is finite.
Since $K$ is a finite simplicial set, it has dimension $\leq n$ for some integer $n \geq -1$. We proceed by induction on $n$. If $n = -1$, then both $K$ and $\Phi(K)$ are empty, and there is nothing to prove. We may therefore assume that $n \geq 0$ and that $\rho_{K'}$ is a categorical equivalence for every simplicial set $K'$ of dimension $< n$. We now proceed by induction on the number $m$ of nondegenerate $n$-simplices of $K$. If $m = 0$, then $K$ has dimension $\leq n - 1$ and the desired result holds by virtue of our inductive hypothesis. We may therefore assume that $K$ has at least one nondegenerate $n$-simplex $\sigma : \Delta^n \to K$. Using Proposition 1.1.3.13 we see that there is a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \Delta^n \\
\downarrow & & \downarrow \sigma \\
K' & \rightarrow & K,
\end{array}
$$

where $S'$ is a simplicial set of dimension $\leq n$ with exactly $(m - 1)$-nondegenerate $m$-simplices. We then have a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\Phi(\partial \Delta^n) & \rightarrow & \Phi(\Delta^n) \\
\downarrow \rho_{\partial \Delta^n} & & \downarrow \rho_{\Delta^n} \\
\partial \Delta^n & \rightarrow & \Delta^n \\
\downarrow \Phi(K') & & \downarrow \Phi(K) \\
\downarrow \rho_{K'} & & \downarrow \rho_K \\
K' & \rightarrow & K.
\end{array}
$$

where the front and back faces are pushout squares (Corollary 5.2.6.18) in which the horizontal maps are monomorphisms (Remark 5.2.6.12), and are therefore categorical pushout squares (Example 4.5.3.8). Our inductive hypotheses guarantees that the maps $\rho_{K'}$ and $\rho_{\partial \Delta^n}$ are categorical equivalences. Consequently, to show that $\rho_K$ is a categorical equivalence, it will suffice to show that $\rho_{\Delta^n}$ is a categorical equivalence (Proposition 4.5.3.5). We may therefore replace $K$ by $\Delta^n$ and thereby reduce to the case where $K$ is a standard simplex.
If \( n = 0 \), then the map \( \rho_{\Delta^n} : \Phi(\Delta^n) \to \Delta^n \) is an isomorphism (Example 5.2.6.9). We may therefore assume without loss of generality that \( n > 0 \), so that Lemma 5.2.6.26 supplies an isomorphism of simplicial sets \( \Phi(\Delta^n) \simeq \Phi(\Delta^{n-1}) \diamond \Delta^0 \). Using this isomorphism, we can identify \( \rho_{\Delta^n} \) with the composite map

\[
\Phi(\Delta^{n-1}) \diamond \Delta^0 \xrightarrow{\rho_{\Delta^{n-1}} \diamond \text{id}} \Delta^{n-1} \diamond \Delta^0 \xrightarrow{c} \Delta^{n-1} \star \Delta^0 \simeq \Delta^n,
\]

where \( c \) is the comparison map of Notation 4.5.5.3 (to check this, it suffices to observe that they agree on vertices). Our inductive hypothesis guarantees that \( \rho_{\Delta^{n-1}} \) is a categorical equivalence of simplicial sets, so that the induced map \( \Phi(\Delta^{n-1}) \diamond \Delta^0 \xrightarrow{\rho_{\Delta^{n-1}} \diamond \text{id}} \Delta^{n-1} \diamond \Delta^0 \) is also a categorical equivalence by virtue of Remark 4.5.5.6. We are therefore reduced to showing that \( c \) is a categorical equivalence, which is a special case of Proposition 4.5.5.11. 

\section{Localization}

Let \( C \) be a category and let \( W \) be a collection of morphisms in \( C \). One can then construct a new category by formally adjoining an inverse to each morphism of \( W \).

\begin{definition}
Let \( F : C \to D \) be a functor between categories and let \( W \) be a collection of morphisms of \( C \). We say that \( F \) exhibits \( D \) as a strict localization of \( C \) with respect to \( W \) if, for every category \( E \), precomposition with \( F \) induces a bijection

\[
\{ \text{Functors } D \to E \} \xrightarrow{\text{Precomposition with } F} \{ \text{Functors } C \to E \text{ carrying each } w \in W \text{ to an isomorphism in } E \}.
\]

\end{definition}

\begin{remark}[Existence and Uniqueness]
Let \( C \) be a category and let \( W \) be a collection of morphisms in \( C \). Then there exists a category \( C[W^{-1}] \) and a functor \( F : C \to C[W^{-1}] \) which exhibits \( C[W^{-1}] \) as a strict localization of \( C \) with respect to \( W \). Moreover, the category \( C[W^{-1}] \) (and the functor \( F \)) are determined uniquely up to isomorphism. In what follows, we will sometimes abuse terminology by referring to \( C[W^{-1}] \) as the strict localization of \( C \) with respect to \( W \). Explicitly, the category \( C[W^{-1}] \) can be constructed from \( C \) by adjoining a new morphism \( w^{-1} : Y \to X \) for each morphism \( w : X \to Y \) of \( W \), and imposing the relations \( w^{-1} \circ w = \text{id}_X \) and \( w \circ w^{-1} = \text{id}_Y \). From this description, we see that the functor \( F \) induces a bijection \( \text{Ob}(C) \simeq \text{Ob}(C[W^{-1}]) \).

\end{remark}

\begin{warning}
Let \( C \) be a category and let \( W \) be a collection of morphisms of \( C \). If \( C \) is small (Definition [?]), then the strict localization \( C[W^{-1}] \) is also small. Beware that if \( C \)
is only assumed to be locally small (Definition [?]), then $\mathcal{C}[W^{-1}]$ need not be locally small. However, one can often ensure that $\mathcal{C}[W^{-1}]$ is small by imposing additional assumptions on the collection of morphisms $W$: we will return to this point (in the setting of $\infty$-categories) in Chapter [?].

**Remark 5.3.0.4.** Let $\mathcal{C}$ be a category, let $W$ be a collection of morphisms of $\mathcal{C}$, and let $F: \mathcal{C} \to \mathcal{C}[W^{-1}]$ be a functor which exhibits $\mathcal{C}[W^{-1}]$ as a strict localization of $\mathcal{C}$ with respect to $W$. Then, for every category $\mathcal{E}$, the precomposition functor $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \xrightarrow{\circ F} \text{Fun}(\mathcal{C}, \mathcal{E})$ induces an isomorphism from $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ to the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those functors $\mathcal{C} \to \mathcal{E}$ which carry each element $w \in W$ to an isomorphism in $\mathcal{E}$. Bijectivity at the level of objects follows immediately from the definition. At the level of morphisms, it follows from the bijectivity of the map

$$\{\text{Functors } \mathcal{C}[W^{-1}] \to \text{Fun}([1], \mathcal{E})\} \xrightarrow{\circ F} \{\text{Functors } \mathcal{C} \to \text{Fun}([1], \mathcal{E}) \text{ carrying } W \text{ to isomorphisms}\}.$$

Beware that Definition [5.3.0.1] is not invariant under equivalence. If $\mathcal{C}$ is a category, $W$ is a collection of morphisms in $\mathcal{C}$, and $\mathcal{D}$ is a category which is equivalent but not isomorphic to the strict localization $\mathcal{C}[W^{-1}]$, then $\mathcal{D}$ is not a strict localization of $\mathcal{C}$ with respect to $W$. We can remedy the situation by introducing a more liberal notion of localization.

**Definition 5.3.0.5.** Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $W$ be a collection of morphisms of $\mathcal{C}$. We will say that $F$ exhibits $\mathcal{D}$ as a classical localization of $\mathcal{C}$ with respect to $W$ if, for every category $\mathcal{E}$, precomposition with $F$ induces a fully faithful functor $\text{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \text{Fun}(\mathcal{C}, \mathcal{E})$, whose essential image consists of those functors $\mathcal{C} \to \mathcal{E}$ which carry each $w \in W$ to an isomorphism in $\mathcal{E}$.

**Example 5.3.0.6.** Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between categories. If $F$ exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to $W$, then $F$ exhibits $\mathcal{D}$ as a classical localization of $\mathcal{C}$ with respect to $W$ (see Remark 5.3.0.4). The converse is false (except in the trivial case where $\mathcal{C}$ is empty).

**Remark 5.3.0.7.** Let $\mathcal{C}$ be a category, let $W$ be a collection of morphisms in $\mathcal{C}$, and let $F: \mathcal{C} \to \mathcal{C}[W^{-1}]$ be a functor which exhibits $\mathcal{C}[W^{-1}]$ as a strict localization of $\mathcal{C}$ with respect to $W$. Let $G: \mathcal{C} \to \mathcal{D}$ be another functor. Then $G$ exhibits $\mathcal{D}$ as a classical localization of $\mathcal{C}$ with respect to $W$ if and only if the following conditions are satisfied:

- The functor $G$ carries each $w \in W$ to an isomorphism in $\mathcal{C}$, and therefore factors uniquely as a composition $\mathcal{C} \xrightarrow{F} \mathcal{C}[W^{-1}] \xrightarrow{G'} \mathcal{D}$. 


5.3. LOCALIZATION

- The functor $G': \mathcal{C}[W^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Our goal in this section is to adapt the notion of localization to the setting of ∞-categories. We begin in §5.3.1 by introducing an ∞-categorical counterpart of Definition 5.3.0.5. Given an ∞-category $\mathcal{C}$ and a collection $W$ of morphisms of $\mathcal{C}$, we say that a functor of ∞-categories $F: \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$ if, for every ∞-category $\mathcal{E}$, precomposition with $F$ induces a fully faithful functor of ∞-categories $\text{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \text{Fun}(\mathcal{C}, \mathcal{E})$, whose essential image consists of those functors which carry each element of $W$ to an isomorphism in $\mathcal{E}$ (Definition 5.3.1.9). In §5.3.2, we show that such a localization always exists (Proposition 5.3.2.1) and is uniquely determined up to equivalence (Remark 5.3.2.2).

**Warning 5.3.0.8.** One could also consider a version of Definition 5.3.0.1 in the ∞-categorical setting. Let $\mathcal{C}$ be an ∞-category and let $W$ be a collection of morphisms of $\mathcal{C}$. Let us say that a functor of ∞-categories $F: \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to $W$ if, for every ∞-category $\mathcal{E}$, precomposition with $F$ induces a bijection

$$
\{\text{Functors } \mathcal{D} \to \mathcal{E}\} \xrightarrow{\circ F} \{\text{Functors } \mathcal{C} \to \mathcal{E} \text{ carrying each } w \in W \text{ to an isomorphism in } \mathcal{E}\}.
$$

Beware that this definition has very limited utility: one can show that $\mathcal{C}$ admits a strict localization with respect to $W$ only in the trivial case where every element of $W$ is already an isomorphism in $\mathcal{C}$ (in which case we can take $F$ to be the identity functor $\text{id}_\mathcal{C}: \mathcal{C} \to \mathcal{C}$). Roughly speaking, the issue is that if $u: X \to Y$ is an isomorphism in an ∞-category, then the inverse isomorphism $u^{-1}: Y \to X$ is only well-defined up to homotopy (or at best up to a contractible space of choices), in contrast with classical category theory where the inverse isomorphism $u^{-1}$ is uniquely determined.

Let $\mathcal{C}$ be an ordinary category, and let $W$ be a collection of morphisms of $\mathcal{C}$. Then $W$ can also be regarded as a collection of morphisms of the ∞-category $N_\bullet(\mathcal{C})$. By virtue of Proposition 5.3.2.1, there exists a functor of ∞-categories $F: N_\bullet(\mathcal{C}) \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $N_\bullet(\mathcal{C})$ with respect to $W$. In this case, it is not hard to see that the induced map $\mathcal{C} \simeq hN_\bullet(\mathcal{C}) \xrightarrow{hF} h\mathcal{D}$ exhibits the homotopy category $h\mathcal{D}$ as a classical localization of $\mathcal{C}$ with respect to $W$, in the sense of Definition 5.3.0.5 (Example 5.3.1.16). Beware that, in this situation, the unit map $\mathcal{D} \to N_\bullet(h\mathcal{D})$ is generally not an equivalence. In other words, the formation of localizations (in the ∞-categorical setting) generally does not carry ordinary categories to ordinary categories, even up to equivalence. In fact, we show in
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§5.3.4 that every ∞-category $\mathcal{D}$ can be obtained by localizing (the nerve of) an ordinary category with respect to a suitably chosen collection of morphisms (Proposition 5.3.4.1). The proof will make use of some basic stability properties for the class of localizations, which we establish in §5.3.3.

Warning 5.3.0.9. Throughout most of this book, we will often employ the following conventions:

- If $\mathcal{C}$ is an ∞-category and $W$ is a collection of morphisms of $\mathcal{C}$, then we let $\mathcal{C}[W^{-1}]$ denote a localization of $\mathcal{C}$ with respect to $W$ (Remark 5.3.2.2).
- If $\mathcal{C}$ is an ordinary category, we abuse terminology by identifying $\mathcal{C}$ with the associated ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$.

Beware that these conventions are in conflict with one another. If $W$ is a collection of morphisms in an ordinary category $\mathcal{C}$ and $\mathcal{C}[W^{-1}]$ denotes the strict localization of Remark 5.3.0.2, then the ∞-categorical localization $\mathrm{N}_{\bullet}(\mathcal{C})[W^{-1}]$ is generally not equivalent to the nerve $\mathrm{N}_{\bullet}(\mathcal{C}[W^{-1}])$. To avoid confusion, we will henceforth use the notation $\mathcal{C}[W^{-1}]$ only for the ∞-categorical notion of localization, unless otherwise specified.

5.3.1 Localizations of ∞-Categories

We begin by introducing some terminology.

Notation 5.3.1.1. Let $\mathcal{C}$ be a simplicial set, let $W$ be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an ∞-category. We let $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those morphisms $F : \mathcal{C} \to \mathcal{E}$ that carry each edge of $W$ to an isomorphism in $\mathcal{E}$.

Remark 5.3.1.2. In the context of Notation 5.3.1.1, we will usually be interested in the situation where the simplicial set $\mathcal{C}$ is an ∞-category (as suggested by the notation). However, it will be technically convenient to allow more general simplicial sets as well.

Example 5.3.1.3. Let $\mathcal{C}$ be a simplicial set and let $W$ be a collection of degenerate edges of $\mathcal{C}$. Then, for every ∞-category $\mathcal{E}$, we have $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

Example 5.3.1.4. Let $\mathcal{C}$ be a simplicial set and let $W$ be a collection of edges of $\mathcal{C}$. If $\mathcal{E}$ is a Kan complex, then $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ (see Proposition 1.3.6.11).

Example 5.3.1.5. Let $W = \{\mathrm{id}_{\Delta^1}\}$ consist of the single nondegenerate edge of the standard 1-simplex $\Delta^1$. For every ∞-category $\mathcal{E}$, $\mathrm{Fun}(\Delta^1[W^{-1}], \mathcal{E})$ is the full subcategory $\mathrm{Isom}(\mathcal{E}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{E})$ spanned by the isomorphisms in $\mathcal{E}$ (Example 4.4.1.13).
Example 5.3.1.6. Let $\mathcal{C}$ be a simplicial set and let $\text{hC}$ denote its homotopy category (Definition 1.2.5.1). Let $W$ be a collection of edges of $\mathcal{C}$, let $[W]$ denote the collection of morphisms in $\text{hC}$ which belong to the image of $W$, and let $F : \text{hC} \to \mathcal{D}$ be a functor of ordinary categories which exhibits $\mathcal{D}$ as a strict localization of $\text{hC}$ with respect to $[W]$ (Definition 5.3.0.1). If $\mathcal{E}$ is an ordinary category, then we have a canonical isomorphism of simplicial sets

$$\text{Fun}(\mathcal{C}[W^{-1}], N_{\bullet}(\mathcal{E})) \simeq N_{\bullet}(\text{Fun}(\mathcal{D}, \mathcal{E})).$$

Remark 5.3.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets and let $W$ be a collection of edges of $\mathcal{C}$. For every $\infty$-category $\mathcal{E}$, the canonical isomorphism $\text{Fun}(\mathcal{C}, \text{Fun}(\mathcal{D}, \mathcal{E})) \simeq \text{Fun}(\mathcal{D}, \text{Fun}(\mathcal{C}, \mathcal{E}))$ restricts to an isomorphism of full subcategories

$$\text{Fun}(\mathcal{C}[W^{-1}], \text{Fun}(\mathcal{D}, \mathcal{E})) \simeq \text{Fun}(\mathcal{D}, \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})).$$

This follows immediately from the criterion of Theorem 4.4.4.4.

Remark 5.3.1.8. Let $\mathcal{C}$ be a simplicial set, let $W$ be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an $\infty$-category. Then the full subcategory $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{E})$ is replete. That is, if $F, F' : \mathcal{C} \to \mathcal{E}$ are isomorphic objects of $\text{Fun}(\mathcal{C}, \mathcal{E})$, then $F$ carries edges of $W$ to isomorphisms in $\mathcal{E}$ if and only if $F'$ carries edges of $W$ to isomorphisms in $\mathcal{E}$ (see Example 4.4.1.13).

Definition 5.3.1.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let $W$ be a collection of edges of $\mathcal{C}$. We say that $F$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$ if, for every $\infty$-category $\mathcal{E}$, the precomposition map $\text{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \text{Fun}(\mathcal{C}, \mathcal{E})$ is fully faithful, and its essential image is the full subcategory $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{E})$.

Remark 5.3.1.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. If $F$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$, then, for every $\infty$-category $\mathcal{E}$ and every morphism $G : \mathcal{D} \to \mathcal{E}$, the composite map $(G \circ F) : \mathcal{C} \to \mathcal{E}$ carries each element of $W$ to an isomorphism in $\mathcal{E}$. In particular, if $\mathcal{D}$ itself is an $\infty$-category, then $F$ carries each element of $W$ to an isomorphism in $\mathcal{D}$.

Example 5.3.1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let $W$ be a collection of degenerate edges of $\mathcal{C}$. Then $F$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$ if and only if it is a categorical equivalence of simplicial sets (see Proposition 4.5.2.8).

Proposition 5.3.1.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let $W$ be a collection of edges of $\mathcal{C}$. The following conditions are equivalent:

1. The morphism $F$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$ (Definition 5.3.1.9).
(2) For every ∞-category \( \mathcal{E} \), the functor \( \text{Fun}(\mathcal{D}, \mathcal{E}) \overset{\circ F}{\to} \text{Fun}(\mathcal{C}, \mathcal{E}) \) factors through the full subcategory \( \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \) and induces an equivalence of ∞-categories \( \text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \).

(3) For every ∞-category \( \mathcal{E} \), the functor \( \text{Fun}(\mathcal{D}, \mathcal{E}) \overset{\circ F}{\to} \text{Fun}(\mathcal{C}, \mathcal{E}) \) factors through the full subcategory \( \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \) and induces a homotopy equivalence of Kan complexes \( \text{Fun}(\mathcal{D}, \mathcal{E}) \simeq \to \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \simeq \).

(4) For every ∞-category \( \mathcal{E} \), the functor \( \text{Fun}(\mathcal{D}, \mathcal{E}) \overset{\circ F}{\to} \text{Fun}(\mathcal{C}, \mathcal{E}) \) factors through the full subcategory \( \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \) and induces a bijection of sets \( \pi_0(\text{Fun}(\mathcal{D}, \mathcal{E}) \simeq) \to \pi_0(\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \simeq) \).

Proof. The equivalence (1) ⇔ (2) follows from Corollary 5.2.2.17 (and the repleteness of the full subcategory \( \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{E}) \)). The implication (2) ⇒ (3) follows from Remark 4.5.1.18 and the implication (3) ⇒ (4) from Remark 3.1.5.4. We will complete the proof by showing that (4) ⇒ (2). Assume that \( F : \mathcal{C} \to \mathcal{D} \) satisfies condition (4), and let \( \mathcal{E} \) be an ∞-category; we wish to show that the precomposition functor \( \text{Fun}(\mathcal{D}, \mathcal{E}) \overset{\circ F}{\to} \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \) is an equivalence of ∞-categories. For this, it will suffice to show that for every ∞-category simplicial set \( \mathcal{B} \), the induced map

\[
\theta : \pi_0(\text{Fun}(\mathcal{B}, \text{Fun}(\mathcal{D}, \mathcal{E}) \simeq)) \to \pi_0(\text{Fun}(\mathcal{B}, \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \simeq))
\]

is a bijection. Using Remark 5.3.1.7 we can identify \( \theta \) with the map

\[
\pi_0(\text{Fun}(\mathcal{D}, \text{Fun}(\mathcal{B}, \mathcal{E}) \simeq)) \to \pi_0(\text{Fun}(\mathcal{C}[W^{-1}], \text{Fun}(\mathcal{B}, \mathcal{E}) \simeq))
\]

which is bijective by virtue of assumption (4). \( \square \)

Example 5.3.1.13. Let \( W = \{\text{id}_{\Delta^1}\} \) consist of the single nondegenerate edge of the standard 1-simplex \( \Delta^1 \). Then the projection map \( \Delta^1 \to \Delta^0 \) exhibits \( \Delta^0 \) as a localization of \( \Delta^1 \) with respect to \( W \). To prove this, it will suffice to show that for every ∞-category \( \mathcal{E} \), the construction \( X \mapsto \text{id}_X \) induces a bijection of sets \( \pi_0(\mathcal{E} \simeq) \to \pi_0(\text{Isom}(\mathcal{E}) \simeq) \). The injectivity of this map is clear (the diagonal embedding \( \mathcal{E} \to \text{Isom}(\mathcal{E}) \subseteq \text{Fun}(\Delta^1, \mathcal{E}) \) has a left inverse, given by evaluation at either vertex of \( \Delta^1 \)). To establish surjectivity, it will suffice to show that every isomorphism \( f : X \to Y \) in the ∞-category \( \mathcal{E} \) is isomorphic, when viewed as an object of \( \text{Fun}(\Delta^1, \mathcal{E}) \), to the identity morphism \( \text{id}_X \). This follows from the observation that there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & X \\
\downarrow{\text{id}_X} & & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}
\]
in the ∞-category E, which we can view as an isomorphism from \( \text{id}_X \) to \( f \) in the ∞-category \( \text{Fun}(\Delta^1, E) \) (see Theorem 4.4.4.4).

**Remark 5.3.1.14.** Let \( F : C \to D \) be a morphism of simplicial sets which exhibits \( D \) as a localization of \( C \) with respect to a collection of edges \( W \). Then, for every Kan complex \( E \), precomposition with \( F \) induces a homotopy equivalence of Kan complexes

\[
\text{Fun}(D, E) \xrightarrow{\circ F} \text{Fun}(C[W^{-1}], E) = \text{Fun}(C, E)
\]

(see Example 5.3.1.4). It follows that \( F \) is a weak homotopy equivalence of simplicial sets.

**Remark 5.3.1.15.** Let \( F : C \to D \) be a morphism of simplicial sets which exhibits \( D \) as a localization of \( C \) with respect to a collection of edges \( W \). Let \([W]\) denote the collection of morphisms in the homotopy category \( \text{h}C \) which belong to the image of \( W \). Then the induced functor \( \text{h}F : \text{h}C \to \text{h}D \) exhibits the homotopy category \( \text{h}D \) as a classical localization of \( \text{h}C \) with respect to \([W]\), in the sense of Definition 5.3.0.5. This follows immediately from Example 5.3.1.6.

**Example 5.3.1.16.** Let \( C \) be an ordinary category and let \( W \) be a collection of morphisms of \( C \), which we identify with edges of the simplicial set \( N_\bullet(C) \). Let \( F : N_\bullet(C) \to D \) be a morphism of simplicial sets which exhibits \( D \) as a localization of \( N_\bullet(C) \) with respect to \( W \). Then the induced functor \( \text{h}C \simeq \text{h}N_\bullet(C) \xrightarrow{\text{h}F} \text{h}D \) exhibits the homotopy category \( \text{h}D \) as a classical localization of \( \text{h}C \) with respect to \([W]\), in the sense of in the sense of Definition 5.3.0.5.

**Remark 5.3.1.17.** Let \( F : C \to D \) and \( G : D \to E \) be morphisms of simplicial sets, and let \( W \) be a collection of edges of \( C \). If any two of the following three conditions is satisfied, then so is the third:

- The morphism \( F \) exhibits \( D \) as a localization of \( C \) with respect to \( W \).
- The morphism \( G \circ F \) exhibits \( E \) as a localization of \( C \) with respect to \( W \).
- The morphism \( G \) is a categorical equivalence of simplicial sets.

**Proposition 5.3.1.18.** Let \( F : C \to D \) be a morphism of simplicial sets, where \( D \) is an ∞-category, and let \( W \) be the collection of all edges of \( C \). The following conditions are equivalent:

1. The morphism \( F \) exhibits \( D \) as a localization of \( C \) with respect to \( W \).
2. The ∞-category \( D \) is a Kan complex and \( F \) is a weak homotopy equivalence of simplicial sets.
Proof. We first prove that (2) implies (1). Assume that \( \mathcal{D} \) is a Kan complex and that \( F : \mathcal{C} \to \mathcal{D} \) is a weak homotopy equivalence; we wish to show that \( F \) exhibits \( \mathcal{D} \) as a localization of \( \mathcal{C} \) with respect to \( W \). By virtue of Proposition 5.3.1.12, it will suffice to show that for every \( \infty \)-category \( \mathcal{E} \), composition with \( F \) induces a homotopy equivalence of Kan complexes \( \theta : \text{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\cong} \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \). Since \( \mathcal{D} \) is a Kan complex, Proposition 4.4.3.16 allows us to identify \( \theta \) with the canonical map \( \text{Fun}(\mathcal{D}, \mathcal{E}) \cong \circ F \to \text{Fun}(\mathcal{C}, \mathcal{E}) \), which is a homotopy equivalence by virtue of our assumption that \( F \) is a weak homotopy equivalence.

We now show that (1) implies (2). Assume that \( F \) exhibits \( \mathcal{D} \) as a localization of \( \mathcal{C} \) with respect to \( W \). Invoking Remark 5.3.1.14, we deduce that \( F \) is a weak homotopy equivalence. We wish to show that \( \mathcal{D} \) is a Kan complex. Choose a weak homotopy equivalence \( G : \mathcal{D} \to \mathcal{E} \), where \( \mathcal{E} \) is a Kan complex (Corollary 3.1.6.2). Then the composite map \( (G \circ F) : \mathcal{C} \to \mathcal{E} \) is also a weak homotopy equivalence (Remark 3.1.5.16). Invoking the implication (2) \( \Rightarrow \) (1), we conclude that \( G \circ F \) exhibits \( \mathcal{E} \) as a localization of \( \mathcal{C} \) with respect to \( W \). It follows from Remark 5.3.1.17 that \( G \) is an equivalence of \( \infty \)-categories. Since \( \mathcal{E} \) is a Kan complex, it follows that the \( \infty \)-category \( \mathcal{D} \) is also a Kan complex (Remark 4.5.1.20). \( \square \)

5.3.2 Existence of Localizations

Our goal in this section is to prove the following:

Proposition 5.3.2.1 (Existence of Localizations). Let \( \mathcal{C} \) be a simplicial set and let \( W \) be a collection of edges of \( \mathcal{C} \). Then there exists an \( \infty \)-category \( \mathcal{D} \) and a morphism of simplicial sets \( F : \mathcal{C} \to \mathcal{D} \) which exhibits \( \mathcal{D} \) as a localization of \( \mathcal{C} \) with respect to \( W \).

Remark 5.3.2.2 (Uniqueness of Localizations). Let \( \mathcal{C} \) be a simplicial set and let \( W \) be a collection of edges of \( \mathcal{C} \). Proposition 5.3.2.1 asserts that there exists an \( \infty \)-category \( \mathcal{D} \) and a morphism \( F : \mathcal{C} \to \mathcal{D} \) which exhibits \( \mathcal{D} \) as a localization of \( \mathcal{C} \) with respect to \( W \). In this case, for every \( \infty \)-category \( \mathcal{E} \), composition with \( F \) induces a bijection

\[
\text{Hom}_{\text{hCat}_\infty}(\mathcal{D}, \mathcal{E}) = \pi_0(\text{Fun}(\mathcal{D}, \mathcal{E}) \cong) \to \pi_0(\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \cong)
\]

(Proposition 5.3.1.12). In other words, the \( \infty \)-category \( \mathcal{D} \) corepresents the functor

\[
\text{hCat}_\infty \to \text{Set} \quad \mathcal{E} \mapsto \pi_0(\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \cong).
\]

It follows that \( \mathcal{D} \) is uniquely determined (up to canonical isomorphism) as an object of the homotopy category \( \text{hCat}_\infty \). We will sometimes emphasize this uniqueness by referring to \( \mathcal{D} \) as the localization of \( \mathcal{C} \) with respect to \( W \), and denoting it by \( \mathcal{C}[W^{-1}] \). Beware that
the localization $\mathcal{C}[W^{-1}]$ is not well-defined up to isomorphism as a simplicial set: in fact, any equivalent $\infty$-category can also be regarded as a localization of $\mathcal{C}$ with respect to $W$ (Remark 5.3.1.17).

**Warning 5.3.2.3.** Let $\mathcal{C}$ be a simplicial set, let $W$ be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an $\infty$-category. We have now given two different definitions for the $\infty$-category $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$:

1. According to Notation 5.3.1.1, $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ denotes the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those diagrams $F : \mathcal{C} \to \mathcal{E}$ which carry each edge of $W$ to an isomorphism in $\mathcal{E}$.

2. By the convention of Remark 5.3.2.2, $\mathcal{C}[W^{-1}]$ denotes an $\infty$-category equipped with a diagram $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to $W$. We can then consider the $\infty$-category of functors from $\mathcal{C}[W^{-1}]$ to $\mathcal{E}$, which we will temporarily denote by $\text{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E})$.

Beware that these $\infty$-categories are not identical. However, they are equivalent: if $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to $W$, then composition with $F$ induces an equivalence of $\infty$-categories $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \to \text{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E})$ (Proposition 5.3.1.12). Note that the $\infty$-category $\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ does not depend on any auxiliary choices: it is well-defined up to equality as a simplicial subset of $\text{Fun}(\mathcal{C}, \mathcal{E})$. By contrast, the $\infty$-category $\text{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E})$ depends on the choice of the functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ (and is therefore well-defined up to equivalence, but not up to isomorphism).

Our proof of Proposition 5.3.2.1 will make use of the following:

**Lemma 5.3.2.4.** Let $Q$ be a contractible Kan complex, let $e : \Delta^1 \hookrightarrow Q$ be a monomorphism of simplicial sets, and let $W = \{\text{id}_{\Delta^1}\}$ consist of the single nondegenerate edge of $\Delta^1$. Then, for any $\infty$-category $\mathcal{E}$, precomposition with $e$ induces a trivial Kan fibration of simplicial sets

$$\theta : \text{Fun}(Q, \mathcal{E}) \to \text{Fun}(\Delta^1[W^{-1}], \mathcal{E}) = \text{Isom}(\mathcal{E}).$$

**Proof.** Since $e$ is a monomorphism, Corollary 4.4.5.3 immediately implies that $\theta$ is an isofibration when regarded as a functor from $\text{Fun}(Q, \mathcal{E})$ to $\text{Fun}(\Delta^1, \mathcal{E})$. Using the pullback diagram

$$
\begin{CD}
\text{Fun}(Q, \mathcal{E}) @>>> \text{Fun}(Q, \mathcal{E}) \\
@V\theta VV @V\theta VV \\
\text{Isom}(\mathcal{E}) @>>> \text{Fun}(\Delta^1, \mathcal{E}),
\end{CD}
$$

we have a pullback square of Kan fibrations, which implies that $\theta$ is a trivial Kan fibration.
we deduce that $\theta$ is also an isofibration when regarded as a functor from $\text{Fun}(Q, \mathcal{E})$ to $\text{Isom}(\mathcal{E})$. Consequently, to show that $\theta$ is a trivial Kan fibration, it will suffice to show that it is an equivalence of $\infty$-categories (Proposition 4.5.7.14). In other words, we are reduced to proving that the morphism $e$ exhibits $Q$ as a localization of $\Delta^1$ with respect to $W$. Let $q : Q \to \Delta^0$ denote the projection map. Since $Q$ is contractible, the morphism $q$ is an equivalence of $\infty$-categories. By virtue of Remark 5.3.1.17, we are reduced to proving that the composite map $\Delta^1 \xrightarrow{i} Q \xrightarrow{q} \Delta^0$ exhibits $\Delta^0$ as a localization of $\Delta^1$ with respect to $W$, which follows from Example 5.3.1.13.

We will deduce Proposition 5.3.2.1 from the following more precise result:

**Proposition 5.3.2.5.** Let $F : C \to D$ be a morphism of simplicial sets, where $D$ is an $\infty$-category. Let $W$ be a collection of edges of $C$ such that, for each $w \in W$, the image $F(w)$ is an isomorphism in $D$. Then $F$ factors as a composition

$$C \xrightarrow{G} [W^{-1}] \xrightarrow{H} D,$$

where $G$ exhibits $[W^{-1}]$ as a localization of $C$ with respect to $W$ and $H$ is an inner fibration (so that $[W^{-1}]$ is also an $\infty$-category). Moreover, this factorization can be chosen to depend functorially on the diagram $F : C \to D$ and the collection of edges $W$, in such a way that the construction $(F : C \to D, W) \mapsto [W^{-1}]$ commutes with filtered colimits.

**Proof.** For each element $w \in W$, the image $F(w)$ can be regarded as a morphism from $\Delta^1$ to the core $D^\simeq$. By virtue of Proposition 3.1.6.1, we can (functorially) choose a factorization of this morphism as a composition

$$\Delta^1 \xrightarrow{i_w} Q_w \xrightarrow{q_w} D^\simeq,$$

where $i_w$ is anodyne and $q_w$ is a Kan fibration. Since $D^\simeq$ is a Kan complex, $Q_w$ is also a Kan complex, which is contractible by virtue of the fact that $i_w$ is anodyne. Form a pushout diagram of simplicial sets

$$\begin{array}{ccc}
\coprod_{w \in W} \Delta^1 & \xrightarrow{i} & C \\
\downarrow \left(\coprod_{w \in W} i_w\right) & & \downarrow i \\
\coprod_{w \in W} Q_w & \xrightarrow{i} & C'.
\end{array}$$

We first claim that $i : C \to C'$ exhibits $C'$ as a localization of $C$ with respect to $W$. Let $\mathcal{E}$ be an $\infty$-category. Note that if $G : C \to \mathcal{E}$ is a morphism of simplicial sets which factors through $C'$, then for each $w \in W$ the morphism $G(w)$ belongs to the image of a functor $Q_w \to \mathcal{E}$, and is therefore an isomorphism in $\mathcal{E}$. It follows that composition with $i$ induces a
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functor \( \theta : \text{Fun}(\mathcal{C}', \mathcal{E}) \to \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \), and we wish to show that \( \theta \) is an equivalence of \( \infty \)-categories. This follows by inspecting the commutative diagram

\[
\array{\text{Fun}(\mathcal{C}', \mathcal{E}) & \xrightarrow{\theta} & \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) & \to & \text{Fun}(Q_w, \mathcal{E}) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{w \in W} \text{Fun}(Q_w, \mathcal{E}) & \xrightarrow{\theta'} & \prod_{w \in W} \text{Isom}(\mathcal{E}) & \to & \prod_{w \in W} \text{Fun}(\Delta^1, \mathcal{E}).}
\]

The outer rectangle is a pullback square by the definition of \( \mathcal{C}' \), and the right square is a pullback by the definition of \( \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \). It follows that the left square is also a pullback. Lemma [5.3.2.4] implies that \( \theta' \) is a trivial Kan fibration, so that \( \theta \) is also a trivial Kan fibration (hence an equivalence of \( \infty \)-categories by Proposition [4.5.2.9]).

Note that the morphism \( F : \mathcal{C} \to \mathcal{D} \) and the collection of morphisms \( \{ q_w : Q_w \to \mathcal{D} \subseteq \mathcal{D} \}_{w \in W} \) can be amalgamated to a single morphism of simplicial sets \( F' : \mathcal{C}' \to \mathcal{D} \). Applying Proposition [4.1.3.2], we can (functorially) factor \( F' \) as a composition \( \mathcal{C}' \xrightarrow{G'} \mathcal{C}[W^{-1}] \xrightarrow{H} \mathcal{D} \), where \( G' \) is inner anodyne and \( H \) is an inner fibration. We conclude by observing that the composite map \( G = (G' \circ i) : \mathcal{C} \to \mathcal{C}[W^{-1}] \) exhibits \( \mathcal{C}[W^{-1}] \) as a localization of \( \mathcal{C} \) with respect to \( W \), by virtue of Remark [5.3.1.17].

Proof of Proposition [5.3.2.1] Apply Proposition [5.3.2.5] in the special case \( \mathcal{D} = \Delta^0 \).}

5.3.3 Stability Properties of Localizations

Our goal in this section is to record some basic formal properties of the localization construction \( \mathcal{C} \mapsto \mathcal{C}[W^{-1}] \) introduced in §5.3.2. We first show that localization commutes with the formation of filtered colimits. More precisely, we have the following:

**Proposition 5.3.3.1.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a morphism of simplicial sets which is given as the colimit (in the arrow category \( \text{Fun}([1], \text{Set}_\Delta) \)) of a filtered diagram of morphisms \( \{ F_\alpha : \mathcal{C}_\alpha \to \mathcal{D}_\alpha \} \). Assume that:

- Each morphism \( F_\alpha \) exhibits \( \mathcal{D}_\alpha \) as a localization of \( \mathcal{C}_\alpha \) with respect to some collection of edges \( W_\alpha \).
- Each of the transition maps \( \mathcal{C}_\alpha \to \mathcal{C}_\beta \) of the diagram carries \( W_\alpha \) into \( W_\beta \).

Let us regard \( W = \lim W_\alpha \) as a collection of edges of the simplicial set \( \mathcal{C} \). Then \( F \) exhibits \( \mathcal{D} \) as a localization of \( \mathcal{C} \) with respect to \( W \).
Proof. Using Corollary 4.1.3.3, we can choose a compatible family of inner anodyne morphisms $G_\alpha : \mathcal{D}_\alpha \to \mathcal{E}_\alpha$, where each $\mathcal{E}_\alpha$ is an $\infty$-category. Set $\mathcal{E} = \lim_{\alpha} \mathcal{E}_\alpha$, so that the morphisms $G_\alpha$ determine a map of simplicial sets $G : \mathcal{D} \to \mathcal{E}$. Since each $G_\alpha$ is a categorical equivalence of simplicial sets, each of the composite maps $(G_\alpha \circ F_\alpha) : \mathcal{C}_\alpha \to \mathcal{E}_\alpha$ exhibits $\mathcal{E}_\alpha$ as a localization of $\mathcal{C}_\alpha$ with respect to $W_\alpha$. In particular, each of the morphisms $G_\alpha \circ F_\alpha$ carries edges of $W_\alpha$ to isomorphisms in the $\infty$-category $\mathcal{E}_\alpha$ (Remark 5.3.1.10). Applying Proposition 5.3.2.5, we can (functorially) factor each of the morphisms $G_\alpha \circ F_\alpha$ as a composition

$$\mathcal{C}_\alpha \xrightarrow{G'_\alpha} \mathcal{C}_\alpha[W^{-1}_\alpha] \xrightarrow{F'_\alpha} \mathcal{E}_\alpha,$$

where each $\mathcal{C}_\alpha[W^{-1}_\alpha]$ is an $\infty$-category, each of the morphisms $G'_\alpha$ exhibits $\mathcal{C}_\alpha[W^{-1}_\alpha]$ as a localization of $\mathcal{C}_\alpha$ with respect to $W_\alpha$, and the colimit map $G' : \mathcal{C} \to \lim_{\alpha} \mathcal{C}_\alpha[W^{-1}_\alpha]$ exhibits $\mathcal{C}[W^{-1}] = \lim_{\alpha} \mathcal{C}_\alpha[W^{-1}_\alpha]$ as a localization of $\mathcal{C}$ with respect to $W$. We then have a filtered diagram of commutative squares

$$\begin{array}{ccc}
\mathcal{C}_\alpha & \xrightarrow{F_\alpha} & \mathcal{D}_\alpha \\
\downarrow G'_\alpha & & \downarrow G_\alpha \\
\mathcal{C}[W^{-1}_\alpha] & \xrightarrow{F'_\alpha} & \mathcal{E}_\alpha
\end{array}$$

having colimit

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow G' & & \downarrow F' \\
\mathcal{C}[W^{-1}] & \xrightarrow{F'} & \mathcal{E}
\end{array}$$

Applying Remark 5.3.1.17, we deduce that each of the morphisms $F'_\alpha$ is a categorical equivalence of simplicial sets. Since the collection of categorical equivalences is stable under filtered colimits (Corollary 4.5.4.2), the morphism $F'$ is also a categorical equivalence of simplicial sets. Applying Remark 5.3.1.17 again, we deduce that $F' \circ G'$ exhibits $\mathcal{E}$ as a localization of $\mathcal{C}$ with respect to $W$. Since each $G_\alpha$ is a categorical equivalence, Corollary 4.5.4.2 also guarantees that $G$ is a categorical equivalence. Using the equality $G \circ F = F' \circ G'$ and applying Remark 5.3.1.17 again, we conclude that $F$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W$, as desired.

We now show that localization of compatible with the formation of categorical pushout squares.
Proposition 5.3.3.2. Suppose we are given a commutative diagram of simplicial sets

with the following properties:

(a) The back face

is a categorical pushout square of simplicial sets.

(b) The morphism of simplicial sets $F_{01} : \mathcal{C}_{01} \to \mathcal{D}_{01}$ exhibits $\mathcal{D}_{01}$ as a localization of $\mathcal{C}_{01}$ with respect to some collection of edges $W_{01}$.

(c) The morphism of simplicial sets $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ exhibits $\mathcal{D}_0$ as a localization of $\mathcal{C}_0$ with respect to some collection of edges $W_0$ containing $G(W_{01})$.

(d) The morphism of simplicial sets $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ exhibits $\mathcal{D}_1$ as a localization of $\mathcal{C}_1$ with respect to some collection of edges $W_1$ containing $H(W_{01})$.

Then the following conditions are equivalent:
(1) The front face

\[
\begin{array}{c}
D_{01} \rightarrow D_0 \\
\downarrow \downarrow \\
D_1 \rightarrow D
\end{array}
\]

is a categorical pushout square of simplicial sets.

(2) The morphism of simplicial sets \( F : C \rightarrow D \) exhibits \( D \) as a localization of \( C \) with respect to the collection of edges \( W = H'(W_0) \cup G'(W_1) \).

Proof. Let \( \mathcal{E} \) be an \( \infty \)-category. Assumption (a) guarantees that the diagram of Kan complexes

\[
\begin{array}{c}
\text{Fun}(\mathcal{C}, \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{C}_0, \mathcal{E})^\simeq \\
\downarrow \downarrow \\
\text{Fun}(\mathcal{C}_1, \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{C}_0, \mathcal{E})^\simeq
\end{array}
\]

is a homotopy pullback square. Applying Proposition 3.4.1.11, we deduce that the diagram of summands

\[
\begin{array}{c}
\text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{C}_0[W_0^{-1}], \mathcal{E})^\simeq \\
\downarrow \downarrow \\
\text{Fun}(\mathcal{C}_1[W_1^{-1}], \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{C}_0[W_0^{-1}], \mathcal{E})^\simeq
\end{array}
\]

is a homotopy pullback square. Applying Proposition 3.4.1.11, we deduce that the diagram of summands

\[
\begin{array}{c}
\text{Fun}(\mathcal{D}, \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{D}_0, \mathcal{E})^\simeq \\
\downarrow \downarrow \\
\text{Fun}(\mathcal{D}_1, \mathcal{E})^\simeq \rightarrow \text{Fun}(\mathcal{D}, \mathcal{E})^\simeq
\end{array}
\]

is a homotopy pullback square.

(2\(\mathcal{E} \)) Precomposition with \( F \) induces a homotopy equivalence of Kan complexes

\[
\text{Fun}(\mathcal{D}, \mathcal{E})^\simeq \xrightarrow{\circ F} \text{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^\simeq.
\]
We now observe that condition (1) is equivalent to the requirement that \( (1_E) \) holds for every \( \infty \)-category \( E \) (by definition), and condition (2) is equivalent to the requirement that \( (2_E) \) holds for every \( \infty \)-category \( E \) (Proposition 5.3.1.12). \( \square \)

### 5.3.4 The Category of Simplices

Our goal in this section is to show that, up to equivalence, every \( \infty \)-category can be realized as a localization \( C[W^{-1}] \), where \( C \) is (the nerve of) an ordinary category. More precisely, we have the following:

**Proposition 5.3.4.1.** Let \( D \) be an \( \infty \)-category. Then there exists an ordinary category \( C \), a collection \( W \) of morphisms in \( C \), and a functor \( F : N(C) \to D \) which exhibits \( D \) as a localization of \( N(C) \) with respect to \( W \).

We will deduce Proposition 5.3.4.1 from a more precise assertion (Theorem 5.3.4.4), which gives an explicit construction of the category \( C \) and the class of morphisms \( W \).

**Construction 5.3.4.2 (The Last Vertex Map).** Let \( S \) be a simplicial set, let \( \Delta_S \) denote the category of simplices of \( S \) (Construction 1.1.8.19), and let \( \tau \) be a \( k \)-simplex of the nerve \( N_\bullet(\Delta_S) \), which we identify with a commutative diagram of simplicial sets

\[
\begin{array}{cccccc}
\Delta^0 & \to & \Delta^1 & \to & \cdots & \to & \Delta^{k-1} & \to & \Delta^k \\
\sigma_0 & & \sigma_1 & & & & \sigma_{k-1} & & \sigma_k \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
& & & & & & & \downarrow & & \\
& & & & & & & \uparrow & & \\
& & & & & & & S, & & \\
\end{array}
\]

We let \( \lambda_\tau^+ : \Delta^k \to S \) denote the map given by the composition

\[
\Delta^k \xrightarrow{f} \Delta^{n_k} \xrightarrow{\sigma_k} X,
\]

where \( f \) carries each vertex \( \{i\} \subseteq \Delta^k \) to the image of the last vertex \( \{n_i\} \subseteq \Delta^{n_k} \) under the composite map \( \Delta^{n_i} \to \Delta^{n_{i+1}} \to \cdots \to \Delta^{n_k} \). The construction \( \tau \mapsto \lambda_\tau^+ \) determines a morphism of simplicial sets \( \lambda_\tau^+ : N_\bullet(\Delta_S) \to S \), which we will refer to as the last vertex map.

By construction, it carries each object \((|n|, \sigma) \in \Delta_S\) to the vertex \( \sigma(n) \in S \).

**Warning 5.3.4.3.** Let \( S \) be a simplicial set. We have now assigned two different meanings to the term “last vertex map”:

- The last vertex map \( \lambda_S \) of Construction 3.3.4.3, which is a morphism of simplicial sets from the subdivision \( \text{Sd}(S) \) to \( S \).
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- The last vertex map $\lambda^+_S$ of Construction 5.3.4.2, which is a morphism of simplicial sets from the nerve $N_\bullet(\Delta_S)$ to $S$.

These meanings are closely related. If the simplicial set $S$ is braced (Definition 3.3.1.1), then we can identify the subdivision $Sd(S)$ with the nerve of the full subcategory $\Delta^{nd}_S \subseteq \Delta_S$ spanned by the nondegenerate simplices of $S$ (Proposition 3.3.3.15). Under this identification, the last vertex map $\lambda_S$ of Construction 3.3.4.3 is given by the composition

$$Sd(S) \simeq N_\bullet(\Delta^{nd}_S) \hookrightarrow N_\bullet(\Delta_S) \xrightarrow{\lambda^+_S} S,$$

where $\lambda^+_S$ is the last vertex map of Construction 5.3.4.2 (Example 3.3.4.7).

Proposition 5.3.4.1 is an immediate consequence of the following:

**Theorem 5.3.4.4.** Let $S$ be a simplicial set, and let $W_S$ be the collection of all morphisms $([n], \sigma) \to ([n'], \sigma')$ in the category of simplices $\Delta_S$ for which the underlying map of linearly ordered sets $\alpha : [n] \to [n']$ satisfies $\alpha(n) = n'$. Then the last vertex map $\lambda^+_S : N_\bullet(\Delta_S) \to S$ exhibits $S$ as a localization of $N_\bullet(\Delta_S)$ with respect to $W_S$.

The rest of this section is devoted to the proof of Theorem 5.3.4.4. We begin by treating the situation where $S = \Delta^n$ is a standard simplex. In this case, we will use the following general observation:

**Proposition 5.3.4.5.** Let $F : C \to D$ be a functor of ∞-categories and let $W$ be a collection of morphisms of $C$. Suppose that there exists a functor $G : D \to C$ with the following properties:

(a) The functor $F$ carries each element of $W$ to an isomorphism in the ∞-category $D$.

(b) The composition $F \circ G$ is isomorphic to the identity functor $\text{id}_D$ (as an object of the ∞-category $\text{Fun}(D, D)$).

(c) There exists a natural transformation $u : \text{id}_C \to G \circ F$ with the property that, for every object $C \in C$, the morphism $u_C : C \to (G \circ F)(C)$ belongs to $W$.

Then $F$ exhibits $D$ as a localization of $C$ with respect to $W$.

**Proof.** It follows from (a) that, for every ∞-category $E$, precomposition with $F$ induces a map

$$\theta_E : \pi_0(\text{Fun}(D, E)^\sim) \to \pi_0(\text{Fun}(C[W^{-1}], E)^\sim)$$

By virtue of Proposition 5.3.1.12, it will suffice to show that $\theta_E$ is bijective for every ∞-category $E$. Let $\rho : \pi_0(\text{Fun}(C[W^{-1}], E)^\sim) \to \pi_0(\text{Fun}(D, E)^\sim)$ be the map given by precomposition with $G$. It follows from (b) that the composition $\rho \circ \theta_E$ is the identity. We
will complete the proof by showing that $\theta_E \circ \rho$ is also the identity. Let $H : C \to E$ be a functor which carries each element of $W$ to an isomorphism in $E$. By virtue of Theorem 4.4.4.4, $H$ carries the natural transformation $u$ to an isomorphism $H = H \circ \text{id}_C \to H \circ G \circ F$ in the $\infty$-category $\text{Fun}(C, E)$, so that the isomorphism class of $H$ is fixed by the composition $\theta_E \circ \rho$.

\hspace{1em} \square

**Corollary 5.3.4.6.** For each nonnegative integer $n$, the last vertex map

$$\lambda_{\Delta^n}^+ : N_\bullet(\Delta_{\Delta^n}) \to \Delta^n$$

exhibits $\Delta^n$ as a localization of $N_\bullet(\Delta_{\Delta^n})$ with respect to the collection of morphisms $W_{\Delta^n}$ appearing in the statement of Theorem 5.3.4.4.

**Proof.** Note that we can identify $\lambda_{\Delta^n}^+$ with a functor of ordinary categories $\Delta_{\Delta^n} \to [n]$, which carries each simplex $\sigma : \Delta^m \to \Delta^n$ to its last vertex $\sigma(m)$. This functor has a right adjoint $G : [n] \to \Delta_{\Delta^n}$, which carries each element $m \in [n]$ to the map $\sigma : \Delta^m \to \Delta^n$ which is the identity on vertices. We observe that the composition $\lambda_{\Delta^n}^+ \circ G$ is equal to the identity, and the unit transformation $u : \text{id}_{\Delta_{\Delta^n}} \to G \circ \lambda_{\Delta^n}^+$ carries each object of $\Delta_{\Delta^n}$ to a morphism which belongs to $W_{\Delta^n}$. Applying Proposition 5.3.4.5, we conclude that $\lambda_{\Delta^n}^+$ exhibits $\Delta^n$ as a localization of $N_\bullet(\Delta_{\Delta^n})$ with respect to $W_{\Delta^n}$. \hspace{1em} \square

We will prove Theorem 5.3.4.4 for a general simplicial set $S$ by writing $S$ as a colimit of simplices, for which the desired result holds by virtue of Corollary 5.3.4.6. For this, we need the following observation:

**Lemma 5.3.4.7.** The functor

$$\text{Set}_\Delta \to \text{Set}_\Delta \quad S \mapsto N_\bullet(\Delta_S)$$

commutes with the formation of colimits.

**Proof.** Fix an integer $n \geq 0$; we wish to show that the set-valued functor $S \mapsto N_n(\Delta_S)$ commutes with colimits. For every $n$-simplex $\tau$ of $N_\bullet(\Delta)$, let us write $N_n(\Delta_S)_\tau$ for the inverse image of $\tau$ under the natural map $N_n(\Delta_S) \to N_n(\Delta)$, so that we have a decomposition

$$N_n(\Delta_S) \simeq \coprod_{\tau \in N_n(\Delta)} N_n(\Delta_S)_\tau$$

depending functorially on $S$. It will therefore suffice to show that, for each $\tau$, the functor

$$\text{Set}_\Delta \to \text{Set} \quad S \mapsto N_n(\Delta_S)_\tau$$

commutes with colimits. We now observe that, if $\tau$ corresponds to a diagram $[m_0] \to [m_1] \to \cdots \to [m_n]$ in the category $\Delta$, then $N_n(\Delta_S)_\tau$ can be identified with the set of $m_n$-simplices of $S$. \hspace{1em} \square
Proof of Theorem 5.3.4.4. Let $S$ be a simplicial set and let $W_S$ be the collection of morphisms in $\Delta_S$ appearing in the statement of Theorem 5.3.4.4. We wish to show that the last vertex map $\lambda^+_S : N_\bullet(\Delta_S) \to S$ exhibits $S$ as a localization of $N_\bullet(\Delta_S)$ with respect to $W_S$. Using Lemma 5.3.4.7, we can realize $\lambda^+_S$ as a filtered colimit of morphisms $\lambda^+_{S_\alpha}$ (and $W_S$ with a filtered colimit of $W_{S_\alpha}$), where $\{S_\alpha\}$ is the collection of all finite simplicial subsets of $S$ (Remark 3.5.1.8). By virtue of Proposition 5.3.3.1, it will suffice to show that each $\lambda^+_{S_\alpha}$ exhibits $S_\alpha$ as a localization of $N_\bullet(\Delta_{S_\alpha})$ with respect to $W_{S_\alpha}$. We may therefore replace $S$ by $S_\alpha$, and thereby reduce to the case where the simplicial set $S$ is finite.

Since $S$ is a finite simplicial set, it has dimension $\leq n$ for some integer $n \geq -1$. We proceed by induction on $n$. If $n = -1$, then both $S$ and $N_\bullet(\Delta_S)$ are empty, and the desired result is clear. We may therefore assume that $n \geq 0$ and that Theorem 5.3.4.4 holds for all finite simplicial sets of dimension $< n$. We now proceed by induction on the number $m$ of nondegenerate $n$-simplices of $S$. If $m = 0$, then $S$ has dimension $\leq n - 1$ and the desired result holds by virtue of our first inductive hypothesis. We may therefore assume that $S$ has at least one nondegenerate $n$-simplex $\sigma : \Delta^n \to S$. Using Proposition 1.1.3.13, we see that there is a pushout diagram of simplicial sets

\[
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & \Delta^n \\
\downarrow & & \downarrow \\
S' & \rightarrow & S,
\end{array}
\]

where $S'$ is a simplicial set of dimension $\leq n$ with exactly $(m - 1)$-nondegenerate $m$-simplices. It follows from our inductive hypotheses together with Corollary 5.3.4.6 that the simplicial sets $\partial \Delta^n$, $\Delta^n$, and $S'$ satisfy the conclusion of Theorem 5.3.4.4. Moreover, the last vertex
map $\lambda_S^+$ fits into a commutative diagram

$$
\begin{array}{ccc}
N_\bullet(\Delta_{\partial\Delta^n}) & \longrightarrow & N_\bullet(\Delta_{\Delta^n}) \\
\downarrow^{\lambda^+_{\partial\Delta^n}} & & \downarrow^{\lambda^+_{\Delta^n}} \\
\partial\Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
N_\bullet(\Delta_{S'}) & \longrightarrow & N_\bullet(\Delta_S) \\
\downarrow^{\lambda^+_{S'}} & & \downarrow^{\lambda^+_S} \\
S' & \longrightarrow & S
\end{array}
$$

where the front and back faces are pushout squares (Lemma 5.3.4.7) in which the horizontal maps are monomorphisms, hence categorical pushout squares (Example 4.5.3.8). Since $W_S$ is the union of the images of $W_{S'}$ and $W_{\Delta^n}$, it follows from Proposition 5.3.3.2 that $\lambda_S^+$ exhibits $S$ as a localization of $N_\bullet(\Delta_S)$ with respect to $W_S$.

**Exercise 5.3.4.8.** Let $S$ be a simplicial set and let $\text{Sd}(S)$ denote the subdivision of $S$. Let $W$ be the collection of all edges $w$ of $\text{Sd}(S)$ with the following property: there exists a simplex $\sigma : \Delta^n \to S$ such that $w = \text{Sd}(\sigma)(\overline{w})$, where $\overline{w}$ is an edge of the subdivision $\text{Sd}(\Delta^n) \simeq N_\bullet(\text{Chain}[n])$ corresponding to a pair of subsets $\emptyset \neq P \subseteq Q \subseteq [n]$ satisfying $\max(P) = \max(Q)$. Show that the last vertex map $\lambda_S : \text{Sd}(S) \to S$ of Construction 3.3.4.3 exhibits $S$ as a localization of $\text{Sd}(S)$ with respect to $W$. 


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